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Prime Numbers, Atomic Nuclei, Symmetries and Superconductivity

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Abstract. The distribution of prime numbers is directly related to the statistical distribution of the nontrivial zeros of the Riemann Zeta function that closely resembles that of energy levels of atomic nuclei. Moreover, Riemann Zeta function plays a fundamental role in many areas of mathematics, from number theory to geometry and theory of dynamical systems and in physics from quantum chaos to the theory of quantum fields and of quasicrystals. Unfortunately, no proof exists of the so-called Riemann hypothesis stating that all its nontrivial zeros lie on the critical line $\Re(s) = ½$. A new method is proposed to prove the Riemann hypothesis based on the Hilbert–Pólya conjecture and a superconducting-type Hamiltonian in the Hilbert-Fock L_2 space.

INTRODUCTION

The Riemann Zeta function is defined as [1]:

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

when $\Re(s) > 1$, while elsewhere $\varsigma(s)$ is provided by analytic continuation being a meromorphic function whose only singularity in the complex plane is a simple pole at s=1, with residue 1. This function has both trivial zeros corresponding to negative even integers, and nontrivial ones. The nontrivial zeros are known to all lie in the so-called critical strip $0 < \Re(s) < 1$, and always come in complex conjugate pairs. All known calculated nontrivial zeros lie on the critical line $\Re(s) = 1/2$. The Riemann hypothesis (RH) states that they all lie on this line:

RH: All the zeros $\rho = \beta + i\gamma$ of function (1) in the critical strip, $0 < \Re(s) < 1$, lie on the line s = 1/2. (2)

Further properties of Riemann Zeta function and definition and properties of main related functions (Hurwitz Zeta function, dilogarithm and polylogarithm or Jonquière's functions, Lerch's transcendent and the Dirichlet L-functions) as well as applications and relevant methods of computation are given in ref. [1] and herein quoted papers.

Many tables on nontrivial zeros were compiled by Odlyzko and coworkers [2]. The first 2,001,052 zeros of the Riemann zeta function, accurate to within 4*10^(-9) (see http://www.dtc.umn.edu/~odlyzko/index.html, home page of Andrew M. Odlyzko, University of Minnesota, School of Mathematics) confirm the RH in eq.(2). Calculations performed up to now have found no nontrivial zeros off the critical line.

Understanding the distribution of the prime numbers is directly related to understanding the zeros of the Riemann Zeta function. The distribution of prime numbers is most simply expressed as a (discontinuous) step function $\pi(x)$, where $\pi(x)$ is the number of primes less than or equal to x. Function (1) is then asymptotic to function $\pi(x)$, so providing a particularly good estimate for the distribution of primes. Therefore, the Riemann Zeta function can be thought of as describing the "average" behaviour of the primes. The correction terms which are defined using the zeros of the zeta function collectively describe the local fluctuations. In other words, with the single function (1) and the set of nontrivial Zeta zeros (2), we can exactly reconstruct the prime counting function $\pi(x)$. Consequently, the sequence of nontrivial zeta zeros is sometimes described as being "dual" to the sequence of primes.

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THE HILBERT – PÓLYA CONJECTURE AND ITS AFTERMATH

A suggestion is traditionally attributed to Hilbert and Pólya that the zeros of the Riemann Zeta function are eigenvalues of some operator, and the Riemann hypothesis is then true if that operator is Hermitian. Therefore, in mathematics, the Hilbert–Pólya conjecture represents a possible approach to the solution of the Riemann hypothesis (2), by means of spectral theory.

However, as clarified in ref.[3], David Hilbert did not work in the central areas of analytic number theory and his name has become known for the Hilbert–Pólya conjecture for reasons that are anecdotal, possibly because Hilbert listed the RH among the 23 most important unsolved problems in mathematics on the occasion of his celebrated lecture delivered at the International Congress of Mathematicians at Paris in 1900. The Hilbert's problem no. 8 is really devoted to "problems of prime numbers" and specifically to the desirable proof of eq. (2) [4].

