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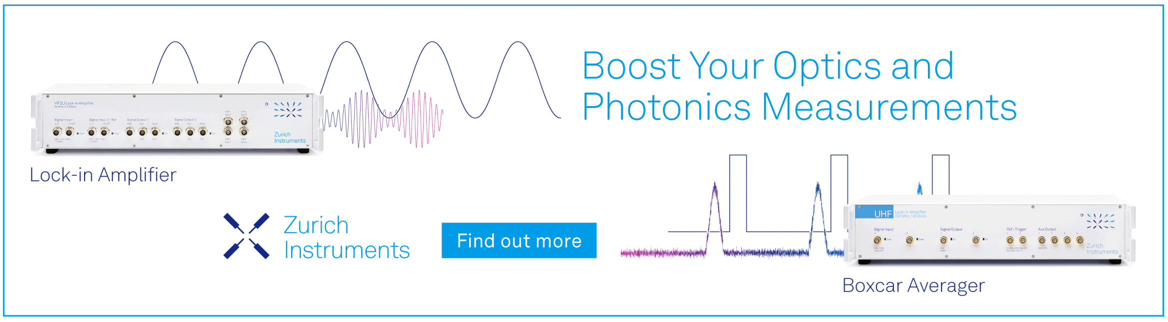

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Optimal Control of a Coupled Tanks System with Model-Reality Differences

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Abstract. In this paper, an efficient computational approach is proposed to optimize and control a coupled tanks system. Since the dynamics of the coupled tanks system is nonlinear, determination of the optimal water level in the tanks could be formulated as an optimal control problem for a useful operation decision. For simplicity, the linear model of the coupled tanks system is suggested to give the true operating height of the coupled tanks. In our approach, the adjustable parameter is added into the model used. The aim is to measure the differences between the real plant and the model used repeatedly during the computation procedure. In this way, the optimal solution of the model used can be updated iteratively. On this basis, system optimization and parameter estimation are integrated. At the end of the iteration procedure, the converged solution approximates to the correct optimal solution of the original optimal control problem, in spite of model-reality differences. For illustration, the numerical parameters of a coupled tank system are studied and the applicable of the approach proposed is shown. In conclusion, the efficiency of the approach proposed in achieving the desired water level of the coupled tanks is highly presented.

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INTRODUCTION

In engineering process, the coupled tanks system is an important component. The applications of the coupled tanks system are widely discussed in the engineering community, for more examples see [1]–[4]. Specifically, the practical aspects of the coupled tanks system are ranged from chemical process, control system to mechanical system [5]–[9]. In essence, the basic concept of the coupled tanks system is based on the mass balance equation. Particularly, the equation of the coupled tanks system is formulated as a set of ordinary differential equations (ODEs). Mathematically, the set of ODEs is well-defined and the solutions of the set of ODEs, both for the analytic and the numeric, are fruitful results [10]–[12].

In this paper, a simple coupled tanks system with two joint tanks, two inflows and two outflows is considered [1]–[2]. Two nonlinear ODEs are formulated to represent the nonlinear system dynamics of the coupled tanks. Further, the determination of the water level in the tanks is defined as an optimal control problem. In our approach, the linear model, which is simplified from the original optimal control problem, is suggested [13]–[14]. Then, the adjusted parameter is added into the model used. The aim is to measure the differences between the real plant and the model used repeatedly during the computation procedure. Follow from this, the optimal solution of the model used can be updated iteratively. As a result, system optimization and parameter estimation are integrated. At the end of the iteration procedure, the converged solution approximates to the correct optimal solution of the original optimal control problem, in spite of model-reality differences [15]–[18], and the desired water level in the tanks would be achieved.

The rest of the paper is organized as follows. In Section 2, the optimal control problem of the coupled tanks system and the simplified linear optimal control model are described. In Section 3, an expanded optimal control

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problem, which integrates system optimization and parameter estimation, is introduced. Then, an iterative algorithm is derived for solving the optimal control problem of the coupled tanks system. In Section 4, the numerical study of the coupled tanks system is illustrated by using the algorithm proposed to achieve the desired water level in the tanks. Finally, some concluding remarks are made.

