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Oscillation Criteria of a Class of Generalized Lienard Equation

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Abstract. In this paper the oscillatory behavior of solutions of a class of a generalized Lienard equation is studded. We give some new criteria for the oscillation of a class of generalized Lienard equation. Several oscillation criteria are presented that improve the results obtained in the literature by using the refined integral averaging technique are extended. Moreover, three examples are given to illustrate the theoretical analysis.

1. INTRODUCTION

In this paper we are concerned with the oscillation criteria of a class of a generalized Lienard equation of the form

$$\ddot{x}(t) + f(x(t))(\dot{x}(t))^2 + g(x(t)\dot{x}(t) + h(x(t)) = 0, \qquad (1,1)$$

where f(x(t)),g(x(t)) and h(x(t)) are continuously differentiable functions on R .

During the past few decades, the oscillation of differential equations has attracted a great deal of interest in various fields due to its theoretical and practical applications in natural sciences and technology. For instance, the oscillation of a building or a machine, the beam vibration in a synchrotron accelerator, the complicated oscillation in a chemical reaction, and so on; (for example see [6] and [18]. Many criteria have been found which involves the behavior of the integral of a combination of the coefficients of second order nonlinear differential. This approach has been motivated by many authors (for example see [1] - [15] and the authors therein).Which often studied by reducing the problem to the estimate of suitable first integral Our attention is concentrated to such solutions x(t) of (1.1) which exists on the interval [α,∞) .

We will consider only nontrivial solutions x(t) of equation (1.1) which exists on some interval [β,∞) , for β ≥ α.

Definition1: A solution x(t) of the differential equation (1.1) is said to be nontrivial if x(t)≠0 for at least one t∈[α,∞).

Definition2: A nontrivial solution x(t) of differential equation (1.1) is said to be oscillatory if it has arbitrarily large zeros on [β,∞), for β> α otherwise it said to be " nonoscillatory.

It is well known (see [15] Reid) that either all solutions of (1.1) are nonoscillatory, or all the solutions are oscillatory . In the former case, we call the differential (1.1) nonoscillatory and in the later case is oscillatory.

This paper is organized as follows: After this introduction, we present the main results in section 2, followed by illustrative examples in section3.

2. MAIN RESULTS

In this section, we present our main results which provide conditions for every solution of (1.1) to be oscillatory on [α,∞).

We prove the following theorems

Theorem 1: If

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(h(y)f(y) + \left(\frac{dh(y)}{dy}\right)\right) ds = \infty, \qquad (2.1)$$

and

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$$\lim_{t\to\infty}\int_\beta^t\left(1-\frac{(g(y))^2}{4\left(h(y)f(y)+\frac{dh(y)}{dy}\right)}\right)dy=\infty. \quad (2.2)$$

Then the differential equation (1.1) is oscillatory.

Proof: Suppose to the contrary that equation (1.1) has a nonoscillatory solution x(t) on the interval [α,∞), , without loss of generality, we may assume x(t) is eventually positive. We define

$w(t)=\dot{x}(t)h^{-1}(x(t))$.

Then w(t) is well defined and satisfies the equation

$$\dot{w}(t)=-(h(x(t)f(x(t)+((dh(x(t))/(dx)))(w(t))^2-g(x(t))w(t)-1. \quad (2.3)$$

Rewriting equation (2.3) we have

$$w(t)=\left\{\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}w(t)+\frac{g(y)}{2\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2+\frac{(g(y))^2}{4\left(h(y)f(y)+\frac{dh(y)}{dy}\right)}-1. \quad (2.4)$$

Integrating both sides of the above equation from α to t we get

$$w(t)=w(\alpha)-\int_\alpha^t\left\{\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}w(t)+\frac{g(y)}{2\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2dy+\int_\alpha^t\left(\frac{(g(y))^2}{4\left(h(y)f(y)+\frac{dh(y)}{dy}\right)}-1\right)dy. \quad (2.5)$$

Using the hypothesis (2.1) of the theorem 1 there exist β > α such that

$$w(t)\geq-\int_\alpha^t\left\{\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}w(t)+\frac{g(y)}{2\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2dy.$$

Define

$$H(t)=-\int_\alpha^t\left\{\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}w(t)+\frac{g(y)}{2\left(h(y)f(y)+\frac{dh(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2dy. \quad (2.6)$$

Thus w(t) ≥ H(t).

