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New Examples of $\mathbb{Z}_2$-symmetric Spinor Models in Two Dimensions

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Abstract. New examples of integrable spinor models are derived and analyzed. They generalize the well known ones of: i) Nambu–Jona-Lasinio–Vaks–Larkin models, related to $SU(N)$; ii) the Gross–Neveu models – $SP(2N,\mathbb{R})$; and the iii) Zakharov–Mikhailov models – $SO(N,\mathbb{R})$. We propose a method for constructing their Lax representations and comment on the spectral properties of their Lax operators.

INTRODUCTION

It is well known that the spinor models derived by Nambu–Jona-Lasinio–Vaks–Larkin [12, 14] and Gross–Neveu [10, 13] have been proposed initially as models for describing the strong interactions of elementary particles. Later [17] it was proven that the two-dimensional versions of these models are integrable by the inverse scattering method (ISM) [15, 2]. In the same paper [17] Zakharov and Mikhailov proposed a third class of spinor models related to the orthogonal groups.

The aim of the present report is to extend the results of [4, 5]. Namely we derive new types of integrable spinor models by applying additional $\mathbb{Z}_2$-symmetries to their Lax representations. In doing this we will be using Mikhailov's reduction group [11].

We start in Section 2 by some preliminaries concerning the spinor models and the reduction group of Mikhailov [11]. In Section 3 we remind the derivation of the type 1 $\mathbb{Z}_2$-symmetric spinor models see [4, 5]. Sections 4–6 are devoted to type 2 $\mathbb{Z}_2$-symmetric spinor models of NJLVL, GN and ZM models respectively. We end by discussions and conclusions.

PRELIMINARIES

The integrability of the 2-dimensional versions of the Nambu–Jona-Lasinio–Vaks–Larkin (NJLVL) and the Gross–Neveu model (GN) was discovered by Zakharov and Mikhailov in [17]. The showed that NJLVL models are related to $su(N)$ algebras, while the Gross–Neveu models are related to the $sp(N)$. In the same paper an additional type of spinor models related to the algebras $so(N)$ was discovered; we will call them Zakharov–Mikhailov (ZM) models.

Let us first recall the Lax representations of these models [17] using a bit more general form.

$$\begin{aligned} \Psi_\xi &= U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), & \Psi_\eta &= U(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \\ U(\xi,\eta,\lambda) &= \frac{U_1(\xi,\eta)}{\lambda-a}, & V(\xi,\eta,\lambda) &= \frac{V_1(\xi,\eta)}{\lambda-b}, \end{aligned} \tag{1}$$

where $\eta = t + x$, $\xi = t - x$ and a and b are real numbers. The well known representation in [17] is obtained by putting $b = -a$. We also impose the $\mathbb{Z}_2$-symmetry:

$$U^\dagger(x,t,\lambda) = -U(x,t,\lambda^*), \qquad V^\dagger(x,t,\lambda) = -V(x,t,\lambda^*). \tag{2}$$

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The compatibility condition of the above linear problems reads:

$$\frac{\partial U}{\partial \eta} - \frac{\partial V}{\partial \xi} + [U, V] = 0, \tag{3}$$

which is equivalent to

$$\frac{\partial U_1}{\partial \eta} + \frac{1}{a-b}[U_1, V_1(\xi,\eta)] = 0, \qquad \frac{\partial V_1}{\partial \xi} - \frac{1}{a-b}[U_1, V_1(\xi,\eta)] = 0. \tag{4}$$

Fixing up properly the gauge, (see [17]) there follows that we can choose U_1 and V_1 in the form:

$$U_1(\xi,\eta) = -i\phi J_1^0 \phi^{-1}, \qquad V_1(\xi,\eta) = i\psi I_1^0 \psi^{-1}, \tag{5}$$

where J_1^0 and I_1^0 are properly chosen constant elements (choice of the gauge) of the corresponding simple Lie algebra $\mathfrak{g} \simeq su(N)$. In what follows we fix up $J_1^0 = -I_1^0 = J$ and choose J for each of the above mentioned models accordingly. The matrix valued functions $\phi(\xi,\eta)$ and $\psi(\xi,\eta)$ take values in the corresponding simple Lie group $SU(N)$ and are solutions of the following ODE's:

$$i\frac{\partial \psi}{\partial \xi} + \frac{\phi J\hat{\phi}(\xi,\eta)}{a-b}\psi(\xi,\eta) = 0, \qquad i\frac{\partial \phi}{\partial \eta} - \frac{\psi J\hat{\psi}(\xi,\eta)}{a-b}\phi(\xi,\eta) = 0. \tag{6}$$

Here and below by 'hat' we will denote the inverse matrix, i.e. $\hat{\psi} \equiv (\psi)^{-1}$.

In this way we get three classes of spinor models. Below, following [17] we briefly outline their derivation.

Nambu-Jona-Lasinio-Vaks-Larkin models

Here we choose $\mathfrak{g} \simeq su(N)$. Then $\psi(\xi,\eta)$ and $\phi(\xi,\eta)$ are elements of the group $SU(N)$ and by definition $\hat{\psi}(\xi,\eta) = \psi^\dagger(\xi,\eta)$, $\hat{\phi}(\xi,\eta) = \phi^\dagger(\xi,\eta)$. We choose $J = \mathrm{diag}\,(1,0,\dots,0)$ and as a result only the first columns ϕ, ψ and the first rows $\hat{\phi}$, $\hat{\psi}$ enter into the systems (6). If we introduce the notations:

$$\phi^\alpha(\xi,\eta) = \phi_{\alpha,1}, \qquad \psi^\alpha(\xi,\eta) = \psi_{\alpha,1}, \tag{7}$$

then the explicit form of the system is:

$$i\frac{\partial \phi^\alpha}{\partial \eta} - \frac{i}{a-b}\psi^\alpha \sum_{\beta=1}^{N} \psi^{*,\beta}\phi^\beta = 0, \qquad i\frac{\partial \psi^\alpha}{\partial \xi} + \frac{i}{a-b}\phi^\alpha \sum_{\beta=1}^{N} \phi^{*,\beta}\psi^\beta = 0. \tag{8}$$