PROBLEM STATEMENT

Consider the coupled tanks system with three valves as shown in Fig. 1. The dynamic equation for each tank is given by

$$\text{Tank 1:} \quad A_1 \frac{dH_1}{dt} = Q_{i1} - Q_{o1} - Q_3 \quad \text{and} \quad \text{Tank 2:} \quad A_2 \frac{dH_2}{dt} = Q_{i2} - Q_{o2} + Q_3 \tag{1}$$

where H_1 and H_2 are height of water in Tanks 1 and 2, respectively, A_1 and A_2 are cross sectional area of Tanks 1 and 2, respectively, Q_{i1} and Q_{i2} are pump flow rate into Tanks 1 and 2, respectively, Q_{o1} and Q_{o2} are flow rate of water out of Tanks 1 and 2, respectively, and Q_3 is flow rate of water between Tanks 1 and 2.

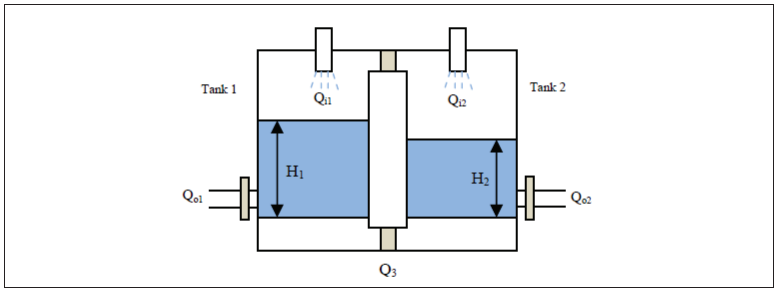

FIGURE 1. Schematic model of coupled tanks system

Each outlet drain is modelled as a simple orifice. According to Bernoulli's equation for steady, non-viscous, incompressible fluid, the outlet flow in each tank is proportional to the square root of the head of water in the tank, and the flow between the tanks is proportional to the square root of the head of differential [1]–[5]. That is,

$$Q_{o1} = a_1\sqrt{H_1}\,, \quad Q_{o2} = a_2\sqrt{H_2} \quad \text{and} \quad Q_3 = a_3\sqrt{H_1 - H_2} \tag{2}$$

where a_1, a_2 and a_3 are proportionality constants, which their values depend on the coefficients of discharge, the cross sectional area of each orifice and the gravitational constant.

Suppose that for the set of inflows Q_{i1} and Q_{i2}, the water level in the tanks is at some steady state levels H_1 and H_2. Consider small variations in each inflow, which are represented by q_1 in Q_{i1}, q_2 in Q_{i2}, h_1 in H_1 and h_2 in H_2. Then, the system dynamics based on these small variations is given by

$$\frac{dh_1}{dt} = \frac{q_1}{A_1} - \frac{a_1}{A_1}(\sqrt{H_1 + h_1} - \sqrt{H_1}) - \frac{a_3}{A_1}(\sqrt{H_1 - H_2 + h_1 - h_2} - \sqrt{H_1 - H_2}) \tag{3}$$

$$\frac{dh_2}{dt} = \frac{q_2}{A_2} - \frac{a_2}{A_2}(\sqrt{H_2 + h_2} - \sqrt{H_2}) - \frac{a_3}{A_2}(\sqrt{H_1 - H_2 + h_1 - h_2} - \sqrt{H_1 - H_2}) \tag{4}$$

Thus, this problem of controlling the water level in Tank 2, which is defined as an optimal control problem and is referred to as Problem (P), is described below:

Problem (P): Find the optimal small variations of the flow rate $q = (q_1 \quad q_2)^{\mathrm{T}}$ in order to minimize the cost function

$$J_0 = \tfrac{1}{2}\int_{t_0}^{t_p} ((h_2 - h_2^s)^2 + (q_1 - q_1^s)^2 + (q_2 - q_2^s)^2)\ dt \tag{5}$$

subject to the system dynamics (3) and (4) with the output measurement $y = h_2$, where h_2^s, q_1^s and q_2^s are the steady states of the value, and t_0 and t_p are, respectively, the initial time and the fixed terminal time.

