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Properties of C -Metric Spaces

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Abstract. The subject of this paper belongs to the theory of approximate metrics [23]. An approximate metric on X is a real application defined on $X \times X$ that satisfies only a part of the metric axioms. In a recent paper [23], we introduced a new type of approximate metric, named C -metric, that is an application which satisfies only two metric axioms: symmetry and triangular inequality. The remarkable fact in a C -metric space is that a topological structure induced by the C -metric can be defined. The innovative idea of this paper is that we obtain some convergence properties of a C -metric space in the absence of a metric. In this paper we investigate C -metric spaces. The paper is divided into four sections. Section 1 is for Introduction. In Section 2 we recall some concepts and preliminary results. In Section 3 we present some properties of C -metric spaces, such as convergence properties, a canonical decomposition and a C -fixed point theorem. Finally, in Section 4 some conclusions are highlighted.

INTRODUCTION

In the last decades, the theories of non-linear spaces, fuzziness, non-additivity and generalized metrics have been developed intensively thanks to their multiple applications in many domains, such as decision making theory, statistics, uncertainty theory, computer science, economy, imaging processes, best approximation problems, medicine, theory of cluster validation (e.g., [1-5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]).

Approximate metrics [23] (i.e. applications that satisfy only a part of the metric axioms) have a lot of uses in fixed point problems for single and multi-valued operators, category problems, cooperative game theory with non-transferable utility, probabilities, approximation theory, optimization, computer science, variational and linear inequalities (e.g., [16], [17], [18], [19], [20], [21], [22]).

Different types of approximate metrics are known in the literature, such as pseudometrics, generalized metrics (in the sense of Luxemburg [24], Lawvere [19] and Branciari [25]), quasimetrics, ultrametrics, C -metrics [23] and A -metrics [26].

In this paper we study a new kind of approximate metric, named C -metrics (introduced in [23]), that is an application which satisfies only two metric axioms: symmetry and triangular inequality. The remarkable fact in a C -metric space is that a topological structure induced by the C -metric can be defined. The innovative idea of this paper is that we obtain some convergence properties in a C -metric space in the absence of a metric. So, we present some convergence properties, a canonical decomposition and a C -fixed point theorem.

The structure of the paper is the following: Section 1 is for Introduction. In Section 2 we recall some concepts and preliminary results. In Section 3 we present some properties of C -metric spaces, such as convergence properties, a canonical decomposition and a C -fixed point theorem. The final Section 4 contains some concluding remarks.

PRELIMINARIES

We denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. For a non-empty set X , $\mathcal{P}_0(X)$ is the family of all non-empty subsets of X and $\mathcal{S}(X)$ is the family of all singletons of X . If X is a metric space, then $\mathcal{P}_b(X)$ will denote the set of all non-empty bounded subsets of X . If X is a real linear space, then for every $E, F \in \mathcal{P}_0(X)$ and $\alpha \in \mathbb{R}$, denote $E + F = \{x + y; x \in E, y \in F\}$ and $\alpha E = \{\alpha x; x \in E\}$.

Definition 2.1. [27] Let X be a real linear space and let \mathcal{A} be a subset of $\mathcal{P}_0(X)$. A function $|\cdot| : \mathcal{A} \rightarrow [0, +\infty]$ is called a set-norm on \mathcal{A} if:

- (i) $|E| = 0 \Leftrightarrow E = \{0\}$, for $E \in \mathcal{A}$;
- (ii) $|\alpha E| = |\alpha| \cdot |E|$, $\forall \alpha \in \mathbb{R}, \forall E \in \mathcal{A}$ (with the convention $0 \cdot (+\infty) = 0$);
- (iii) $|E + F| \leq |E| + |F|$, $\forall E, F \in \mathcal{A}$.

Example 2.2. I. If $(X, \|\cdot\|)$ is a normed space, then the function defined by $|E| = \sup_{x \in E} \|x\|$ for every $E \in \mathcal{P}_0(X)$ is a set-norm on $\mathcal{P}_0(X)$.

