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CYCLICAL PROPERTIES OF BAXTER-KING FILTERED TIME SERIES

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Abstract—This note demonstrates that the Baxter-King (1999) filter, and in general any bandpass filter, does not isolate the cycle in an unobserved-components model with a stochastic trend. The first difference of the trend passes through the filter, and as a result, the spectral properties of the filtered series depend on the trend in the unfiltered series. It is demonstrated that for postwar U.S. real GDP, the spectral properties of the BK-filtered series are primarily due to the stochastic trend in output.

I. Introduction

In a recent paper, Baxter and King (1999) propose an approximate bandpass filter to extract the business cycle component from a time series with deterministic or stochastic trends. This note analyzes the relationship between the output from the Baxter-King (BK) filter and the cycle in an unobserved-components model, where the cycle is the stationary deviation from a stochastic trend. I demonstrate that the BK filter, and more generally any bandpass filter, does not isolate the cycle when the trend component of the series to be filtered is integrated. The first difference of the trend passes through the filter, and as a result, the spectral properties of the BK-filtered series depend on the trend in the unfiltered series.


This note is organized as follows. Section II discusses alternative definitions of the business cycle. I define the business cycle as stationary deviations from a stochastic trend. In Section III, I demonstrate that the spectrum of a BK-filtered unobserved-components model comprises three components: one due to the stochastic trend, one due to the cycle, and a covariance term. In addition, the BK filter assigns a higher weight to the trend component than it does to the cyclical component in determining the spectral power of the filtered series. Section IV summarizes and offers concluding remarks.

II. Defining the Business Cycle

Hodrick and Prescott (1980), Baxter and King (1999), and others define the business cycle as stationary deviations from a stochastic trend. In this note, I demonstrate that the spectrum of a BK-filtered unobserved-components model assigns a higher weight to the trend component than it does to the cyclical component in determining the spectral power of the filtered series.

An alternative definition of the business cycle relies on unobserved components (UC) view of output. In this case, (the log of) output is the sum of an unobserved trend and cycle:

\[ y_t = \tau_t + c_t, \]

where \( \tau_t \) is the nonstationary trend and \( c_t \) is the stationary cycle around this trend. The UC model has a rich history in econometrics, and has been analyzed by Harvey (1985), Watson (1986), Clark (1987), Harvey and Jaeger (1993), and Morley, Nelson, and Zivot (2003) among others. See Cogley (2001) for additional discussion on alternative definitions of the business cycle.

In this note, I adopt the UC definition of the business cycle. The purpose of this study is to ascertain whether or not the BK filter can isolate the cyclical component of an integrated series, where the cycle is defined as stationary deviations from trend. In other words, how closely does the output from passing \( y_t \) through the BK filter resemble \( c_t \) over a particular frequency band?

III. Cylcical Properties of Baxter-King-Filtered Time Series

A. The Baxter-King Filter

The BK filter is an approximation to an ideal bandpass filter. This ideal filter has the following two-sided infinite-moving-average representation:

\[ a(L) = \sum_{k=-\infty}^{\infty} a_k L^k, \]

where symmetry \((a_k = a_{-k})\) is imposed so that the filter does not induce a phase shift. The transfer function of a filter determines the extent to which periodic components of the filtered series are related to periodic components of the underlying (unfiltered) series. The BK filter is designed to pass the stationary component of output whose periodicity ranges from 1.5 years to 8 years per cycle. For stationary time series, the transfer function of this ideal filter takes the form:

\[ \alpha(\omega) = \begin{cases} 1 & \text{if } \pi/16 \leq |\omega| \leq \pi/3, \\ 0 & \text{otherwise}. \end{cases} \]

This ideal filter is not feasible, as it requires an infinite amount of data. Baxter and King employ the following truncated version of the ideal filter, which is the optimal approximation:

\[ a_k(L) = \sum_{k=-K}^{K} a_k L^k. \]

This approximate bandpass filter, with corresponding transfer function \( \alpha_k(\omega) \), sacrifices \( 2K \) data points.