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Notice that Problem (P) is a complex problem and solving this kind of problem is computational demanding. However, the optimal solution of Problem (P) could be obtained by solving the simplified problem, which is referred to as Problem (M), given by

$$\min_{q} J_1 = \frac{1}{2}\int_{t_0}^{t_p} ((h_2 - h_2^s)^2 + (q_1 - q_1^s)^2 + (q_2 - q_2^s)^2)\ dt$$

subject to (6)

$$\begin{pmatrix} \dot{h}_1 \\ \dot{h}_2 \end{pmatrix} = \begin{pmatrix} -k_1 - k_3 & k_3 \\ k_4 & -k_2 - k_4 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} + \begin{pmatrix} A_1^{-1} & 0 \\ 0 & A_2^{-1} \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix},\ y = \begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$$

with

$$k_1 = \frac{a_1}{2A_1\sqrt{H_1}},\quad k_2 = \frac{a_2}{2A_2\sqrt{H_2}},\quad k_3 = \frac{a_3}{2A_1\sqrt{H_1 - H_2}},\quad k_4 = \frac{a_3}{2A_2\sqrt{H_1 - H_2}},$$

where $\alpha = (\alpha_1 \quad \alpha_2)^{\mathrm{T}}$ is the adjustable parameter.

SYSTEM OPTIMIZATION WITH PARAMETER ESTIMATION

Now, denote $x = (h_1 \quad h_2)^{\mathrm{T}}$ and $u = (q_1 \quad q_2)^{\mathrm{T}}$ as the state variable and the control input, respectively, and $f : \Re^2 \times \Re^2 \times \Re \to \Re^2$ represents the system dynamics of the coupled tanks system in (3) and (4). The terms of $x^s = (h_1^s \quad h_2^s)^{\mathrm{T}}$ and $u^s = (q_1^s \quad q_2^s)^{\mathrm{T}}$ are the steady states. Let us introduce an expanded optimal control problem, which is referred to as Problem (E), given by

$$\min_{u} J_2 = \frac{1}{2}\int_{t_0}^{t_p} (x(t) - x^s)^{\mathrm{T}} Q(x(t) - x^s) + (u(t) - u^s)^{\mathrm{T}} R(u(t) - u^s))$$
$$+ \frac{1}{2} r_1 \| u(t) - v(t) \|^2 + \frac{1}{2} r_2 \| x(t) - z(t) \|^2\ dt$$

subject to (7)

$$\dot{x}(t) = Ax(t) + Bu(t) + \alpha(t),\ y(t) = Cx(t)$$
$$Az(t) + Bv(t) + \alpha(t) = f(z(t), v(t), t),\ u(t) = v(t),\ x(t) = z(t)$$

where $v(t) \in \Re^2$ and $z(t) \in \Re^2$ are introduced to separate the control variable and the state variable in the optimization problem from the respective signals in the parameter estimation problem. It is important to note that the algorithm is designed in such a way that the constraints $u(t) = v(t)$ and $x(t) = z(t)$ are satisfied due on the termination of iterations, assuming convergence is achieved. The state constraint $z(t)$ and the control constraint $v(t)$ are used for the computation of the parameter estimation and the matching scheme, while the corresponding state constraint $x(t)$ and control constraint $u(t)$ are reserved for optimizing the linear model-based optimal control problem. Hence, system optimization and parameter estimation are mutually interactive.

Define the Hamiltonian function by

$$H(t) = \frac{1}{2}((x(t) - x^s)^{\mathrm{T}} Q(x(t) - x^s) + (u(t) - u^s)^{\mathrm{T}} R(u(t) - u^s)) - \lambda(t)^{\mathrm{T}} u(t) - \beta(t)^{\mathrm{T}} x(t)$$
$$+ \frac{1}{2} r_1 \| u(t) - v(t) \|^2 + \frac{1}{2} r_2 \| x(t) - z(t) \|^2 + p(t)^{\mathrm{T}} (Ax(t) + Bu(t) + \alpha(t)) \quad (8)$$

where $p(t) \in \Re^2$ is the Lagrange multiplier. Then, the augmented cost function becomes

$$J_a = \frac{1}{2}\int_{t_0}^{t_p} H(t) - p(t)^{\mathrm{T}} \dot{x}(t) + \theta(t)^{\mathrm{T}} (y(t) - Cx(t)) + \mu(t)^{\mathrm{T}} (f(z(t), v(t), t) - Az(t) - Bv(t) - \alpha(t))$$
$$+ \lambda(t)^{\mathrm{T}} v(t) + \beta(t)^{\mathrm{T}} z(t))\ dt \quad (9)$$

where $p(t)$, $\theta(t)$, $\mu(t)$, $\lambda(t)$ and $\beta(t)$ are the appropriate multipliers to be determined later.