II. Let $(X, \|\cdot\|)$ be a real normed space, $\alpha \in \mathbb{R} \setminus \{0\}$ and the function defined for every $E \in \mathcal{P}_0(X)$ by

$$|E| = \begin{cases} |\alpha| \max_{x \in E} \|x\|, & \text{if } E \text{ is finite} \\ +\infty, & \text{if } E \text{ is not finite.} \end{cases}$$

Then the function $|\cdot|$ is a set-norm on $\mathcal{P}_0(X)$ (with the convention $0 \cdot (+\infty) = 0$).

In the sequel, let T be a non-empty set, $\mathcal{P}(T)$ the family of all subsets of T and $C \subseteq \mathcal{P}(T)$ a ring of subsets of T (i.e. $C \neq \emptyset$, $A \cup B \in C$ and $A \setminus B \in C$ for every $A, B \in C$).

Definition 2.3. If T, Y are non-empty sets, then we call a set-valued function (also called a multifunction) from T to Y a function $F : T \rightarrow \mathcal{P}_0(Y)$.

Definition 2.4. [2] Let L be a non-empty set. We say that L is a *semi-linear space* if it is endowed with two operations, sum and multiplication by scalars from a field Γ :

$$" + " : L \times L \rightarrow L$$

and

$$" \cdot " : \Gamma \times L \rightarrow L$$

which verify the following axioms :

- (S1) $(x + y) + z = x + (y + z)$, $\forall x, y, z \in L$;
- (S2) $\exists \theta \in L$ such that $x + \theta = \theta + x = x$, $\forall x \in L$ (θ is called the origin of L);
- (S3) $x + y = y + x$, $\forall x, y \in L$;
- (S4) $\lambda(\mu x) = (\lambda\mu)x$, $\forall \lambda, \mu \in \Gamma, \forall x \in L$;
- (S5) $1 \cdot x = x$, $\forall x \in L$;
- (S6) $\lambda(x + y) = \lambda x + \lambda y$, $\forall \lambda \in \Gamma, \forall x, y \in L$;
- (S7) $0 \cdot x = \theta$, $\forall x \in L$.

PROPERTIES OF C-METRIC SPACES

In this section we study C -metric spaces and present some properties regarding canonical decomposition and C -fixed point problem.

We now recall some concepts from [23].

Definition 3.1. [23] Let X be a non-empty set. A function $d : X \times X \rightarrow \overline{\mathbb{R}}$ is called an *approximate metric* if it satisfies only a part of the metric axioms. The pair (X, d) is called an *approximate metric space*.

For example, pseudometrics, generalized metrics (in the sense of Luxemburg [24], Lawvere [19] and Branciari [25]), quasimetrics and ultrametrics are approximate metrics.

In [26] we have introduced and studied another type of approximate metric, namely *A-metric*.

Definition 3.2. [26] Let X be a non-empty set. An application $d : X \times X \rightarrow \overline{\mathbb{R}}$ is called an *A-metric* if it satisfies the following axiom:

$d(x, z) \leq d(x, y) + d(y, z), \quad \forall x, y, z \in X,$
if the addition in the right side of the above inequality makes sense.
The pair (X, d) is called an *A-metric space*.

In the sequel we study another type of approximate metric, namely *C-metric* (introduced in [23]).

Definition 3.3. [23] Let X be a non-empty set. An application $d : X \times X \rightarrow \overline{\mathbb{R}}$ is called a *C-metric* if it satisfies the following properties:

(i) $d(x, y) = d(y, x), \quad \forall x, y \in X.$
(ii) $d(x, z) \leq d(x, y) + d(y, z), \quad \forall x, y, z \in X,$
if the addition in the right side of the above inequality makes sense.
The pair (X, d) is called a *C-metric space*.

A *C-metric* does not generally satisfy the axioms:

(iii) $d(x, x) = 0, \quad \forall x \in X;$
(iv) $d(x, y) = 0 \implies x = y$ (in this case, d and (X, d) are called separated).
So, in a *C-metric space* we can have $d(x, x) \neq 0$ for some $x \in X$.

