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The Significance of Temporal Discretization in the High Order Numerical Scheme for Hyperbolic Conservation Law

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Abstract. Spatial and temporal discretization has a big impact to the overall accuracy of the numerical scheme. The spatial discretization affects the calculation of physical flux or other space-derived variables, which often require high order scheme in the non-linear problem to achieve the desired accuracy. While temporal discretization also has a significant impact on the stability of the numerical scheme and total computational cost. Strong stability preserving Runge Kutta scheme often used as temporal discretization scheme for high order numerical scheme. In this work, we are motivated to study the significance of temporal discretization order (up to fourth order) to the overall numerical scheme accuracy and total computational cost while high order weighted essentially non-oscillatory scheme used as spatial discretization technique in hyperbolic conservation law. We found that the third order SSP Runge Kutta scheme is an effective scheme with respect to accuracy and computational cost.

INTRODUCTION

Space-time dependent partial differential equations (PDEs) require spatial and temporal discretization in their numerical scheme. For practical reason, the second-order accurate scheme for spatial discretization which classified as low order scheme often used in compressible flow calculation because of the simplicity and robustness [1], but it has several limitations [2] such as the occurrence of oscillations in the complicated flow problem. The high order scheme became a feasible choice to overcome this problem. Classical high order scheme sometimes still give dissipative result when fixed stencil interpolation is employed. The problem arises because fixed stencil does not manage which stencil would be used for interpolation but the scheme selects all stencil according to its accuracy order, while adaptive stencil interpolation firstly chooses the right stencil for interpolation, its mean that the first thing to do is to select non-discontinuous or most smooth stencil. For temporal discretization, strong stability preserving (SSP) or also called total variation diminishing (TVD) Runge Kutta is the most popular scheme. The importance of the temporal discretization is the stability during the time stepping and maintain the positivity of some physical variable in the complicated flow that contains shock or discontinuity [3].

A popular high order scheme for spatial discretization is weighted essentially non-oscillatory (WENO) scheme [4] which able to preserve non-oscillatory state in the discontinuous function and give a high-resolution result. The main problem of WENO scheme is the high computational load, but indeed equivalent to the accuracy given by WENO scheme. Some simplification of WENO scheme reconstruction component such as simplifying the smoothness indicator [5] is one of several attempts to reduce the computational cost of standard WENO scheme. Besides the spatial discretization, temporal discretization also has a big impact on the total computational cost. For example, the third order TVD Runge Kutta scheme required three times time evolutions of each time step, for the fourth order, it would increase to five times evolutions since there is no four stage optimal fourth order RK scheme, so that computational cost increased 25%. In this study, we are motivated to study the significance of temporal discretization order (up to fourth order) to the overall numerical scheme accuracy while high order weighted essentially non-oscillatory scheme used as spatial discretization technique in hyperbolic conservation law.

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EULER EQUATIONS

Consider following one-dimensional Euler equations:

$$\begin{cases} \mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = 0, & x \in \mathrm{R},\ t > 0 \\ \mathbf{q}(x,0) = \mathbf{q}_0(x), & x \in \mathrm{R},\ t = 0 \end{cases} \tag{1}$$

where $\mathbf{q}$ is a vector of conserved variable and $\mathbf{f}(\mathbf{q})$ is a flux vector, while x and t denote the spatial location and time respectively.

$$\mathbf{q} = \begin{bmatrix} \rho \\ \rho\mathbf{u} \\ \rho e \end{bmatrix} \quad \mathbf{f}(\mathbf{q}) = \begin{bmatrix} \rho u \\ \rho\mathbf{u}^2 + p \\ (\rho e + p)\mathbf{u} \end{bmatrix}$$

Since the ideal gas relation is used, $\rho e = \rho\mathbf{u}^2 + p/(\gamma - 1)$ with γ denote the specific heat ratio.

Let we evaluate (1) in a small control volume (cell) $I_j = (x_{j+1/2}, x_{j-1/2})$ with a uniform length $\Delta x = x_{i+1/2} - x_{i-1/2}$, integrating (1) over cell I_j we obtain the following equation:

$$\frac{\mathrm{d}\overline{\mathbf{q}}}{\mathrm{d}t} + \frac{1}{\Delta x}\left(\mathbf{f}_{j+1/2} - \mathbf{f}_{j-1/2}\right) = 0 \tag{2}$$

where $\overline{\mathbf{q}}$ is the cell average of $\mathbf{q}$

$$\overline{\mathbf{q}}(t) = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \mathbf{q}(x,t)\,dx \tag{3}$$

and $\mathbf{f}_{j\pm1/2}$ is the spatial average of flux vector at cell boundaries $x_{j\pm1/2}$.