Now differentiating equation (2.6) with respect to t we get

$$\dot{H}(t)\geq-\left(h(y)f(y)+\left(\frac{dh(y)}{dy}\right)\right)(H(t))^2.$$

Therefor

$$-(h(y)f(y)+((dh(y))/(dy)))\leq((H(t))/((H(t))^2)).$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for t > β we get

$$\int_\beta^t\left(h(y)f(y)+\left(\frac{dh(y)}{dy}\right)\right)^{\frac{1}{2}}dy\leq\frac{1}{H(\beta)}-\frac{1}{H(t)}.$$

since H(t) > 0. We conclude that

$$\lim_{t\to\infty}\int_\beta^t\left(h(y)f(y)+\left(\frac{dh(y)}{dy}\right)\right)^{\frac{1}{2}}dy\leq\frac{1}{H(\beta)}.$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory. This completes the proof.

Theorem 2: If

$$\lim_{t\to\infty}\int_\beta^t\left((f(y))^2+\left(\frac{df(y)}{dy}\right)\right)ds=\infty, \quad (2.7)$$

and

$$\lim_{t\to\infty}\int_\beta^t\left(\frac{(g(y))^2}{4((f(y))^2+((df(y))/(dy)))}-\frac{h(y)}{f(y)}\right)dy=\infty. \quad (2.8)$$

Then the differential equation (1.1) is oscillatory.

Proof: Let x(t) be a nonoscillatory solution of (1.1) on the interval [α,∞), , without loss of generality x(t) can be supposed such that x(t)>0 on [α,∞).

We define

$$w(t) = \dot{x}(t) f^{-1}(x(t)).$$

Then w(t) is well defined and satisfies the equation

$$w(t) = -((f(x(t))^2 + \left(\left(\frac{df(x(t))}{dx}\right)\right)(w(t))^2 - g(x(t))w(t) - \left(\frac{h(x(t))}{f(x(t))}\right). \quad (2.9)$$

Rewriting equation (2.9) we have

$$\dot{w}(t) = -\left\{\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(x(t))}{2\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2 + \frac{(g(x(t)))^2}{4\left((f(y))^2 + \frac{df(y)}{dy}\right)} - \frac{h(x(t))}{f(x(t))}\ . \quad (2.10)$$

Integrating both sides of equation (2.10) from α to t we get

$$w(t) = w(\alpha) - \int_\alpha^t \left\{\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(x(t))}{2\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2 ds + \int_\alpha^t \left(\frac{(g(x(t)))^2}{4\left((f(y))^2 + \frac{df(y)}{dy}\right)} - \frac{h(x(t))}{f(x(t))}\right) ds. \quad (2.11)$$

Using the hypothesis (2.8) of the theorem 2 there exist β > α such that

$$w(t) \geq -\int_\alpha^t \left\{\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(x(t))}{2\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2 ds.$$

Define

$$R(t) = -\int_\alpha^t \left\{\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}} w(t) + \frac{g(x(t))}{2\left((f(y))^2 + \frac{df(y)}{dy}\right)^{\frac{1}{2}}}\right\}^2 ds. \quad (2.12)$$

Thus $w(t) \geq R(t)$.

Now differentiating equation (2.12) with respect to t we get

$$\dot{R}(t) \geq -\left((f(y))^2 + \left(\frac{df(y)}{dy}\right)\right)(R(t))^2.$$

Therefor

$$-\left((f(y))^2 + \left(\frac{df(y)}{dy}\right)\right) \leq \left(\frac{\dot{R}(t)}{(R(t))^2}\right).$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for t > β we get

$$-\int_\beta^t \left((f(y))^2 + \left(\frac{df(y)}{dy}\right)\right) ds \leq \left(\frac{1}{H(\beta)}\right) - \left(\frac{1}{H(t)}\right),$$

Since $R(t) > 0$. We conclude

$$-\int_\beta^t \left((f(y))^2 + \left(\frac{df(y)}{dy}\right)\right) ds \leq \left(\frac{1}{H(\beta)}\right).$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory.

This completes the proof.