There exist *C-metrics* that are not metrics (see the next examples).

Example 3.4. I. Let (X, ρ) be a metric space and let d be defined by $d(A, B) = \sup_{\substack{x \in A \\ y \in B}} \rho(x, y)$, for every $A, B \in \mathcal{P}_b(X)$.

Then d is a *C-metric* on $\mathcal{P}_b(X)$.

II. Let $C^+([0, 1]) = \{f|f : [0, 1] \rightarrow [0, +\infty), f \text{ continuous}\}$ and let d be defined by

$$d(f, g) = \int_0^1 \max\{f(x), g(x)\} dx, \quad \forall f, g \in C^+([0, 1]).$$

Then d is a *C-metric* on $C^+([0, 1])$, with $d(0, 0) = 0$.

III. Let $X = \{a, b, c\}$ and let d be defined by $d(a, a) = d(b, b) = d(c, c) = d(a, b) = d(b, a) = 0, d(a, c) = d(c, a) = d(b, c) = d(c, b) = 2$. Then d is a *C-metric* on X that does not satisfy axiom (iv).

Remark 3.5. Let (X, d) be a *C-metric space*. Consider the family $\tau_d = \{\emptyset\} \cup \{D \subseteq X | D \neq \emptyset \text{ and there is } r \in (0, +\infty) \text{ such that } S(x, r) \subseteq D, \forall x \in D\}$, where $S(x, r) = \{y \in X | d(y, x) < r\} \cup \{x\}$. Then τ_d is a topology on X , named the topology induced by d .

The following concepts are defined in a standard way.

Definition 3.6. [23] Let (X, d) be a *C-metric space*.

I. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called *d-convergent* in X if there exists $a \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, a) = 0$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, it holds $|d(x_n, a)| < \varepsilon$ (denoted by $x_n \xrightarrow{d} a$). The set of all such points a will be denoted by $\lim_{n \rightarrow \infty} x_n$. If $\lim_{n \rightarrow \infty} x_n$ is a singleton $\{a\}$, then we write $\lim_{n \rightarrow \infty} x_n = \{a\}$ as $\lim_{n \rightarrow \infty} x_n = a$. If the set $\lim_{n \rightarrow \infty} x_n$ is empty, then $(x_n)_{n \in \mathbb{N}}$ is called *d-divergent*.

II. A sequence $(x_n)_{n \in \mathbb{N}}$ in X is called a *d-Cauchy sequence* if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|d(x_n, x_m)| < \varepsilon$ for every $n, m \geq n_0$.

III. The *C-metric space* (X, d) is said to be *d-complete* if every *d-Cauchy sequence* in X is *d-convergent* in X .

IV. An application $f : X \rightarrow X$ is said to be a *d-contraction* on X if there exists $L \in (0, 1)$ such that $d(f(x), f(y)) \leq Ld(x, y)$, for all $x, y \in X$.

V. An element $x \in X$ is called a *C-fixed point* of a function $f : X \rightarrow X$ if $d(f(x), x) = 0$.

The following statements are demonstrated as in the classical case.

Theorem 3.7. I. If (X, d) is a separated C -metric space and the sequence $(x_n)_{n \in \mathbb{N}}$ in X is d -convergent in X , then $\lim_{n \rightarrow \infty} x_n \in \mathcal{S}(X)$.

II. Every d -convergent sequence in a C -metric space is d -Cauchy.

III. If (X, d) is a C -metric space, with $d : X \times X \rightarrow [0, +\infty]$ and the sequence $(x_n)_{n \in \mathbb{N}} \subset X$ is d -convergent to $a \in X$, then every subsequence of $(x_n)_{n \in \mathbb{N}}$ is also d -convergent to a .