In a letter to Andrew Odlyzko, dated January 3, 1982, George Pólya said that while he was in Göttingen around 1912 to 1914 he was asked by Edmund Landau for a physical reason that the Riemann hypothesis should be true, and he suggested that this would be the case if the imaginary parts of the zeros of the Riemann zeta function corresponded to eigenvalues of an unbounded self-adjoint operator [3].

The earliest published statement of the conjecture seems to be in [5], where Hugh Montgomery discusses the pair correlation of zeros of the Zeta function. Previously, Selberg proved a duality between the length spectrum of a Riemann surface and the eigenvalues of its Laplacian [6]: the Selberg's trace formula is formally similar to the explicit formulas relating the zeros of the Riemann Zeta function to prime numbers, with the Zeta zeros corresponding to eigenvalues of the Laplacian, and the primes corresponding to geodesics. Montgomery's analysis shows that the statistical distribution of the zeros on the critical line tend not to cluster too closely together, but to repel each other [5], a behavior analogous to the quasi-crossing of energy levels in dynamical systems, a signature of quantum chaos [7].

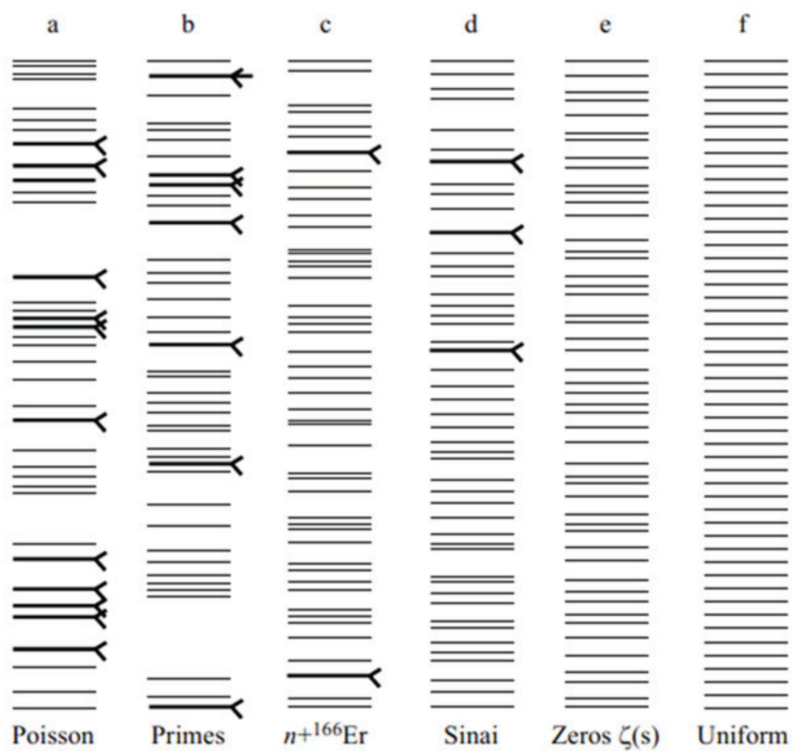


FIGURE 1. The statistics of nearest-neighbor spacings range from random to uniform behavior as shown in this graphic elaborated starting from the image of ref. [8]. From left to right, there are a Poisson distribution, that of the primes from 7,791,097 to 7,791,877, then the energy levels for an excited heavy nucleus by neutron capture. The fourth column is a "length spectrum" of periodic trajectories for Sinai billiards and, finally, the spectrum of zeroes of the Riemann zeta function and a uniform distribution are shown.

In April 1972, during a visit to Selberg at the Institute for Advanced Study in Princeton, at a teatime Montgomery showed his result to Freeman Dyson, one of the founders of the theory of random matrices. Dyson saw that the statistical distribution found by Montgomery appeared to be the same as the pair correlation distribution for the eigenvalues of a random Hermitian matrix (fig. 1). These distributions are of importance in physics — the eigenstates of a Hamiltonian, for example the energy levels of an atomic nucleus, satisfy such statistics. Subsequent work has strongly borne out the connection between the distribution of the zeros of the Riemann zeta function and the eigenvalues of a random Hermitian matrix drawn from the Gaussian unitary ensemble, and both are now believed to obey the same statistics, thus providing an important foundation to the Hilbert–Pólya conjecture [9].

