Computational capabilities of small-world Boolean networks

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Abstract

We discuss an ensemble investigation of the computational capabilities of small-world networks as compared to ordered and random topologies, using random Boolean functions to provide dynamics of the nodes. We find that the ordered phase of the dynamics (low activity in dynamics) and topologies with low randomness are dominated by information storage, while the chaotic phase (high activity in dynamics) and topologies with high randomness are dominated by information transfer. Information storage and information transfer are somewhat balanced near the small-world regime, providing quantitative evidence that small-world networks do indeed have a propensity to "combine" comparably large information storage and transfer capacity.

Introduction

It is often suggested that the prevalence of small-world networks in nature is due to an inherent capability to store and transfer information efficiently (Watts and Strogatz, 1998; Latora and Marchiori, 2001). Yet while these claims are all based on quantitative results, they are not based on direct measurement of the relevant dynamic information quantities, either relying on measurements of topological features or on equating perturbation or damage-spreading type results to information transfer. A recently published framework (Lizier et al., 2008, 2010) affords the opportunity to directly measure these computational properties or *information dynamics*.

We discuss our previously published ensemble investigation (Lizier et al., 2011) of the *information dynamics* of small-world Boolean networks, from the perspective of the distributed computation undertaken by the nodes of the network in the transient computation of their attractor. We show that small-world networks exhibit something of a balance between information storage and transfer capabilities, with the capability for apparent (or coherent) information transfer being maximized near the small-world state.

Small-world Boolean Networks

The small-world network model (Watts and Strogatz, 1998) specifies how to tune networks with N nodes (with $\bar{K}$ nearest neighbors each) from ordered, lattice-like structures, to fully-random topologies using a level of random rewiring of edges γ. There is a significant intermediate range of values γ for which networks exhibit both high clustering (typical of ordered networks) and small average path length (typical of randomized networks); networks in this range are labeled *small-world networks*.

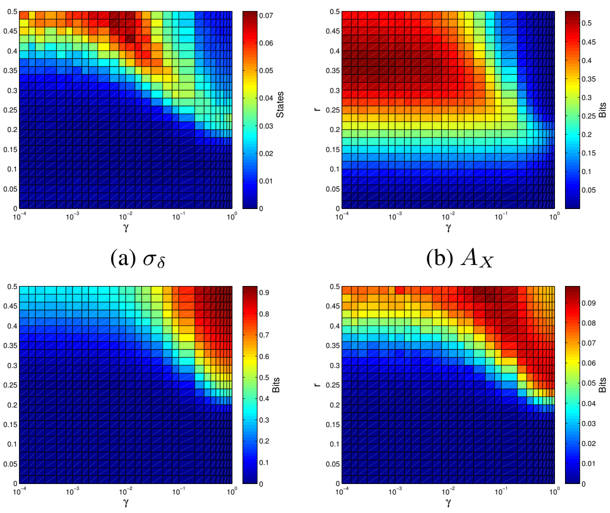
We generate time-series activity for the networks by assigning synchronous random Boolean functions to the nodes (with a bias probability r for each input configuration of each node to produce a "1" output). This equates to combining *random Boolean networks* (RBNs) (Kauffman, 1993; Gershenson, 2004) with small-world topologies. We select RBNs due to the very large sample space they provide, and their use as models of Gene Regulatory Networks (GRNs). They display a well-known phase transition from *ordered dynamics* (at low connectivity $\bar{K}$ and activity r) to *chaotic dynamics* (at high connectivity and activity), as measured by damage spreading with the normalized Hamming distance δ (Gershenson, 2004). We identify the critical state in *finite* networks where the standard deviation σ_δ of δ is maximized. Finally, other recent studies combine RBNs with small-world topologies, e.g. (Lu and Teuscher, 2009).

Information dynamics

The *active information storage* (Lizier et al., 2010) for a node X is defined as the average mutual information (MI) between its semi-infinite past $x_n^{(k)}$ (as $k \to \infty$) and its next state x_{n+1} at time step $n+1$: $A_X(k) = \langle i(x_n^{(k)}; x_{n+1}) \rangle$.

We note that the local entropy for X is the sum of $A_X(k)$ and the local entropy rate $H_{\mu X}(k) = \langle h(x_{n+1} \mid x_n^{(k)}) \rangle$; i.e. $H_X = A_X(k) + H_{\mu X}(k)$. In a deterministic system such as RBNs, there is no intrinsic uncertainty in $H_{\mu X}(k)$ so it represents the *joint* contribution or transfer from the causal information sources to the destination (Lizier et al., 2010).

The *information transfer* (formulated in the *apparent transfer entropy* (Schreiber, 2000; Lizier et al., 2008)) from *one* source Y to a destination X is the average MI between the previous source state y_n and the next destination state x_{n+1}, *conditioned* on the semi-infinite past of the destination $x_n^{(k)}$ (as $k \to \infty$): $T_{Y \to X}(k) = \langle i(y_n; x_{n+1} \mid x_n^{(k)}) \rangle$.