A BRIEF STANDARD WENO RECONSTRUCTION

Let we freeze at the cell I_j, for all cell j:

1. Calculate the averaged value of $\overline{q}_{j+1/2}$ using Roe averaging
2. Form the left eigenvector L and the right eigenvector R with the averaged value
3. Transform the conservative variable to the characteristic variable using left eigenvector
$$\overline{v} = L\overline{q}$$
4. Perform WENO 5th reconstruction
   Compute the third order polynomial $k = 3$ ($r = 0,1,...,k-1$):
$$v_{j+1/2}^{(r)} = \sum_{j=0}^{k-1} c_{rj}\overline{v}_{i-r+j}$$
$$v_{j-1/2}^{(r)} = \sum_{j=0}^{k-1} \tilde{c}_{rj}\overline{v}_{i-r+j}$$
   Calculate the smoothness indicator:
$$\beta_0 = \frac{13}{12}\left(\overline{v}_j - 2\overline{v}_{j+1} + \overline{v}_{j+2}\right)^2 + \frac{1}{4}\left(3\overline{v}_j - 4\overline{v}_{j+1} + \overline{v}_{j+2}\right)^2$$
$$\beta_1 = \frac{13}{12}\left(\overline{v}_{j-1} - 2\overline{v}_j + \overline{v}_{j+1}\right)^2 + \frac{1}{4}\left(3\overline{v}_{j-1} - \overline{v}_{j+1}\right)^2$$
$$\beta_2 = \frac{13}{12}\left(\overline{v}_{j-2} - 2\overline{v}_{j-1} + \overline{v}_j\right)^2 + \frac{1}{4}\left(\overline{v}_{j-2} - 4\overline{v}_{j-1} + 3\overline{v}_j\right)^2$$

Calculate the non-linear weight:

$$\omega_r = \frac{\alpha_r}{\sum_{r=0}^{k-1}\alpha_r}, \quad \alpha_r = \frac{d_r}{\left(\varepsilon + \beta_r\right)^2}$$

d_r denote the optimal weight and ε is a small number to prevent denominator zero valued, the value of ε is set 10^{-6} suggested by [4]. Finally,

$$v_{j+1/2}^{-} = \sum_{r=0}^{k-1}\omega_r v_{j+1/2}^{(r)}$$

Reconstruction of $v_{j-1/2}^{+}$ is same as above with reverse d_r.

5. Transform back to the conservative variable

$$q_{j\pm1/2}^{\mp} = Rv_{j\pm1/2}^{\mp}$$

6. Compute the cell boundary flux using HLLC approximate Riemann solver [6]

$$F = h\left(q_{j-1/2}^{+}, q_{j+1/2}^{-}\right)$$

TEMPORAL DISCRETIZATION

Since the spatial derivatives of equation (2) are obtained, it leaves ordinary differential time-dependent equation:

$$\frac{\mathrm{d}\overline{\mathbf{q}}}{\mathrm{d}t} = L\left(\mathbf{q}\right) \tag{4}$$

We would use a different order of SSP Runge Kutta scheme [7] as temporal discretization for equations (4). The following equations are the SSP RK scheme, (n, m) denotes the n-stage m[th] order.

1. SSP RK (2,2)

$$q^{(1)} = q^n + \Delta t L\left(q^n\right)$$

$$q^{n+1} = \frac{1}{2}\left(q^n + q^{(1)} + \Delta t L\left(q^{(1)}\right)\right)$$

with Courant Friedrich-Levy (CFL) number $c = 1$.

2. SSP RK (3,3)

$$q^{(1)} = q^n + \Delta t L\left(q^n\right)$$

$$q^{(2)} = \frac{3}{4}q^n + \frac{1}{4}q^{(1)} + \frac{1}{4}\Delta t L\left(q^{(1)}\right)$$

$$q^{n+1} = \frac{1}{3}q^n + \frac{2}{3}q^{(2)} + \frac{2}{3}\Delta t L\left(q^{(2)}\right)$$

with CFL number $c = 1$.