The corresponding action functional is:

$$A_{\mathrm{NJLVL}} = \int_{-\infty}^{\infty} dx\, dt \left(i\sum_{\alpha=1}^{N}\left(\phi^{*,\alpha}\frac{\partial \phi^\alpha}{\partial \eta} + \psi^{*,\alpha}\frac{\partial \psi^\alpha}{\partial \xi}\right) - \frac{1}{a-b}\left|\sum_{\alpha=1}^{N}(\psi^{*,\alpha}\phi^\alpha)\right|^2 \right). \tag{9}$$

Gross-Nevew models

Here we choose $\mathfrak{g} \simeq sp(2N,\mathbb{R})$; then $\psi(\xi,\eta)$ and $\phi(\xi,\eta)$ are elements of the group $\mathfrak{G} \simeq SP(2N,\mathbb{R})$. Following [17] we use the standard definition of symplectic group elements: We introduce $U_1(\xi,\eta), V_1(\xi,\eta) \in sp(2N,\mathbb{R})$ by:

$$U_1(\xi,\eta) = \phi U_1^0 \phi^{-1}(\xi,\eta), \qquad V_1(\xi,\eta) = \psi U_1^0 \psi^{-1}(\xi,\eta), \qquad U_1^0 = V_1^0 = E_{1,N+1}. \tag{10}$$

$$U_1^T = -\mathcal{J}U_1\hat{\mathcal{J}}, \qquad V_1^T = -\mathcal{J}V_1\hat{\mathcal{J}}, \qquad \hat{\phi} = \mathcal{J}\phi^T\hat{\mathcal{J}}, \qquad \hat{\psi} = \mathcal{J}\psi^T\hat{\mathcal{J}}, \qquad \mathcal{J} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \tag{11}$$

Then the corresponding Lie algebraic elements acquire the following block-matrix structure:

$$U_1(\xi,\eta) = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \tag{12}$$

where A, B, C are real $N \times N$ matrices such that $B = B^T$ and $C = C^T$. As a consequence again only the first columns $\phi^{(1)}$, $\psi^{(1)}$ and the first rows $\hat{\phi}^{(1)}$, $\hat{\psi}^{(1)}$ enter into the systems (6). If we introduce the N-component complex vectors:

$$\phi^\alpha(\xi,\eta) = \frac{i}{c}(\phi_{\alpha,1} + i\phi_{N+\alpha,1}), \qquad \psi^\alpha(\xi,\eta) = \frac{1}{c}(\psi_{\alpha,1} + i\psi_{N+\alpha,1}) \tag{13}$$

then the explicit form of the system is:

$$i\frac{\partial\phi^\alpha}{\partial\eta} = \frac{c^2}{2(a-b)}\psi^\alpha \sum_{\beta=1}^{N}(\psi_\beta\phi^{*,\beta} + \psi^{*,\beta}\phi_\beta), \qquad i\frac{\partial\psi^\alpha}{\partial\xi} = \frac{c^2}{2(a-b)}\phi^\alpha \sum_{\beta=1}^{N}(\phi^\beta\psi^{*,\beta} + \phi^{*,\beta}\psi^\beta). \tag{14}$$

The functional of the action is:

$$A_{\rm GN} = \int_{-\infty}^{\infty} dx\, dt\, \left(i\sum_{\alpha=1}^{N}\left(\phi_\alpha^*\frac{\partial\phi_\alpha}{\partial\eta} + \psi_\alpha^*\frac{\partial\psi_\alpha}{\partial\xi}\right) - \frac{c^2}{2(a-b)}\left(\sum_{\alpha=1}^{N}(\psi_\alpha^*\phi_\alpha + \phi_\alpha^*\psi_\alpha)\right)^2\right). \tag{15}$$

Zakharov–Mikhailov models

The ZM models are related to the $SO(N,\mathbb{R})$ group and $U_1(\xi,\eta)$ and $V_1(\xi,\eta)$ are taking values in the algebra $so(N,\mathbb{R})$. Therefore

$$U_1(\xi,\eta) = \phi U_1^0\phi^{-1}(\xi,\eta), \qquad V_1(\xi,\eta) = \psi U_1^0\psi^{-1}(\xi,\eta), \qquad U_1^0 = V_1^0 = E_{1,N} - E_{N,1}. \tag{16}$$

Obviously

$$U_1^T = -U_1, \qquad \phi^{-1} = \phi^T, \qquad V_1^T = -V_1, \qquad \psi^{-1} = \psi^T. \tag{17}$$

Here the $N \times N$ matrices E_{kp} are defined by $(E_{kp})_{nm} = \delta_{kn}\delta_{pm}$. As a consequence now the first and the last columns $\phi^{(1)}, \phi^{(N)}$, $\psi^{(1)}, \psi^{(N)}$ and the first and the last rows $\hat{\phi}^{(1)}, \hat{\phi}^{(N)}$, $\hat{\psi}^{(1)}, \hat{\psi}^{(N)}$ enter into the systems (6). If we introduce the N-component complex vectors:

$$\phi^\alpha(\xi,\eta) = \frac{1}{c}(\phi_{\alpha,1} + i\phi_{\alpha,N}), \quad \psi^\alpha(\xi,\eta) = \frac{1}{c}(\psi_{\alpha,1} + i\psi_{\alpha,N}) \tag{18}$$

then the explicit form of the system becomes:**vectors**

$$i\frac{\partial\psi^\alpha}{\partial\xi} = \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}(\phi^\alpha\phi^{*,\beta} - \phi^{*,\alpha}\phi^\beta)\psi^\beta), \qquad i\frac{\partial\phi^\alpha}{\partial\eta} = \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}(\psi^\alpha\psi^{*,\beta} - \psi^{*,\alpha}\psi^\beta)\phi^\beta, \tag{19}$$