(a) σ_δ (b) A_X

(c) $H_{\mu X}$ (d) $T_{Y \to X}$

Figure 1: Measures of dynamics versus r and γ for $K = 4$ (color online). Ordered dynamics occur for low r and γ (bottom left), and chaotic dynamics for high r and γ (top right); the critical regime at the maxima in (a) separates these.

Results and discussion

We examine ensembles of networks of size $N = 264$ as a function of $\bar{K}$, r and γ (sources of links only rewired; other details in (Lizier et al., 2011)) using extensions to RBNLab (Gershenson, 2003). We use $\bar{K} = 4$: the small-world region then occurs approximately for $0.03 \leq \gamma \leq 0.1$.

Using σ (not shown) and σ_δ (Fig. 1a, Fig. 2) we see that (i) for fixed γ an ordered phase exists for low r, with the chaotic phase for large r. Crucially, a similar transition occurs (ii) with respect to γ for fixed r, with ordered dynamics for small γ (more ordered networks), and chaotic dynamics for large γ (more randomized networks). The critical region in dynamics has much similarity to the small-world regions of γ (however this is highly dependent on activity r).

The ordered phase of these dynamics (low r and γ) is dominated by information storage A_X (Fig. 1b), while the chaotic phase of the dynamics (high r and γ) is dominated by information transfer (captured in total in $H_{\mu X}$ in Fig. 1c). The critical regime exhibited a balance between these two operations (Fig. 2), and since this was near to the small-world topology regime, it could be said that **small-world networks have a propensity to combine comparably large information storage and transfer capabilities**.

This balance can be explained by considering how the topological features related to the information dynamics. Information storage is strongly correlated to the clustering coefficient: locally clustered structure appears to strongly support storage operations. In contrast, information transfer was anti-correlated with average path length: long links appear to be a crucial facilitator of transfer across the network.

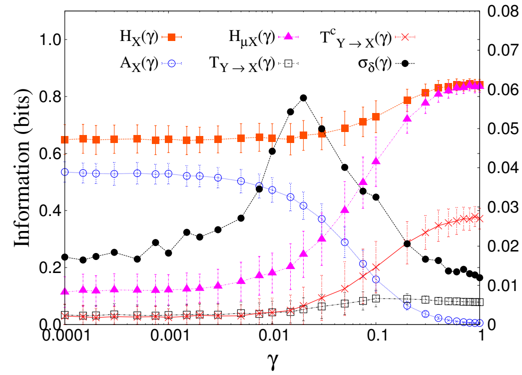


Figure 2: Measures of dynamics versus γ, for $K = 4$ and $r = 0.36$. σ_δ is plotted against the right y-axis. Error bars indicate *standard deviation* across 250 sampled networks.

Additionally, Fig. 1d shows apparent information transfer $T_{Y \to X}$ is maximized slightly inside the chaotic phase of dynamics (near to the small-world regime). The capacity for coherent computation is eroded as too many random links promote interactions and make the dynamics more chaotic.

These results add evidence that small-world networks hold computational advantages over other topologies.

References

Gershenson, C. (2003). RBNLab. Online software: http://rbn.sourceforge.net.

Gershenson, C. (2004). Introduction to random Boolean networks. In Bedau, M., Husbands, P., Hutton, T., Kumar, S., and Suzuki, H., editors, *Proceedings of the Workshops and Tutorials of the Ninth International Conference on the Simulation and Synthesis of Living Systems (ALife IX), Boston, USA*, pages 160–173.

Kauffman, S. A. (1993). *The Origins of Order: Self-Organization and Selection in Evolution*. Oxford University Press, New York.

Latora, V. and Marchiori, M. (2001). Efficient behavior of small-world networks. *Physical Review Letters*, 87(19):198701.

Lizier, J. T., Pritam, S., and Prokopenko, M. (2011). Information dynamics in small-world Boolean networks. *Artificial Life*. to appear.

Lizier, J. T., Prokopenko, M., and Zomaya, A. Y. (2008). Local information transfer as a spatiotemporal filter for complex systems. *Physical Review E*, 77(2):026110.

Lizier, J. T., Prokopenko, M., and Zomaya, A. Y. (2010). Information modification and particle collisions in distributed computation. *Chaos*, 20(3):037109.

Lu, Q. and Teuscher, C. (2009). Damage spreading in spatial and small-world random Boolean networks. arXiv:0904.4052.

Schreiber, T. (2000). Measuring information transfer. *Physical Review Letters*, 85(2):461–464.

Watts, D. J. and Strogatz, S. (1998). Collective dynamics of 'small-world' networks. *Nature*, 393:440–442.