3. SSP RK (5,4)

$$q^{(1)} = q^n + 0.391752226571890\Delta t L\left(q^n\right)$$

$$q^{(2)} = 0.444370493651235\,q^n + 0.555629506348765\,q^{(1)} + 0.368410593050371\Delta t L\left(q^{(1)}\right)$$

$$q^{(3)} = 0.620101851488403\,q^n + 0.379898148511597\,q^{(2)} + 0.251891774271694\,\Delta t L\left(q^{(2)}\right)$$

$$q^{(4)} = 0.178079954393132\, q^{n} + 0.821920045606868\, q^{(3)} + 0.544974750228521\, \Delta t L\left(q^{(3)}\right)$$

$$q^{n+1} = 0.517231671970585\, q^{(2)} + 0.096059710526147\, q^{(3)} + 0.063692468666290\, \Delta t L\left(q^{(3)}\right) + 0.386708617503269\, q^{(4)} + 0.226007483236906\, \Delta t L\left(q^{(4)}\right)$$

with CFL number $c = 1.508$ and the optimal coefficient is $c = 0.377$.

NUMERICAL STUDIES

We test the numerical scheme using common Sod shock tube case with the following initial condition:

$$(\rho, u, p) = \begin{cases} (1, 0, 1) & x \le 0.5 \\ (0.125, 0, 0.1) & x > 0.5 \end{cases}$$

To ensure the stability of the scheme, the CFL number set to 80% of its maximum. The final time is 0.1 and the errors (absolute error) are calculated from the known exact solution.

Fig. 1 shows the density profile at $t = 0.1$ using different RK scheme with 200 elements. The WENO-RK(2,2) scheme gives worst monotonicity in the shock region while the other scheme give monotonicity.

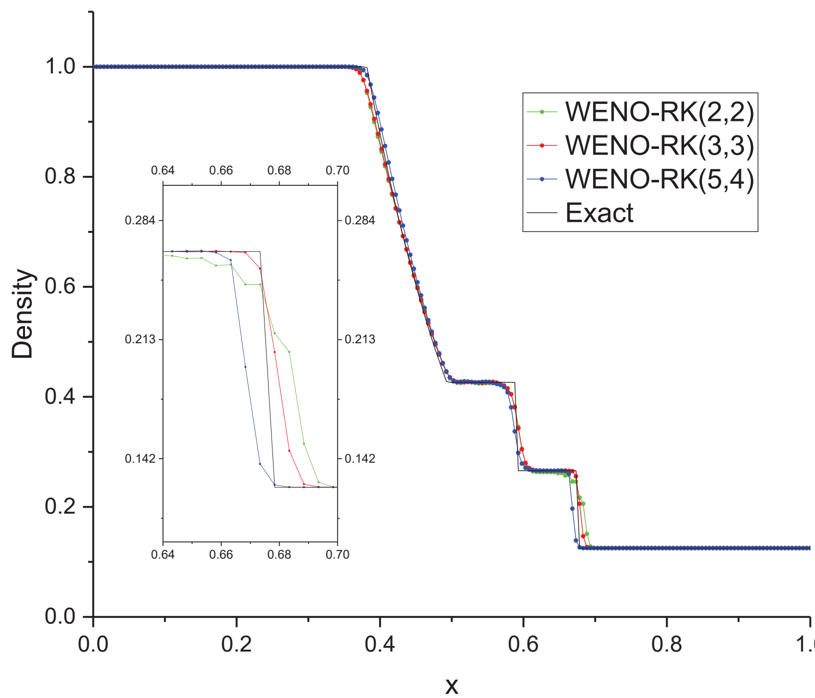

FIGURE 1. Density profile at $t = 0.1$ and $N = 200$ elements

Fig. 2 shows the density errors with respect to the number of elements. The WENO-RK(2,2) and WENO-RK(3,3) schemes decrease the errors consistent with the number of elements, but the WENO-RK(5,4) scheme does not agree with that and we notice that using optimal coefficient for RK(5,4) scheme give the same result.

As shown in Fig. 3, the computational cost increase due to the number of stages for each time step in the temporal discretization scheme. Compared to the other schemes, the WENO-RK(3,3) scheme give good accuracy with moderate computational cost, also the monotonicity is preserved.