Applying the calculus of variation [13]–[14] to the augmented cost function (9), the following necessary conditions for optimality are obtained:

$$0 = \nabla_u H = R(u(t) - u^s) + B^{\mathrm{T}} p(t) + r_1(u(t) - v(t)) - \lambda(t) \quad (10)$$

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$$-\dot{p}(t) = \nabla_x H = Q(x(t) - x^s) + A^{\mathrm{T}} p(t) + r_2(x(t) - z(t)) - \beta(t) \tag{11}$$

$$\dot{x}(t) = \nabla_p H = Ax(t) + Bu(t) + \alpha(t) \tag{12}$$

$$y(t) = Cx(t) \tag{13}$$

$x(0)$ and $p(t_p)$ are given

$$f(z(t), v(t), t) = Az(t) + Bv(t) + \alpha(t) \tag{14}$$

$$\lambda(t) = -\left(\frac{\partial f}{\partial v} - B\right)^{\mathrm{T}} \hat{p}(t) \quad \text{and} \quad \beta(t) = -\left(\frac{\partial f}{\partial z} - A\right)^{\mathrm{T}} \hat{p}(t). \tag{15}$$

$$z(t) = x(t),\ v(t) = u(t),\ \hat{p}(t) = p(t)$$

where $\mu(t) = \hat{p}(t)$ and $\theta(t) = 0$.

Refer to the necessary conditions (10)–(13), a modified optimal control problem, which is referred to as Problem (MM), is defined by

$$\min_u J_3 = \tfrac{1}{2}\int_{t_0}^{t_p} (x(t) - x^s)^{\mathrm{T}} Q(x(t) - x^s) + (u(t) - u^s)^{\mathrm{T}} R(u(t) - u^s))$$

$$-\lambda(t)^{\mathrm{T}} u(t) - \beta(t)^{\mathrm{T}} x(t) + \tfrac{1}{2} r_1 \| u(t) - v(t) \|^2 + \tfrac{1}{2} r_2 \| x(t) - z(t) \|^2 \, dt$$

subject to (16)

$$\dot{x}(t) = Ax(t) + Bu(t) + \alpha(t),\ y(t) = Cx(t)$$

with the specified $\alpha(t)$, $\lambda(t)$, $\beta(t)$, $v(t)$ and $z(t)$, where the boundary conditions $x(0)$ and $p(t_p)$ are given.

The optimal control law for Problem (MM), which is the expanded optimal control policy, is a feedback control law. This control law is explained in the following theorem.

Theorem 1 (Expanded optimal control policy): Assume that the expanded optimal control policy exists. Then, this optimal control law is given by

$$u(t) = -K(t)x(t) + u_{ff}(t) + R_a^{-1} R u^s \tag{17}$$

where

$$u_{ff}(t) = -R_a^{-1} B^{\mathrm{T}} s(t) + R_a^{-1} \lambda_a(t) \tag{18}$$

$$K(t) = R_a^{-1} B^{\mathrm{T}} S(t) \tag{19}$$

$$\dot{S}(t) = -S(t)A - A^{\mathrm{T}} S(t) - Q_a + S(t) B R_a^{-1} B^{\mathrm{T}} S(t) \tag{20}$$

$$\dot{s}(t) = -(A - BK(t))^{\mathrm{T}} s(t) - K(t)^{\mathrm{T}} \lambda_a(t) + \beta_a(t) - S(t)\alpha(t) - S(t) B R_a^{-1} R u^s + Q x^s \tag{21}$$

with the boundary conditions $S(t_p) = 0$ and $s(t_p) = 0$, and $R_a = R + r_1 I_2$, $Q_a = Q + r_2 I_2$, $\lambda_a(t) = \lambda(t) + r_1 v(t)$ and $\beta_a(t) = \beta(t) + r_2 z(t)$.