This system has as action functional

$$A_{\rm ZM} = \int_{-\infty}^{\infty} dx\, dt\, \left(i\sum_{\alpha=1}^{N}\left(\phi^{*,\alpha}\frac{\partial\phi^\alpha}{\partial\eta} + \psi^{*,\alpha}\frac{\partial\psi^\alpha}{\partial\xi}\right) - \frac{c^2}{2(a-b)}\left(\sum_{\alpha,\beta=1}^{N}(\phi^{*,\alpha}\phi^\beta - \phi^{*,\beta}\phi^\alpha)(\psi^{*,\alpha}\psi^\beta - \psi^{*,\beta}\psi^\alpha)\right)\right). \tag{20}$$

For more details of deriving the models see [17].

Mikhailov's $\mathbb{Z}_2$ reductions

Each $\mathbb{Z}_2$ -element of Mikhailov's reduction group can be realized as follows:

$$\begin{array}{lll}
1) & A_1U^\dagger(x,t,f(\lambda^*))A_1^{-1} = -U(x,t,\lambda), & A_1V^\dagger(x,t,f(\lambda^*))A_1^{-1} = -V(x,t,\lambda), \\
2) & A_2U^T(x,t,f(\lambda))A_2^{-1} = -U(x,t,\lambda), & A_2V^T(x,t,f_2(\lambda))A_2^{-1} = -V(x,t,\lambda), \\
3) & A_3U^*(x,t,f(\lambda^*))A_3^{-1} = U(x,t,\lambda), & A_3V^*(x,t,f(\lambda^*))A_3^{-1} = V(x,t,\lambda), \\
4) & A_4U(x,t,f(\lambda))A_4^{-1} = U(x,t,\lambda), & A_4V(x,t,f_2(\lambda))A_4^{-1} = V(x,t,\lambda),
\end{array} \tag{21}$$

where A_k, $k = 1, \dots, 4$ are involutive automorphisms of the relevant Lie algebra. Besides $f_1(\lambda^*)$ and $f_2(\lambda)$ are $\mathbb{Z}_2$ elements of the group of conformal transformation of the complex λ-plane. Another constraint on $f_1(\lambda^*)$ comes from the natural requirement that it preserves the analyticity properties of the Lax pair in λ. Typical examples of such transformations are:

$$\text{Type 1} \qquad f(\lambda) = \frac{\epsilon}{\lambda}, \qquad \epsilon = \pm 1. \qquad\qquad \text{Type 2} \qquad f(\lambda) = -\lambda, \tag{22}$$

The consequences of these reductions and the constraints they impose on the spectral data of the Lax operators are well known, see [15, 3, 7, 6].

In what follows we will split the $\mathbb{Z}_2$-reductions into two types according to the realization of the $\mathbb{Z}_2$-group as a conformal map on the complex λ-plane. In [4, 5] the first type of such $\mathbb{Z}_2$-reductions, which involve the mappings $\lambda \to \lambda^{-1}$ or $\lambda \to \lambda^{*,-1}$ were analyzed. Such transformations affects substantially also the spectral properties of the relevant Lax operators, see [4, 5]. .

Remark 1 *For generic potentials $U_1(\xi, \eta)$ and $V_1(\xi, \eta)$ all four reductions (21.1)–(21.4) lead to inequivalent reduced systems. However let us assume that $U_1(\xi, \eta)$ and $V_1(\xi, \eta)$ are invariant with respect to one of the external automorphism, e.g.*

$$U_1 = -B_1 U_1^T B_1^{-1}, \qquad V_1 = -B_1 V_1^T B_1^{-1}, \qquad B_1^2 = \mathbb{1}, \tag{23}$$

then the reductions (21.3) and (21.4) will be equivalent to (21.1) and (21.2) respectively. Similarly, if

$$U_1^\dagger = -C_1 U_1^T C_1^{-1}, \qquad V_1^\dagger = -C_1 V_1^T C_1^{-1}, \qquad C_1^2 = \mathbb{1}, \tag{24}$$

then the reductions (21.1) and (21.3) will be equivalent to (21.2) and (21.4) respectively.

Since each of the examples below satisfies either (23) or (24) it will be enough to study at most two inequivalent reductions for each of them.

Remark 2 *We also assume that the constants a and b are real. For the unreduced cases with $b = -a$ this is no restriction in view of the simple change $\lambda \to \lambda e^{\arg a}$. However such change can not be used when we apply one of the reductions (21,1) or (21,3).*

Type 1 $\mathbb{Z}_2$-SYMMETRIC OF THE SPINOR MODELS [4, 5]

In this section we formulate the results in [4, 5] where we treated the Lax pair (1) with $b = -a$. In addition we want to avoid poles of second order in the compatibility condition. This means, that following [4, 5] we can combine the $\mathbb{Z}_2$-reductions in (21) only with type-1 transformations on the complex λ-plane: $\lambda \to \epsilon\lambda^{-1}$ and $\lambda \to \epsilon\lambda^{*,-1}$.