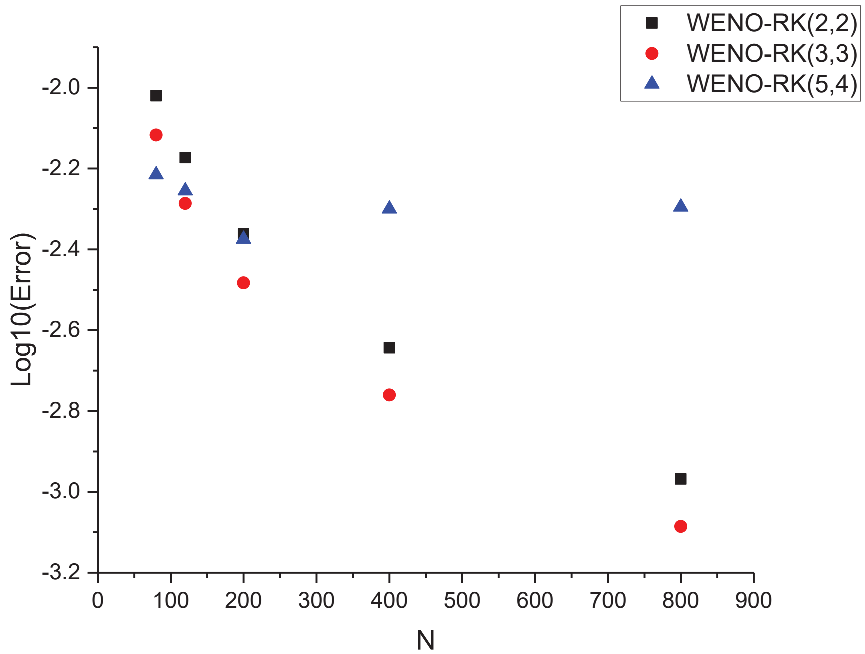

FIGURE 2. Density errors for the different number of elements

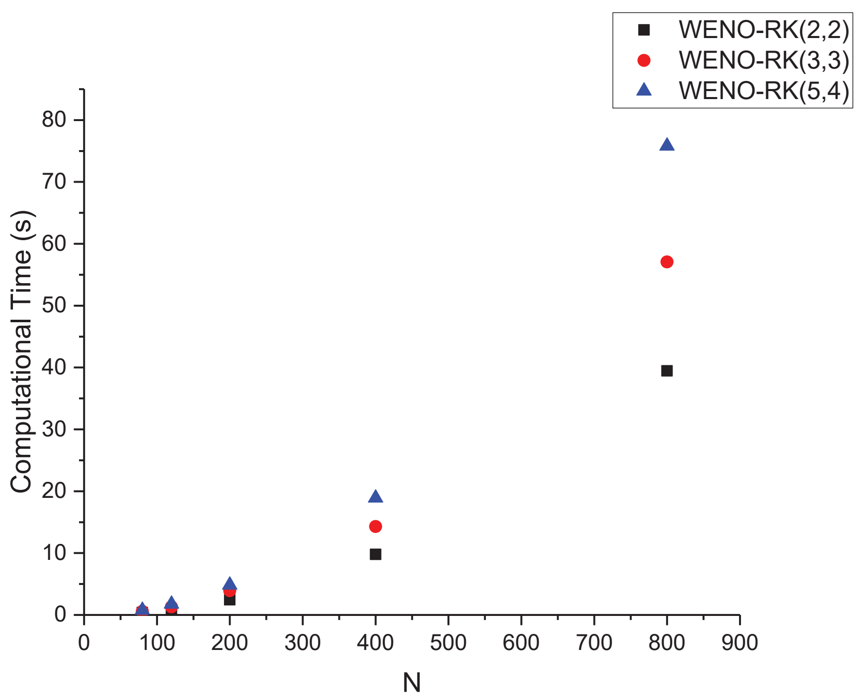

FIGURE 3. Computational time for the different number of elements

CONCLUSIONS

In this work, we have studied the significance of the temporal discretization to the solution of hyperbolic conservation law using second-, third-, and fourth-order SSP Runge Kutta scheme. The numerical errors, the computational time and the monotonicity are considered as the primary variable in this study. The numerical result shows that the temporal discretization scheme has a significant impact to the monotonicity of the numerical result, errors and computational time. We notice that the SSP RK(3,3) is suitable for high order scheme that gives good resolution, moderate computational speed, and gives the monotone solution as shown in the numerical result.

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