The result of Theorem 1 can be proven by applying the sweep method [13]–[16], that is,

$$p(t) = S(t)x(t) + s(t). \tag{22}$$

On the other hand, taking (17) into (12), the state equation becomes

$$\dot{x}(t) = (A - BK(t))x(t) + Bu_{ff}(t) + BR_a^{-1} R u^s + \alpha(t) \tag{23}$$

and the output measurement is given by

$$y(t) = Cx(t). \tag{24}$$

RESULT AND DISCUSSION

Table 1 shows the value of each parameter of the coupled tanks system [1]. The steady state values are set at $x^s = (0.0 \quad 0.6037)^{\mathrm{T}}$ and $u^s = (1.0 \quad 0.7)^{\mathrm{T}}$. The initial state is $x(0) = (0.0 \quad 0.0)^{\mathrm{T}}$, and the time interval is $t \in [0.0, 10.0]$. The algorithm proposed is implemented in MATLAB 12 in order to obtain the results.

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TABLE 1. Parameters of coupled tanks system

Parameter	Value	Unit
H_1	17.00	cm
H_2	15.00	cm
a_1	10.78	$cm^{3/2}$/sec
a_2, a_3	11.03	$cm^{3/2}$/sec
A_1, A_2	32.00	cm^2

The simulation result spent the elapsed time of 0.163674 seconds to obtain the converged solution with two number of iterations. The value of the final cost function is 10.59 units. Moreover, the graphical solutions for the final control input, the final state variable, the final output measurement and the final costate variable are, respectively, shown in Fig. 2(a), 2(b), 2(c) and 2(d). The trajectory of the control input diverges at the beginning and then turns towards to the steady state value after 0.5 second. It stays at the steady state after 2 seconds. With this control input, the water level at Tank 2 is increasing linearly from the empty situation to stay at the steady state value at $x_2 = 0.6037$ cm after 4.0 seconds.

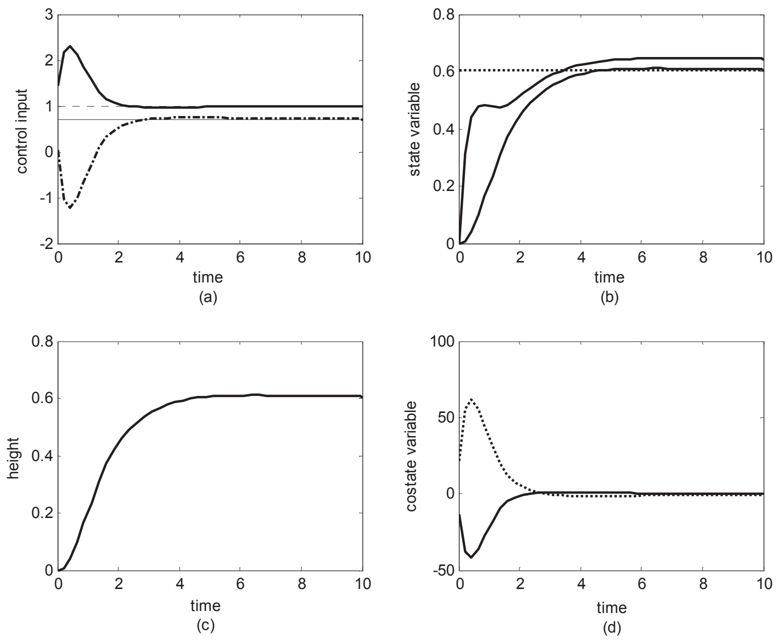

FIGURE 2. (a) Final control trajectory and the steady values, (b) Final state trajectory and the steady value, (c) Final output trajectory and (d) Final costate trajectory

From these simulation results, it is noticed that the water level for both tanks should be non-negativity. By virtue of this, the constraint of $x \geq 0$ can be added for this purpose. However, the algorithm proposed shows the efficiency in controlling the water level at the normal operating height in Tank 2 is achieved. Hence, the applicable of the algorithm proposed is certainly illustrated.

CONCLUDING REMARKS

In this paper, the optimal control policy for the coupled tanks system with three valves was discussed. The mathematical formulation of the coupled tanks system was made according to the mass balance equation. The simplified linear model of the original optimal control problem with the adjusted parameter was solved repeatedly. Because of the different structure, the adjusted parameter could capture the differences between the real plant and the model used in order updating the optimal solution of the model used. After the convergence was achieved, the

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iterative solution approximated to the correct optimal solution of the original optimal control problem, in spite of model-reality differences. From the result obtained, the water level of the second tank was able to be controlled at the desired steady state value by using the algorithm proposed. Hence, the efficiency of the algorithm was shown. In conclusion, it is emphasized that the usefulness of the algorithm proposed in controlling the water level is highly presented.

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