Using the ideas of the reduction group [11] below we construct new spinor models in two dimensions generalizing the ones in [17]. We start with the Lax representation:

$$\begin{aligned} \Psi_\xi &= U_{\rm R}(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \qquad U_{\rm R}(\xi, \eta, \lambda) = \frac{U_1(\xi, \eta)}{\lambda - a} + \frac{CU_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1} - a}, \\ \Psi_\eta &= V_{\rm R}(\xi, \eta, \lambda)\Psi(\xi, \eta, \lambda), \qquad V_{\rm R}(\xi, \eta, \lambda) = \frac{V_1(\xi, \eta)}{\lambda + a} + \frac{CV_1(\xi, \eta)C^{-1}}{\epsilon\lambda^{-1} + a}, \end{aligned} \tag{25}$$

where $\epsilon = \pm 1$, $a \neq 1$ is a real number and C is an involutive automorphism of $\mathfrak{g}$. Obviously this Lax representation along with the typical reduction (2) satisfies also:

$$U_{\rm R}(\xi, \eta, \lambda) = CU_{\rm R}(\xi, \eta, \epsilon\lambda^{-1})C^{-1}, \qquad V_{\rm R}(\xi, \eta, \lambda) = CV_{\rm R}(\xi, \eta, \epsilon\lambda^{-1})C^{-1}, \tag{26}$$

which is automatically compatible with the Lax representation [11].

The new Lax representation is:

$$\frac{\partial U_{\rm R}}{\partial \eta} - \frac{\partial V_{\rm R}}{\partial \xi} + [U_{\rm R}, V_{\rm R}] = 0, \tag{27}$$

which is equivalent to

$$U_{1,\eta} + [U_1, V_{\rm R}(\xi,\eta,a)] = 0, \qquad V_{1,\xi} + [V_1, U_{\rm R}(\xi,\eta,-a)] = 0. \tag{28}$$

Next we apply the same way of deriving the models as in Section II; obviously, due to the additional terms in $U_{\rm R}$ and $V_{\rm R}$ we get additional terms in the models. In what follows we also list some typical choices for the automorphism C. Skipping the details we get:

$\mathbb{Z}_2$-Nambu-Jona-Lasinio-Vaks-Larkin models

Here $\mathfrak{G} \simeq SU(N)$ and the system takes the form:

$$i\frac{\partial\vec{\phi}}{\partial\eta} + \frac{1}{2a}\vec{\psi}(\vec{\psi}^{\,\dagger}\vec{\phi}) + \frac{1}{\epsilon a^{-1}+a}C\vec{\psi}(\vec{\psi}^{\,\dagger}\hat{C}\vec{\phi})(\xi,\eta) = 0, \qquad i\frac{\partial\vec{\psi}}{\partial\xi} + \frac{1}{2a}\vec{\phi}(\vec{\phi}^{\,\dagger}\vec{\psi}) + \frac{1}{\epsilon a^{-1}+a}C\vec{\phi}(\vec{\phi}^{\,\dagger}\hat{C}\vec{\psi})(\xi,\eta) = 0. \tag{29}$$

where $\vec{\psi} = (\psi_{\alpha,1},\dots,\psi_{\alpha,N})^T$ and $\vec{\phi} = (\phi_{\alpha,1},\dots,\phi_{\alpha,N})^T$.

For the automorphism C of the $SU(N)$ group we may have

$$\text{a)} \quad C_N = \operatorname{diag}(\epsilon_1,\epsilon_2,\dots,\epsilon_N), \qquad \text{b)} \quad C' \in \mathcal{W}, \qquad (C')^2 = 1\!\!1, \tag{30}$$

where $\epsilon_j = \pm 1$ and $\mathcal{W}$ is the Weyl group of $SU(N)$ and is such that $(C')^2 = 1\!\!1$. These two special choices of C are such that $\lim_{\xi\to\pm\infty} U_R(\xi,\eta) = \lim_{\xi\to\pm\infty} CU_R(\xi,\eta)\hat{C}$.

$\mathbb{Z}_2$-Gross-Nevew models

Here $\mathfrak{G} \simeq SP(2N,\mathbb{R})$ and $U_1, V_1 \in sp(2N,\mathbb{R})$ as chosen as in (10). We consider two typical choices for the automorphism C:

$$\text{a)} \quad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_1 \end{pmatrix}, \qquad \text{b)} \quad C' = \begin{pmatrix} 0 & C_2 \\ C_2 & 0 \end{pmatrix}, \tag{31}$$

where $C_1^2 = C_2^2 = 1\!\!1$. This leads to two different systems of GN-type. Using the N-component vectors $\vec{\psi}$ and $\vec{\phi}$ we can write them down in compact form:

$$\begin{aligned} \frac{\partial\vec{\phi}}{\partial\eta} &= -\frac{i}{a}\vec{\psi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) - \frac{2i}{a+\epsilon a^{-1}}C_1\vec{\psi}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right), \\ \frac{\partial\vec{\psi}}{\partial\xi} &= \frac{i}{a}\vec{\phi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}}C_1\vec{\phi}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right). \end{aligned} \tag{32}$$

The corresponding action can be written as follows:

$$A_{\mathbb{Z}_2,\text{GNa}} = \int_{-\infty}^{\infty} dx\,dt\,\left(i\left(\vec{\phi}^{\,\dagger}\frac{\partial\vec{\phi}}{\partial\eta} + \vec{\psi}^{\,\dagger}\frac{\partial\vec{\psi}}{\partial\xi}\right) - \frac{1}{2a}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right)^2 - \frac{1}{\epsilon a^{-1}+a}\left((\vec{\psi}^{\,\dagger}C_1\vec{\phi}) - (\vec{\phi}^{\,\dagger}C_1\vec{\psi})\right)^2\right). \tag{33}$$

The second $\mathbb{Z}_2$-reduced GN-system is:

$$\begin{aligned} \frac{\partial\vec{\phi}}{\partial\eta} &= -\frac{i}{a}\vec{\psi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}}C_2\vec{\psi}^{\,*}\left((\vec{\psi}^T C_2\vec{\phi}) + (\vec{\psi}^{\,\dagger}C_2\vec{\phi}^{\,*})\right), \\ \frac{\partial\vec{\psi}}{\partial\xi} &= \frac{i}{a}\vec{\phi}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}}C_2\vec{\phi}^{\,*}\left((\vec{\phi}^T C_2\vec{\psi}) + (\vec{\phi}^{\,\dagger}C_2\vec{\psi}^{\,*})\right). \end{aligned} \tag{34}$$