Let X be a real linear space and $\mathcal{F} = \{F|F : T \rightarrow \mathcal{A}\}$ where \mathcal{A} is a subset of $\mathcal{P}_0(X)$ satisfying the following conditions:

- (a) $\{0\} \in \mathcal{A}$;
- (b) $E_1 + E_2 \in \mathcal{A}$ for every $E_1, E_2 \in \mathcal{A}$;
- (c) $\alpha E \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$ and $E \in \mathcal{A}$.

Remark 3.8. I. Such subfamily $\mathcal{A} \subset \mathcal{P}_0(X)$ can be $\mathcal{P}_0(X)$ or $\mathcal{P}_b(X)$ or the family $\mathcal{P}_k(X)$ of all non-empty compact subsets of X , where X is a real normed space.

II. \mathcal{F} is a semi-linear space with respect to the operations of sum and multiplication by real scalars defined by:

$$(F + G)(t) = F(t) + G(t), \forall t \in T,$$

$$(\alpha F)(t) = \alpha F(t), \forall t \in T, \alpha \in \mathbb{R}.$$

The origin is the multifunction θ defined by $\theta(t) = \{0\}$ for every $t \in T$.

Let $|\cdot|$ be a finite set-norm on \mathcal{A} .

Definition 3.9. A multifunction $F : T \rightarrow \mathcal{A}$ is called *sn-bounded* if there exists $\alpha \in [0, +\infty)$ such that $|F(t)| \leq \alpha$ for every $t \in T$.

Example 3.10. Let $\mathcal{M} = \{F; F : T \rightarrow \mathcal{A}, F \text{ is sn-bounded}\} \subseteq \mathcal{F}$. For every $F, G \in \mathcal{M}$, denote

$$d(F, G) = \sup_{t \in T} |F(t) - G(t)|.$$

Then d is a C -metric on \mathcal{M} .

Definition 3.11. Let (X, d) be a C -metric space with $d : X \times X \rightarrow [0, +\infty]$ and let " \sim " be a relation on X defined by: $x \sim y$ if and only if $d(x, y) < +\infty$ or $x = y$.

It follows that the relation " \sim " is an equivalence relation on X and let \tilde{X} be the set of all equivalence classes, $\tilde{X} = \{X_a; a \in X\}$. The writing $X = \bigcup_{a \in X} X_a$ is called the *canonical decomposition* of X .

Theorem 3.12. Let (X, d) be a C -metric space, with $d : X \times X \rightarrow [0, +\infty]$ and let $\rho : X \times X \rightarrow [0, +\infty]$ be defined for every $x, y \in X$ by $\rho(x, y) = d(x, y)$ if $x \neq y$ and $\rho(x, y) = 0$ if $x = y$. Then ρ is also a C -metric on X , that satisfies axiom (iii). If d is separated, then ρ is separated, too. And if d has its values in $[0, +\infty)$, then (X, ρ) is a metric space.

As in Jung [27], we obtain the following properties.

Theorem 3.13. Suppose that (X, d) is a C -metric space with $d : X \times X \rightarrow [0, +\infty]$, $X = \bigcup_{a \in X} X_a$ is the canonical decomposition of X and let $d_a = d|_{X_a \times X_a}$ for any $a \in X$.

Then the following properties hold:

- I. (X_a, d_a) is a C -metric space for every $a \in X$.
- II. (X, d) is d -complete if and only if (X_a, d_a) is d_a -complete for all $a \in X$.

Proof. I. It follows from the definitions of X_a and d_a .

II. Let (X, d) be a d -complete C -metric space. Consider $a \in X$ and $(x_n)_{n \in \mathbb{N}} \subset X_a$ a d_a -Cauchy sequence. From the definition of d_a it results that $(x_n)_{n \in \mathbb{N}}$ is d -Cauchy in X . By the hypothesis, it is d -convergent in X . So there exists $b \in X$ such that $\lim_{n \rightarrow \infty} d(x_n, b) = 0$. But $(x_n)_{n \in \mathbb{N}} \subset X_a$, that is $d(x_n, a) = 0$ or $x_n = a$, for each $n \in \mathbb{N}$.