\(^1\)At quarterly frequencies, the desired band is 6 quarters to 32 quarters per cycle. Since \( \omega = 2\pi/P \), this translates into a frequency band of \( \pi/16 \leq |\omega| \leq \pi/3. \)

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The BK filter is designed so that it renders trending series stationary. This is achieved by constraining the frequency response of the filter to be zero at zero frequency. The BK filter may thus be factored as

$$ a_k(L) = -(1 - L)(1 - L^{-1})\psi_k(L) $$

$$ = L^{-1}(1 - L)^2\psi_k(L), $$

where

$$ \psi_k(L) = \sum_{h=-/(k-1)}^{K-1} \psi_h L^h $$

and the coefficients of $\psi_k(L)$ are given by

$$ \psi_{h0} = \sum_{j=|h|+1}^{K} (j - |h|)a_j. $$

Since the BK filter contains two differencing operators, it removes linear and quadratic time trends, and up to two unit roots.

**B. Properties of BK-Filtered Series When the Trend Is Integrated**

Consider the following unobserved-components model:

$$ y_t = \tau_t + c_t, $$

$$ \tau_t = \mu + \tau_{t-1} + \eta_t, $$

where $c_t$ and $\eta_t$ are both stationary processes. It is instructive to factor the BK filter as follows:

$$ a_k(L) = (1 - L)b_k(L), $$

where $b_k(L) = -(1 - L^{-1})\psi_k^{-1}(L).$

Define $x_t$ as the output from passing $y_t$ through the BK filter:

$$ x_t = a_k(L)y_t. $$

For the above UC model, the BK-filtered series is

$$ x_t = a_k(L)\tau_t + a_k(L)c_t = b_k(L)\eta_t + a_k(L)c_t. $$

The appearance of $a_k(L)c_t$ in the above expression demonstrates that the BK filter approximately passes the UC cycle through business-cycle frequencies. However, the BK filter also allows the first difference of the trend, $\eta_t$, to pass through. In addition, the periodic components of $\eta_t$ and $c_t$ are passed through with different weights. When the BK filter is applied to an integrated process, it passes the trend, rendering it stationary, and then filters the resulting stationary series. Transforming the series “uses up” one of the difference operators in $a_k(L)$, and the asymmetric filter $b_k(L)$ is then applied to the first difference of the trend. Therefore, although the BK filter removes unit roots, it does not remove stochastic trends.

The spectrum of the BK filtered series is

$$ f_\omega(\omega) = |\beta_\omega(\omega)|^2f_\omega(\omega) + |\alpha_\omega(\omega)|^2f_\omega(\omega) + 2|\alpha_\omega(\omega)|^2 \Re \left[ \frac{f_\omega(\omega)}{\Delta(\omega)} \right], $$

where $\beta_\omega(\omega)$ is the transfer function of $b_k(L)$; $f_\omega(\omega)$ and $f_\omega(\omega)$ are the spectra of $\eta_t$ and $c_t$, respectively; $f_\omega(\omega)$ is the cross-spectrum between $\eta_t$ and $c_t$, and $\Delta(\omega)$ is the transfer function of the difference operator. The last term in the above expression can be further simplified as

$$ 2|\alpha_\omega(\omega)|^2 \Re \left[ \frac{f_\omega(\omega)}{\Delta(\omega)} \right] = |\alpha_\omega(\omega)|^2 c_{\omega}(\omega) $$

$$ - \frac{\sin \omega}{1 - \cos \omega} |\alpha_\omega(\omega)|^2 q_{\omega}(\omega), $$

where $c_{\omega}(\omega)$ and $q_{\omega}(\omega)$ are the cospectra and quadrature spectra between $\eta_t$ and $c_t$, respectively.

The periodicity displayed by the BK-filtered UC model thus arises from three sources: the first difference of the stochastic trend, the UC cycle, and their covariance. However, in most of the existing literature, such as Harvey (1985) and Clark (1987), the trend and cycle are assumed to be uncorrelated. In this case, the term involving the cross-spectrum vanishes.