These equations can be obtained from the action:

$$A_{\mathbb{Z}_2,\text{GNb}} = \int_{-\infty}^{\infty} dx\,dt\,\left(i\left(\vec{\phi}^{\,\dagger}\frac{\partial\vec{\phi}}{\partial\eta} + \vec{\psi}^{\,\dagger}\frac{\partial\vec{\psi}}{\partial\xi}\right) - \frac{1}{2a}\left((\vec{\psi}^{\,\dagger},\vec{\phi}) - (\vec{\phi}^{\,\dagger},\vec{\psi})\right)^2 - \frac{1}{\epsilon a^{-1}+a}\left((\vec{\phi}^{\,\dagger}C_2\vec{\psi}^{\,*}) + (\vec{\phi}^T C_2\vec{\psi})\right)^2\right). \tag{35}$$

$\mathbb{Z}_2$-Zakharov-Mikhailov models

We introduce again U_1 and V_1 as in (16) and use N-component vectors to cast the $\mathbb{Z}_2$-reduced ZM systems in the form:

$$\begin{aligned} \frac{\partial \vec{\psi}}{\partial \xi} &= \frac{i}{a}\left(\vec{\phi}^{\,*}(\vec{\phi}^T,\vec{\psi}) - \vec{\phi}(\vec{\phi}^{\,\dagger},\vec{\psi})\right) + \frac{2i}{a+\epsilon a^{-1}} C\left(\vec{\phi}^{\,*}(\vec{\phi}^T C\vec{\psi}) - \vec{\phi}(\vec{\phi}^{\,\dagger} C\vec{\psi})\right), \\ \frac{\partial \vec{\phi}}{\partial \eta} &= \frac{i}{a}\left(\vec{\psi}^{\,*}(\vec{\psi}^T,\vec{\phi}) - \vec{\psi}(\vec{\psi}^{\,\dagger},\vec{\phi})\right) + \frac{2i}{a+\epsilon a^{-1}} C\left(\vec{\psi}^{\,*}(\vec{\psi}^T \hat{C}\vec{\phi}) - \vec{\psi}(\vec{\psi}^{\,\dagger} \hat{C}\vec{\phi})\right), \end{aligned} \tag{36}$$

where the involutive automorphism C can be chosen as one of the type:

$$\text{a)} \qquad C = \operatorname{diag}(\epsilon_1, \epsilon_2, \dots, \epsilon_2, \epsilon_1), \qquad \text{b)} \qquad C' = \begin{pmatrix} 1 & 0 \\ 0 & C_3 \end{pmatrix}, \tag{37}$$

with $\epsilon_j = \pm 1$ and $C_3^2 = 1\!\!1$. For this choices of C we have $\lim_{\xi\to\pm\infty} U_R(\xi,\eta) = \lim_{\xi\to\pm\infty} CU_R(\xi,\eta)\hat{C}$. The action for the reduced ZM models is provided by:

$$\begin{aligned} A_{\mathbb{Z}_2,\mathrm{ZM}} = \int_{-\infty}^{\infty} dx\, dt\, \Bigg(& i\left(\vec{\phi}^{\,\dagger}\frac{\partial \vec{\phi}}{\partial \eta} + \vec{\psi}^{\,\dagger}\frac{\partial \vec{\psi}}{\partial \xi}\right) + \frac{1}{a}(\vec{\psi}^{\,\dagger},\vec{\phi}^{\,*})(\vec{\phi}^T,\vec{\psi}) - \frac{1}{a}(\vec{\phi}^{\,\dagger},\vec{\psi})(\vec{\psi}^{\,\dagger},\vec{\phi}) \\ & + \frac{2i}{a+\epsilon/a}\left((\vec{\psi}^{\,\dagger} C\vec{\phi}^{\,*})(\vec{\phi}^T C\vec{\psi}) - (\vec{\phi}^{\,\dagger} C\vec{\psi})(\vec{\psi}^{\,\dagger} C\vec{\phi})\right)\Bigg). \end{aligned} \tag{38}$$

TYPE 2 $\mathbb{Z}_2$ SYMMETRIC NAMBU-JONA-LASINIO-VAKS-LARKIN MODELS

Again we want to avoid poles of second order in the compatibility condition. That is why we started with the Lax representation (1) with $b \neq -a$. With it we can study also type 2 $\mathbb{Z}_2$-symmetries. In what follows we fix up $f(\lambda) = -\lambda$.

Reduction case (21.1)

The Lax pair invariant with respect to the reduction (21.1) takes the form:

$$\frac{\partial \Psi}{\partial \xi} = \left(\frac{U_1(x,t)}{\lambda - a} + \frac{A_1 U_1^\dagger(x,t)\hat{A}_1}{\lambda + a}\right)\Psi(\xi,\eta,\lambda), \qquad \frac{\partial \Psi}{\partial \eta} = \left(\frac{V_1(x,t)}{\lambda - b} + \frac{A_1 V_1^\dagger(x,t)\hat{A}_1}{\lambda + b}\right)\Psi(\xi,\eta,\lambda). \tag{39}$$

The compatibility condition gives:

$$\frac{\partial U_1}{\partial \eta} + \frac{1}{a-b}[U_1,V_1] + \frac{1}{a+b}[U_1, A_1 V_1^\dagger \hat{A}_1] = 0, \qquad \frac{\partial V_1}{\partial \xi} + \frac{1}{a-b}[U_1,V_1] - \frac{1}{a+b}[A_1 U_1^\dagger \hat{A}_1, V_1] = 0, \tag{40}$$