Theorem 3: The differential equation (1.1) is oscillatory if

$$-\lim_{t\to\infty} \int_\beta^t \left(g(y)f(y) + \frac{dg(y)}{dy}\right) dy = \infty, \quad (2.13)$$

and

$$\lim_{t\to\infty}\int_\beta^t\left(\frac{(g(y))^2}{4\left(g(y)f(y)+\frac{dg(y)}{dy}\right)}-\frac{h(y)}{g(y)}\right)dy=\infty. \tag{2.14}$$

Proof: Suppose to the contrary that equation (1.1) has a nonoscillatory solution x(t) on the interval [α,∞), , without loss of generality, we may assume x(t) is eventually positive

We define

$$\dot{w}(t)=x(t)g^{-1}\big(x(t)\big).$$

Then w(t) is well defined and satisfies the equation

$$\dot{w}(t)=-((f\big(x(t)\big)^2+\left(\left(\frac{df(x(t))}{dx}\right)\right)\big(w(t)\big)^2-g\big(x(t)\big)w(t)-\left(\frac{h(x(t))}{f(x(t))}\right). \tag{2.15}$$

Rewriting equation (2.15) we have

$$w(t)=-\left\{\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}w(t)+\left(\frac{g(y)}{2\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}}\right)\right\}^2+\frac{(g(y))^2}{4\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)}-\frac{h(y)}{g(y)}. \tag{2.16}$$

Integrating both sides of the above equation from α to t we get

$$w(t)=w(\alpha)-\int_\alpha^t\left\{\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}w(s)+\frac{g(y)}{2\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}}\right\}^2ds+\int_\alpha^t\left(\frac{(g(y))^2}{4\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)}-\frac{h(y)}{g(y)}\right)ds.$$

Using the hypothesis (2.13) of the theorem 3 there exist β > α such that

$$w(t)\geq-\int_\beta^t\left\{\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}w(s)+\frac{g(y)}{2\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}}\right\}^2ds. \tag{2.17}$$

Define

$$Z(t)=)\geq-\int_\beta^t\left\{\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}w(s)+\frac{g(y)}{2\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)^{\frac{1}{2}}}\right\}^2ds \tag{2.18}$$

Thus w(t) ≥ Z(t).

Now differentiating equation (2.18) with respect to t we get

$$\dot{Z}(t)\geq-\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)\big(Z(t)\big)^2.$$

Therefor

$$-\left(g(y)f(y)+\left(\frac{dg(y)}{dy}\right)\right)\leq\left(\frac{\dot{Z}(t)}{\big(Z(t)\big)^2}\right).$$

Integrating both sides of this inequality with respect to t (with t replaced by s) from β to t for t > β we get

$$-\int_\beta^t\left(g(y)f(y)+\frac{dg(y)}{dy}\right)ds\leq\left(\frac{1}{Z(\beta)}\right)-\left(\frac{1}{Z(t)}\right).$$

since Z(t) > 0. We conclude that

$$-\lim_{t\to\infty}\int_\beta^t\left(g(y)f(y)+\frac{dg(y)}{dy}\right)ds<\left(\frac{1}{Z(\beta)}\right).$$

Which contradicts the hypothesis of the theorem. Hence the differential equation (1.1) is oscillatory.

This completes the proof.

3. EXAMPLES

The following examples illustrate the applicability of the theorems.

Example 1: The applicability of theorem 1.
Consider the second nonlinear order differential equation

$$\ddot{x}(t) - \left(\frac{1+2x}{1+x^2}\right)\left(\dot{x}(t)\right)^2 - e^{-x}\dot{x}(t) + (1+x^2) = 0. \quad (3.1)$$

For this differential equation we have $f(x) = -\left(\left(\frac{1+2x}{1+x^2}\right)\right), g(x) = -e^{\{-x\}}\ and\ h(x) = (1+x^2).$

To show the applicability the hypothesis (2.1) of theorem 1

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(h(y)f(y) + \left(\frac{dh(y)}{dy}\right)\right) ds = -lim\int_{\beta}^{t}\left(-\left(\left(\frac{1+2x}{1+x^2}\right)\right)(1+x^2) + 2x\right) ds = -lim\int_{\beta}^{t}(-1)\, ds$$

$$= lim[y(s)]_{\beta}^{t} = \infty.$$

And the applicability of the hypothesis (2.2) of theorem 1

$$\lim_{t\to\infty}\int_{\beta}^{t}\left(1 - \frac{\left(g(y)\right)^2}{4\left(h(y)f(y) + \frac{dh(y)}{dy}\right)}\right)dy = \lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(-e^{-y})^2}{4\left(\frac{1-2y}{1+y^2}\right)(1+y^2) + 2y} - 1\right)dy$$

$$=\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{-e^{-2y}}{4} - 1\right)dy = \lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{-e^{-2y}-4}{4}\right)dy = \lim_{t\to\infty}\left[-\frac{e^{-y(s)}}{8} - \left(\frac{y(s)}{4}\right)\right]_{\beta}^{t} = \infty.$$

Therefore the theorem implies that the differential equation is oscillatory.