The extent to which the periodic behavior present in $x_t$ reflects $c_t$ and $\eta_t$ is determined by $|\alpha_\omega(\omega)|^2$ [the squared gain of $a_k(L)$] and $|\beta_\omega(\omega)|^2$ [the squared gain of $b_k(L)$], respectively. To quantify the influence of the stochastic trend and the UC cycle in determining the spectral power of a BK-filtered unobserved-components model, figure 1 plots $|\alpha_\omega(\omega)|^2$ (the solid line) and $|\beta_\omega(\omega)|^2$ (the dashed line) for various values of $K$. In addition, the frequency band of 1.5 years to 8 years per cycle is shaded.

We can clearly see the extent to which the behavior present in $x_t$ depends on the underlying stochastic trend and the UC cycle. For each value of $K$, the BK filter ascribes a higher frequency response to $\eta_t$ than to $c_t$. Furthermore, the contribution of $\eta_t$ to the spectral power of $x_t$ increases as more data are sacrificed. Indeed, when 20 years of quarterly data are sacrificed ($K = 40$), $|\alpha_\omega(\omega)|^2$ is dwarfed by $|\beta_\omega(\omega)|^2$.

Therefore, when the trend component of the underlying series is integrated and is uncorrelated with the UC cycle, the BK filter will overstate the importance of transitory dynamics at business cycle frequencies.

We note that when the trend and cycle are correlated, it is possible for the BK filter to underestimate transitory variation at business cycle frequencies. This will occur if

$$ |\beta_\omega(\omega)|^2 f_\omega(\omega) + 2|\alpha_\omega(\omega)|^2 \Re \left[ \frac{f_\omega(\omega)}{\Delta(\omega)} \right] < 0. $$

Whether or not this condition holds will depend on the properties of $f_\omega(\omega)$ and $f_\omega(\omega)$, as well as $|\alpha_\omega(\omega)|^2$ and $|\beta_\omega(\omega)|^2$.

**C. BK-Filtered Postwar Quarterly U.S. Real GDP**

As an illustration of the potential for the BK filter to overstate transitory dynamics in practice, we employ a parameterization taken from Morley, Nelson, and Zivot (2003). They estimate the following stochastic-trend-plus-cycle model for the postwar quarterly U.S. real GDP, 1947.1–1998.2:

$$ y_t = \tau_t + c_t, $$

$$ \tau_t = 0.82 + \tau_{t-1} + \eta_t, \quad \sigma_\tau = 1.24, $$

$$ (1 - 1.34L + 0.71L^2)c_r = \epsilon_r, \quad \sigma_c = 0.75, $$

where

$$ L = \left( \begin{array}{c} 1 & \frac{1}{1} \\ -1 & \frac{1}{1} \end{array} \right), $$

and $E(\eta_r) = \sigma_\eta = -0.84$. This parameterization is particularly convenient in that $c_r$ has 84% of its spectral power in the BK band.

Figure 2 plots the three spectral components of $x_t$, the BK-filtered series, from this data-generating process. $|\alpha_\omega(\omega)|^2 f_\omega(\omega)$ is represented...
by the solid line, $|\alpha_L(\omega)|^2$ by the dashed line, and $2|\alpha_L(\omega)|^2 \text{Re}(f_0(\omega)/(\Delta(\omega)))$ by the solid-and-dashed line. Again, the contribution of the first difference of the trend in determining the spectral power of the BK-filtered series increases with $K$. Also, since the trend and cycle are negatively correlated, the cross-spectral term is negative. This demonstrates the potential to mitigate the presence of the term involving the stochastic trend. Notice however, that in this case, $2|\alpha_L(\omega)|^2 \text{Re}(f_0(\omega)/(\Delta(\omega)))$ and $|\alpha_L(\omega)|^2 f_0(\omega)$ nearly cancel each other, so that the spectral power of the BK filtered UC model is almost entirely determined by the first difference of the stochastic trend.
IV. Summary and Concluding Remarks