Introducing again U_1 and V_1 as in eq. (5) we find that the fields ϕ and ψ must satisfy:

$$\frac{\partial \phi}{\partial \eta} - \left(\frac{V_1}{a-b} + \frac{A_1 V_1^\dagger \hat{A}_4}{a+b}\right)\phi(\xi,\eta) = 0, \qquad \frac{\partial \psi}{\partial \xi} + \left(\frac{U_1}{a-b} - \frac{A_1 U_1^\dagger \hat{A}_1}{a+b}\right)\psi(\xi,\eta) = 0, \tag{41}$$

or in components:

$$i\frac{\partial \phi^\alpha}{\partial \eta} + \frac{\psi^\alpha}{a-b}\sum_{\beta=1}^N \psi^{*\beta}\phi^\beta - \frac{(A_1\psi)^\alpha}{a+b}\sum_{\beta=1}^N \psi^{*\beta}(A_1\phi)^\beta = 0, \qquad i\frac{\partial \psi^\alpha}{\partial \xi} + \frac{\phi^\alpha}{a-b}\sum_{\beta=1}^N \phi^{*\beta}\psi^\beta + \frac{(A_1\phi)^\alpha}{a+b}\sum_{\beta=1}^N \phi^{*\beta}(\hat{A}_1\psi)^\beta = 0, \tag{42}$$

For the special case $A_1 = \sum_{k=1}^N E_{k,\bar{k}}$ with $\bar{k} = N+1-k$ we get:

$$i\frac{\partial \phi^\alpha}{\partial \eta} + \frac{\psi^\alpha}{a-b}\sum_{\beta=1}^N \psi^{*\beta}\phi^\beta - \frac{\psi^{\bar{\alpha}}}{a+b}\sum_{\beta=1}^N \psi^{*\beta}\phi^{\bar{\beta}} = 0, \qquad i\frac{\partial \psi^\alpha}{\partial \xi} + \frac{\phi^\alpha}{a-b}\sum_{\beta=1}^N \phi^{*\beta}\psi^\beta + \frac{\phi^{\bar{\alpha}}}{a+b}\sum_{\beta=1}^N \phi^{*\beta}\psi^{\bar{\beta}} = 0, \tag{43}$$

Reduction case (21.2)

The Lax pair invariant with respect to the reduction (21.2) takes the form:

$$\frac{\partial\Psi}{\partial\xi} = \left(\frac{U_1(x,t)}{\lambda - a} + \frac{A_2 U_1^T(x,t)\hat{A}_2}{\lambda + a}\right)\Psi(\xi,\eta,\lambda), \qquad \frac{\partial\Psi}{\partial\eta} = \left(\frac{V_1(x,t)}{\lambda - b} + \frac{A_2 V_1^T(x,t)\hat{A}_2}{\lambda + b}\right)\Psi(\xi,\eta,\lambda). \tag{44}$$

The compatibility condition gives:

$$\frac{\partial U_1}{\partial\eta} + \frac{1}{a-b}[U_1, V_1] + \frac{1}{a+b}[U_1, A_2 V_1^T\hat{A}_2] = 0, \qquad \frac{\partial V_1}{\partial\xi} + \frac{1}{a-b}[U_1, V_1] - \frac{1}{a+b}[A_2 U_1^T\hat{A}_2, V_1] = 0, \tag{45}$$

Introducing again U_1 and V_1 as in eq. (5) we find that the fields ϕ and ψ must satisfy:

$$\frac{\partial\phi}{\partial\eta} - \left(\frac{V_1}{a-b} + \frac{A_2 V_1^T\hat{A}_2}{a+b}\right)\phi(\xi,\eta) = 0, \qquad \frac{\partial\psi}{\partial\xi} + \left(\frac{U_1}{a-b} - \frac{A_2 U_1^T\hat{A}_2}{a+b}\right)\psi(\xi,\eta) = 0, \tag{46}$$

or in components ($A_2 = A_2^*$):

$$i\frac{\partial\phi^\alpha}{\partial\eta} + \frac{\psi^\alpha}{a-b}\sum_{\beta=1}^N \psi^{*\beta}\phi_\beta + \frac{(A_2\psi)^{*,\alpha}}{a+b}\sum_{\beta=1}^N \psi^\beta (A_2\phi)^\beta = 0, \qquad i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{\phi^\alpha}{a-b}\sum_{\beta=1}^N \phi^{*\beta}\psi^\beta - \frac{(A_2\phi)^{*,\alpha}}{a+b}\sum_{\beta=1}^N \phi^\beta(\hat{A}_2\psi)^\beta = 0, \tag{47}$$

For the special case $A_2 = \sum_{k=1}^N E_{k,\bar{k}}$ with $\bar{k} = N+1-k$ we get:

$$i\frac{\partial\phi^\alpha}{\partial\eta} + \frac{\psi^\alpha}{a-b}\sum_{\beta=1}^N \psi^{*\beta}\phi_\beta + \frac{\psi^{*,\bar{\alpha}}}{a+b}\sum_{\beta=1}^N \psi^\beta\phi^{\bar{\beta}} = 0, \qquad i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{\phi^\alpha}{a-b}\sum_{\beta=1}^N \phi^{*\beta}\psi^\beta - \frac{\phi^{*,\bar{\alpha}}}{a+b}\sum_{\beta=1}^N \phi^\beta\psi^{\bar{\beta}} = 0, \tag{48}$$

TYPE 2 $\mathbb{Z}_2$ SYMMETRIC GROSS-NEVEW MODELS

We have chosen U_1 and V_1 to be real, see eq. (10). Then the reductions (21.1) and (21.3) become equivalent to (21.2) and (21.4) respectively. In fact, in view of (11) it is enough to consider only the reduction (21.2).

$$\begin{aligned} \frac{\partial\Psi}{\partial\xi} &= U_R(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), & \frac{\partial\Psi}{\partial\eta} &= V_R(\xi,\eta,\lambda)\Psi(\xi,\eta,\lambda), \\ U_R(\xi,\eta,\lambda) &= \frac{U_1}{\lambda-a} + \frac{A_2 U_1^T\hat{A}_2}{\lambda+a}, & V_R(\xi,\eta,\lambda) &= \frac{V_1}{\lambda-b} + \frac{A_2 V_1^T\hat{A}_2}{\lambda+b}. \end{aligned} \tag{49}$$