Example 2: The applicability of theorem 2.
Consider the second nonlinear order differential equation.

$$\ddot{x}(t) + coth\, x(t)\ \left(\dot{x}(t)\right)^2 - e^{x(t)}\dot{x}(t) + cothx(t) = 0.$$

For this differential equation we have $f(x) = cothx(t), g(x) = e^{x(t)}\ and\ h(x) = \coth(x).$

To show the applicability of the hypothesis (2.7) of the theorem 2

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(\left(f(y)\right)^2 + \left(\frac{df(y)}{dy}\right)\right)ds = -\lim_{t\to\infty}\int_{\beta}^{t}((cothy)^2 - (cschy)^2)ds$$

$$= -\lim_{t\to\infty}\int_{\beta}^{t}(-1)ds = \lim_{t\to\infty}[x(s)]_{\beta}^{t} = \infty.$$

And the applicability of the hypothesis (2.8) of theorem 2

$$\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(g(y))^2}{4((f(y))^2 + \frac{df(y)}{dy})} - \frac{h(y)}{f(y)}\right)dy = \lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(-e^{-y})^2}{4((cothy)^2 - (cschy)^2)} - \frac{cothy}{cothy}\right)dy$$

$$= \lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(-e^{-y})^2}{-4} - 1\right)dy = -\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{e^{-2y}+4}{4}\right)dy = -\lim_{t\to\infty}\left[\frac{e^{-2x(t)}}{8} - \frac{x(t)}{4}\right]_{\beta}^{t} = -\infty.$$

Hence the theorem is applicable.

Example 3: The applicability of theorem 3.
Consider the second nonlinear order differential equation.

$$\ddot{x}(t) - \left(\frac{1+2x}{1+x^2}\right)\left(\dot{x}(t)\right)^2 + (1+x^2)\dot{x}(t) + (1+x^2) = 0.$$

For this differential equation we have $f(x) = \left(\frac{1+2x}{1+x^2}\right),\ g(x) = h(x) = (1+x^2).$

To show the applicability of the hypothesis (2.13) of the theorem 3

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(g(y)f(y)+\frac{dg(y)}{dy}\right)ds=-\lim_{t\to\infty}\int_{\beta}^{t}\left(\left(\frac{1+2y}{1+y^2}\right)((1+y^2))+2y\right)ds=-\lim_{t\to\infty}\int_{\beta}^{t}(-1)\,ds$$

$$=\lim_{t\to\infty}[s]_{\beta}^{t}=\infty.$$

And the applicability of the hypothesis (2.14) of theorem 3

$$\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(g(y))^2}{4\left(g(y)f(y)+\frac{dg(y)}{dy}\right)}-\frac{h(y)}{g(y)}\right)dy=\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(1+y^2)^2}{4\left(\left(-\frac{1+2y}{1+y^2}\right)(1+y^2)+2y\right)}-1\right)dy$$

$$-\lim_{t\to\infty}\int_{\beta}^{t}\left(\frac{(1+y^2)^2}{4(-1)}\right)dy=-\lim_{t\to\infty}\left[\left(\frac{5y(s)}{4}\right)+\left(\frac{2\big(y(s)\big)^3}{3}\right)+\left(\frac{\big(y(s)\big)^5}{5}\right)\right]_{\beta}^{s}=-\infty.$$

4. CONCLUSION

In this paper we are concerned with the oscillation criteria of a class of a generalized Lienard equation of the form (1,1), where $f\big(x(t)\big)$, $g\big(x(t)\big)$ *and* $h(x(t))$ are continuously differentiable functions on R . Under certain assumptions, we have derived a complete characterization of an eventually positive solution $x(t)$ of (1.1). By using generalized Riccati techniques, we have proved that under a number of conditions that every solution $x(t)$ of (1.1) is oscillatory. Also, we have given three examples to illustrate the obtained results.

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