This note analyzes the relationship between the output of the BK filter and the cycle in an unobserved components model. I demonstrate that the filter does not isolate the cycle, but rather passes the first difference of the trend through to the filtered series. Furthermore, the weight that the BK filter assigns to the first difference of the trend in determining the spectral power of the BK-filtered series is much larger than the weight it assigns to the UC cycle. This illustrates the potential for the BK filter to overstate the importance of transitory dynamics at business cycle frequencies. An empirical example using postwar...
quarterly U.S. real GDP demonstrates the importance of this phenomenon in practice. The analysis presented in this note is not specific to the BK filter, but indeed applies to all bandpass filters. The simple act of differentiating does not remove a stochastic trend; it merely renders it stationary. Therefore, although bandpass filtering can render an integrated series stationary, the properties of the filtered series will depend on the trend in the unfiltered series.

REFERENCES


THE GENERALIZED COMPOSITE COMMODITY THEOREM: STRONGER SUPPORT IN THE PRESENCE OF DATA LIMITATIONS

George C. Davis*

Abstract—Because of common data limitations, the existing testing framework for the generalized composite commodity theorem (Lewbel, 1996) is incomplete. This note clarifies and strengthens the testing procedure by implementing modified Bonferroni procedures. The conditions are established for consistency between the existing and modified Bonferroni tests. In an empirical application, the Bonferroni tests provide more powerful support for the generalized composite commodity theorem than is obtained from the existing test.

I. Introduction

One of the most problematic areas in all of economics is the consistent aggregation of commodities. For over fifty years there have existed two solutions: the composite commodity theorem (Hicks, 1936; Leontief, 1936) and the separability theorem (Leontief, 1947; Sono, 1961). Rejections of both are common. Recently, Lewbel (1996) proposed a generalization of the Hicks-Leontief composite commodity theorem that is appealing for several reasons. First, it imposes no restrictions on preferences. Second, it allows prices to be highly but imperfectly collinear, a fact frequently observed. Three, it imposes no restrictions on preferences. Second, it allows prices to be highly but imperfectly collinear, a fact frequently observed. Four, in contrast to other aggregation procedures, the early empirical evidence is very encouraging and supports the generalized composite commodity theorem (Davis, Lin, and Shumway, 2000; Lewbel, 1996). Lewbel’s contribution offers the hope of finally basing commodity aggregation on an empirically valid foundation. However, though theoretically attractive, a complete test of the theory has not been conducted and is more difficult than first appears.

The purpose of this note is to clarify the test of the generalized composite commodity theorem and to strengthen the testing in the common situation of short nonstationary time series data. The clarification points out that a complete test of the theory implies using a multivariate cointegration procedure that is simply not reliable given common data limitations. The strengthening is achieved by using multiple applications of Lewbel’s simple test in conjunction with modified Bonferroni procedures. This approach leads to a demonstration that the simple test is a special case of the more powerful Bonferroni procedures. The conditions under which the inferences will be the same are given.

The next section gives a brief review of the generalized composite commodity theorem. The following section discusses how data limitations prevent the theorem from being tested completely with standard procedures. More complete and powerful modified Bonferroni testing procedures are then discussed. The empirical section then applies the modified Bonferroni tests to Lewbel’s data set and finds, with two exceptions, even stronger support for the generalized composite commodity theorem. The note closes with conclusions.

II. The Generalized Composite Commodity Theorem

Drawing freely from Lewbel (1996, pp. 526–527), let \( p_i \) and \( w_i \) be individual prices and budget shares of individual goods \( i = 1, 2, \ldots, n \), and \( p \) and \( w \) be the corresponding vectors. Similarly, let \( P \) and \( W \) represent vectors of group price indices \( P_j \) and group budget shares \( W_j = \sum_{i \in J} w_i \), where \( J \) indexes groups of goods or commodities \( J = \{ 1, 2, \ldots, N \} \) and \( N < n \). Each good \( i \) is an element of a group \( J \). Furthermore, define \( r = \log p_i \), \( R_j = \log P_j \), and let \( r \) and \( R \)