The compatibility condition is:

$$\frac{\partial U_R}{\partial\eta} - \frac{\partial V_R}{\partial\xi} + [U_R(\xi,\eta), V_R(\xi,\eta)] = 0, \tag{50}$$

or

$$\frac{\partial U_1}{\partial\eta} + \left[U_1, \frac{V_1}{a-b} + \frac{A_2 V_1^T\hat{A}_2}{a+b}\right] = 0, \qquad \frac{\partial V_1}{\partial\xi} + \left[\frac{U_1}{a-b} - \frac{A_2 U_1^T\hat{A}_2}{a+b}, V_1\right] = 0, \tag{51}$$

In view of eq. (10) there follows that:

$$\frac{\partial\phi}{\partial\eta} - \left(\frac{V_1}{a-b} + \frac{A_2 V_1^T\hat{A}_2}{a+b}\right)\phi(\xi,\eta) = 0, \qquad \frac{\partial\psi}{\partial\xi} + \left(\frac{U_1}{a-b} - \frac{A_2 U_1^T\hat{A}_2}{a+b}\right)\psi(\xi,\eta) = 0, \tag{52}$$

In what follows we will use for A_2 one of the following choices:

$$A_2 = \begin{pmatrix} a_1 & 0 \\ 0 & \hat{a}_1^T \end{pmatrix}, \qquad \tilde{A}_2 = \begin{pmatrix} 0 & b_1 \\ -\hat{b}_1^T & 0 \end{pmatrix}, \tag{53}$$

In addition we introduce the complex variables as in (13). After some calculations we find that in terms of the new variables the $\mathbb{Z}_2$-reduced GN model takes the form:

$$\begin{aligned} &i\frac{\partial\phi^\alpha}{\partial\eta} - \frac{c^2}{2(a-b)}\psi^\alpha \sum_{\beta=1}^N \left(\psi^{*\beta}\phi^\beta + \psi^\beta\phi^{*\beta}\right) + \frac{c^2}{2(a+b)}(a_1\psi)^\alpha \sum_{\beta=1}^N \left(\psi^{*\beta}(a_1\phi)^\beta - \psi^\beta(a_1\phi)^{*\beta}\right) = 0,\\ &i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{ic^2}{2(a-b)}\phi^\alpha \sum_{\beta=1}^N \left(\psi^{*\beta}\phi^\beta - \psi^\beta\phi^{*\beta}\right) - \frac{c^2}{2(a+b)}(a_1\phi)^\alpha \sum_{\beta=1}^N \left(\phi^{*\beta}(a_1\psi)^\beta - \phi^\beta(a_1\psi)^{*\beta}\right) = 0. \end{aligned} \tag{54}$$

As a_1 we can use any $N\times N$ matrix that has the properties $a_1^2 = \mathbb{1}$ and $a_1 = a_1^T$. For each choice of a_1 eqs. (54) provide a $\mathbb{Z}_2$-symmetric GN model. In particular, if we put $a_1 = \sum_{k=1}^N E_{k,\bar{k}}$, where $\bar{k} = N+1-k$ we find:

$$\begin{aligned} &i\frac{\partial\phi^\alpha}{\partial\eta} - \frac{c^2}{2(a-b)}\psi^\alpha \sum_{\beta=1}^N \left(\psi^{*\beta}\phi^\beta + \psi^\beta\phi^{*\beta}\right) + \frac{c^2}{2(a+b)}\psi^{\bar\alpha} \sum_{\beta=1}^N \left(\psi^{*\beta}\phi^{\bar\beta} - \psi^\beta\phi^{*\bar\beta}\right) = 0,\\ &i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{ic^2}{2(a-b)}\phi^\alpha \sum_{\beta=1}^N \left(\psi^{*\beta}\phi^\beta - \psi^\beta\phi^{*\beta}\right) - \frac{c^2}{2(a+b)}\phi^{\bar\alpha} \sum_{\beta=1}^N \left(\phi^{*\beta}\psi^{\bar\beta} - \phi^\beta\psi^{*\bar\beta}\right) = 0. \end{aligned} \tag{55}$$

TYPE 2 $\mathbb{Z}_2$ SYMMETRIC ZAKHAROV-MIKHAILOV MODELS

We choose $U_1(\xi,\eta)$ and $V_1(\xi,\eta)$ as in (16). Then all reductions (21) are equivalent and we choose to impose (21.4):

$$U_R = \frac{U_1}{\lambda - a} - \frac{A_4 U_1 \hat{A}_4}{\lambda + a}, \qquad V_R = \frac{V_1}{\lambda - b} - \frac{A_4 V_1 \hat{A}_4}{\lambda + b}, \tag{56}$$

The compatibility condition becomes

$$\frac{\partial U_R}{\partial\eta} - \frac{\partial V_R}{\partial\xi} + [U_R, V_R] = 0. \tag{57}$$

The r.h.s of eq. (57) is a rational function of λ with four simple poles located at $\lambda = \pm a$ and $\lambda = \pm b$. Equating to zero the residues at $\lambda = a$ and $\lambda = b$ we get that U_1 and V_1 must satisfy the following equations:

$$\frac{\partial U_1}{\partial\eta} + \left[U_1, \frac{V_1}{a-b} - \frac{A_4 V_1 \hat{A}_4}{a+b}\right] = 0, \qquad \frac{\partial V_1}{\partial\xi} + \left[\frac{U_1}{a-b} + \frac{A_4 U_1 \hat{A}_4}{a+b}, V_1\right] = 0. \tag{58}$$

The residues at the other two poles $\lambda = -a$ and $\lambda = -b$, due to the symmetry condition are equivalent to eqs. (58) and also vanish.