Now, two cases hold:

- (1) There is $n_0 \in \mathbb{N}$ such that $d(x_n, a) = 0$ for any $n \geq n_0$. It results

$$0 \leq d(a, b) \leq d(a, x_n) + d(x_n, b) = d(x_n, b), \forall n \geq n_0$$

and so $d(a, b) = 0$. This implies that $b \in X_a$.

(2) There exists $(x_{n_k})_{k \in \mathbb{N}}$ a subsequence of $(x_n)_{n \in \mathbb{N}}$ such that $x_{n_k} = a$ for every $k \in \mathbb{N}$. Now, $0 = \lim_{k \rightarrow \infty} d(x_{n_k}, b) = d(a, b)$. So $b \in X_a$.

Therefore $(x_n)_{n \in \mathbb{N}}$ is d_a -convergent in X_a , so (X_a, d_a) is d_a -complete.

Conversely, suppose that (X_a, d_a) is d_a -complete for every $a \in X$ and let $(x_n)_{n \in \mathbb{N}}$ be a d -Cauchy sequence. Then there exists $N \in \mathbb{N}$ such that $(x_n)_{n \geq N}$ is d_N -Cauchy in X_N . Since X_N is d_N -complete it follows that there is $a \in X_N$ such that $x_n \xrightarrow{d} a$ and the proof ends. ■

Theorem 3.14. Let (X, d) be a C -metric space, $X = \bigcup_{a \in X} X_a$ the canonical decomposition of X and let $f : X \rightarrow X$ be a function such that if $x, y \in X$, $d(x, y) < +\infty$ then $d(f(x), f(y)) < +\infty$. Then f has a C -fixed point related to d if and only if there exists $a \in X$ such that $f_a = f|_{X_a}$ has a C -fixed point related to d_a .

Proof. Suppose there exists $a \in X$ such that $d(f(a), a) = 0$. It follows that $f(a) \in X_a$ and $d_a(f(a), a) = 0$. So a is a C -fixed point for f_a .

Conversely, suppose that there exists $a \in X$ such that f_a has a C -fixed point $b \in X_a$. So $d_a(f_a(b), b) = 0$. Consequently, $d(f(b), b) = 0$ and b is a C -fixed point for f . ■

CONCLUSIONS

This paper investigate a topic in the theory of approximate metrics [23], i.e. applications that satisfy only a part of the metric axioms. In fact, we study some properties of a new kind of approximate metric, that is a C -metric, introduced in a recent paper [23]. A C -metric on X is a real application defined on $X \times X$ that satisfies only two metric axioms: symmetry and triangular inequality. Using the topology induced by a C -metric, we obtain some properties of C -metric spaces, such as convergence properties, a canonical decomposition and a C -fixed point theorem, which is remarkable since there is no metric on the space. As a direction of our future work, we will study different aspects of the topological structure of C -metric spaces: closed sets, compactness, connection, continuity etc.

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REFERENCES

- [1] G. Apreutesei, *Hausdorff topology and some operations with subsets*, An. Şt. Univ. "Al. I. Cuza" Iaşi 44, 445-454 (1998).
- [2] G. Apreutesei, *Cauchy nets and convergent nets on semilinear topological spaces*, [Topology and its Applications](#) 159, 2922-2931 (2012).
- [3] G. Apreutesei and N. Apreutesei, *On the continuity of special set-valued functions*, An. Şt. Univ. "Dunărea de Jos" Galaţi 23, 65-68 (2005).
- [4] G. Apreutesei, *The hyperspatial topologies and almost linear spaces*, An. Şt. Univ. "Al. I. Cuza" Iaşi 48, 3-18 (2002).
- [5] G. Apreutesei, N. E. Mastorakis, A. Croitoru and A. Gavriluţ, *On the translation of an almost linear topology*, WSEAS Transactions on Mathematics 8, 479-488 (2009).
- [6] S. M. Aseev, *Quasilinear operators and their applications in the theory of multivalued mappings* (in Russian), Tr. MIAN SSSR, Nauka Moskva 167, 25-52 (1985); Proc. Steklov Inst. Math. 2, 23-52 (1986).
- [7] D. Candeloro, A. Croitoru, A. Gavriluţ and A. R. Sambucini, *Atomicity related to non-additive integrability*, [Rend. Circ. Mat. Palermo, II. Ser](#) 65 (3), 435-449 (2016). DOI:10.1007/s12215-016-0244-z
- [8] A. Croitoru and D. M. Borş, *Remarks on weak linear spaces*, Bulletin of the Polytechnic Institute of Iaşi, Tomul LXI (LXV), Fasc. 2, 67-74 (2015).
- [9] G. Godini, *A framework for best simultaneous approximation: normed almost linear spaces*, [Journal of Approximation Theory](#) 43, 338-358 (1985).