An immediate consequence of eqs. (58) and the representations (16) is that $\phi(\xi,\eta)$ and $\psi(\xi,\eta)$ must satisfy:

$$\frac{\partial\phi}{\partial\eta} - \left(\frac{V_1}{a-b} - \frac{A_4 V_1 \hat{A}_4}{a+b}\right)\phi(\xi,\eta) = 0, \qquad \frac{\partial\psi}{\partial\xi} + \left(\frac{U_1}{a-b} + \frac{A_4 U_1 \hat{A}_4}{a+b}\right)\psi(\xi,\eta) = 0. \tag{59}$$

In components these give:

$$\begin{aligned} &\frac{\partial\phi_{\alpha,1}}{\partial\eta} - \frac{1}{a-b}\sum_{\beta=1}^N \left(\psi_{\alpha,1}\psi_{\beta,N} - \psi_{\alpha,N}\psi_{\beta,1}\right)\phi_{\beta,s} + \frac{1}{a+b}\sum_{\beta=1}^N \left((A_4\psi)_{\alpha,1}\psi_{\beta,N} - (A_4\psi)_{\alpha,N}\psi_{\beta,1}\right)(A_4\phi)_{\beta,s} = 0,\\ &\frac{\partial\psi_{\alpha,1}}{\partial\eta} + \frac{1}{a-b}\sum_{\beta=1}^N \left(\phi_{\alpha,1}\phi_{\beta,N} - \phi_{\alpha,N}\phi_{\beta,1}\right)\psi_{\beta,s} + \frac{1}{a+b}\sum_{\beta=1}^N \left((A_4\phi)_{\alpha,1}\phi_{\beta,N} - (A_4\phi)_{\alpha,N}\phi_{\beta,1}\right)(A_4\psi)_{\beta,s} = 0, \end{aligned} \tag{60}$$

where s takes values 1 and N. Again we introduce ψ^α and ϕ^α as in eq. (18). In terms of these new variables eqs. (60) take the form:

$$
\begin{aligned}
&i\frac{\partial\phi^\alpha}{\partial\eta} - \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}\left(\psi^{*,\alpha}\psi^\beta - \psi^\alpha\psi^{*,\beta}\right)\phi^\beta + \frac{c^2}{2(a+b)}\sum_{\beta=1}^{N}\left((A_4\psi)^{*,\alpha}\psi^\beta - (A_4\psi)^\alpha\psi^{*,\beta}\right)(A_4\phi)^\beta = 0,\\
&i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}\left(\phi^{*,\alpha}\phi^\beta - \phi^\alpha\phi^{*,\beta}\right)\psi^\beta + \frac{c^2}{2(a+b)}\sum_{\beta=1}^{N}\left((A_4\phi)^{*,\alpha}\phi^\beta - (A_4\phi)^\alpha\phi^{*,\beta}\right)(A_4\psi)^\beta = 0.
\end{aligned}
\tag{61}
$$

Thus for each choice of the matrix A_4 such that $A_4^2 = 1\!\!1$ and $A_4 = A_4^T$ we get a ZM model with $\mathbb{Z}_2$-symmetry. Below we write down the system (61) for the special case when $A_4 = \sum_{k=1}^N E_{k,\bar{k}}$ where $\bar{k} = N + 1 - k$:

$$
\begin{aligned}
&i\frac{\partial\phi^\alpha}{\partial\eta} - \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}\left(\psi^{*,\alpha}\psi^\beta - \psi^\alpha\psi^{*,\beta}\right)\phi^\beta + \frac{c^2}{2(a+b)}\sum_{\beta=1}^{N}\left(\psi^{*,\bar{\alpha}}\psi^\beta - \psi^{\bar{\alpha}}\psi^{*,\beta}\right)\phi^{\bar{\beta}} = 0,\\
&i\frac{\partial\psi^\alpha}{\partial\xi} + \frac{c^2}{2(a-b)}\sum_{\beta=1}^{N}\left(\phi^{*,\alpha}\phi^\beta - \phi^\alpha\phi^{*,\beta}\right)\psi^\beta + \frac{c^2}{2(a+b)}\sum_{\beta=1}^{N}\left(\phi^{*,\bar{\alpha}}\phi^\beta - \phi^{\bar{\alpha}}\phi^{*,\beta}\right)\psi^{\bar{\beta}} = 0.
\end{aligned}
\tag{62}
$$

DISCUSSION AND CONCLUSIONS

Each of the $\mathbb{Z}_2$-symmetries above in fact includes several inequivalent examples. Indeed, they involve additional involutive automorphism of the relevant Lie algebra. For example, in (30) we can choose C to be an element of the Cartan subgroup (case (30,a)) or to be a Weyl reflection (case (30.b)). Therefore one needs to separate the inequivalent symmetries. Obviously in the second case it is enough to pick up just one element from each of the equivalence classes of the Weyl group. Similar arguments should be applied also to all the other $\mathbb{Z}_2$-symmetries derived above.

The results derived above can be extended in several ways. For example, we can apply the same analyzes to a somewhat more general than (1) Lax pairs, e.g. assuming that the a and b are complex. This will give new results when the symmetry includes complex conjugation.

Another natural extension would be to study the spectral properties of the symmetrized Lax operators. This has been already started in [4, 5] for the type 1 symmetrized Lax operators. Constructing the fundamental analytic solutions one can reduce the inverse scattering problem for L to a Riemann-Hilbert problem. Then using the Zakharov-Shabat dressing method [15] and taking into account its more general form proposed in [17, 16] one can derive the soliton solutions also for the symmetrized models obtained above.

Other important developments are related to the interpretation of the ISM as a generalized Fourier transform [1, 9, 8]. This can be done using the Wronskian relations to analyze the mapping between the potential $U_{\rm R}$ and the scattering data.

Another important problem will be to explore the supersymmetric generalizations of the above models.

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