- [10] Y. Liu, Z. Li, H. Xiong, X. Gao and J. Wu, *Understanding of Internal Clustering Validation Measures*, in Proceedings of 2010 IEEE International Conference on Data Mining, pp. 911-916, DOI 10.1109/ICDM.2010.35.
- [11] N. E. Mastorakis, *Modeling dynamical systems via the Takagi-Sugeno fuzzy model*, WSEAS Transactions on Systems 3, Issue 2, 668-675 (2004).
- [12] A. Croitoru and N. Mastorakis, *Estimations, convergences and comparisons on fuzzy integrals of Sugeno, Choquet and Gould type*, in Proceedings of the 2014 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), July 6-11, 2014, Beijing, China, pp. 1205-1212.
- [13] A. Precupanu, A. Gavriluț and A. Croitoru, "A fuzzy Gould type integral", *Fuzzy Sets and Systems* 161, 661-680 (2010).
- [14] T. Precupanu, *Scalar minimax properties in vectorial optimization*, International Series of Numerical Mathematics, Birkhauser Verlag Basel 107, 299-306 (1992).
- [15] C. Stamate and A. Croitoru, *Convergence results for sequences of non-linear integrals*, Recent Advances In Computer Engineering Series 15, 76-81 (2013).
- [16] A. Edalat and R. Heckmann, *A computational model for metric spaces*, *Theoret. Comput. Sci.* 193, 53-73 (1998).
- [17] S. Hart, *An axiomatization of the consistent non-transferable utility value*, *Int. J. Game Theory* 33, 355-366 (2005).
- [18] A. K. Laha and M. Saha, *Fixed point for some classes of set valued mappings on a metric space endowed with a graph*, *ROMAI Journal* 11, No. 1, 115-129 (2015).
- [19] F. Lawvere, *Metric spaces, generalized logic and closed categories*, *Rend. del Sem. Mat. e Fis. di Milano* 43, 135-166 (1973).
- [20] E. Pap and I. Stajner, *Generalized pseudo-convolution in the theory of probabilistic metric spaces, information, fuzzy numbers, optimization, system theory*, *Fuzzy Sets and Systems* 102, 393-415 (1999).
- [21] M. Turinici, *Functional contractions in local Branciari metric spaces*, *ROMAI Journal* 8 (2), 189-199 (2012).
- [22] S. Vickers, *Localic completion of quasimetrics spaces*, Technical Report Doc 97/2, Department of Computing, Imperial College, London, 1998.
- [23] A. Croitoru and G. Apreutesei, *C-Metric spaces*, submitted for publication.
- [24] W. A. J. Luxemburg, *On the convergences of successive approximations in the theory of ordinary differential equations*, *Indag. Math.* 20, 540-546 (1958).
- [25] A. Branciari, *A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces*, *Publ. Math. Debrecen* 571-2, 31-37 (2000).
- [26] G. Apreutesei and A. Croitoru, *A-Metric spaces*, submitted for publication.
- [27] A. Croitoru, *Set-norm continuity of set multifunctions*, *ROMAI Journal* 6, 47-56 (2010).
- [28] C.F.K. Jung, *On generalized complete metric spaces*, *Bull. A.M.S.* 75, 113116 (1969).