Nuclear spectra and the Zeta function zeros are therefore correlated through their statistical distribution since both spacings between the energy levels of a heavy atomic nucleus and the zeros of the Riemann Zeta function behave like spacings between eigenvalues of a random matrix (Gaussian orthogonal ensemble, GOE) [10].

Further theoretical developments are worth mentioning: Alain Connes [11] has formulated a trace formula that is equivalent to the Riemann hypothesis, so strengthening the analogy with the Selberg trace formula and providing a geometric interpretation of the explicit formula of number theory as a trace formula on noncommutative geometry of Adele classes. Michael Berry and Jonathan Keating suggested [12] that the Hamiltonian H can be expressed by means of some quantization of the one-dimensional classical hyperbolic Hamiltonian $H = xp$, where p is the canonical momentum associated with coordinate x. This refinement of the Hilbert–Pólya conjecture is known as the Berry conjecture (or the Berry–Keating conjecture), but it must still be established on which space this operator should act in order to get the correct dynamics, or how to regularize it in order to get the expected logarithmic corrections.

Recently, following the Berry's approach to the RH problem, a self-adjoint operator was introduced in ref. [13], similar to the quantum analog of the generator of the dilation operator, $A = \left(\widehat{x}\widehat{p} + \widehat{p}\widehat{x}\right)/2$. This paper has been criticized [14] and the authors have responded with clarifications [15]. Finally, another very recent approach is based on a Schrödinger equation [16].

PHYSICAL APPLICATIONS OF THE RIEMANN ZETA FUNCTION

In addition to the connection with nuclear structure, the Zeta function arises in many physical applications [17] such as the calculation of the partition function of ideal Bose-Einstein and Fermi-Dirac quantum gases and the determination of the critical gas temperature and density for the Bose–Einstein condensation phase transition in a dilute gas. In the quantum field theory, formally divergent sums need to be evaluated by means of a process of regularization, by equating them to Zeta functions and associated functions [18] as in the case of the energy of the electromagnetic vacuum in a confined space (Casimir–Polder effect).

A few months ago, the great mathematician, sir Michael Atiyah (London, April 22, 1929 – January 11, 2019), announced in a lecture given at the 2018 Heidelberg Laureate Forum that he has demonstrated the RH problem, through a *reductio ad absurdum* proof [19]. No verification of the demonstration by the scientific community has yet been made. Atiyah's simple proof is based on the Todd function whose properties were used by himself to investigate and understand the fine structure constant, α [20].

Regardless of the verification of the RH demonstration, the fruitful link between the Zeta function and numerous physical phenomena is evident and deserves further studies. Assuming the Hilbert-Pólya conjecture, then random-matrix methods succeed in number theory for essentially the same reason they work in nuclear physics, since the detailed structure of a large operator is less important than its global symmetries, so that any typical matrix with the right symmetries will produce statistically similar results.

At this point, it is worth noticing that both continuous and discrete symmetries are pervasive in nuclear, atomic, molecular and condensed-matter physics, as Iachello's work has clearly elucidated in all these fields, introducing the Interacting Boson Model (IBM) of nuclear structure with its dynamical symmetries [21], the Interacting Boson-Fermion Model (IBFM) and discovering the related supersymmetries, the discrete symmetries of the (algebraic) cluster model, the Vibron Model (VM) of molecular structure, to mention finally his studies of the structure of a physical system at the critical point of a quantum phase transition [22] and the excited-state quantum phase transitions (ESQPTs) that are generalizations of quantum phase transitions to excited levels [23].

In the version 2 of IBM where neutron and proton degrees of freedom are distinguished, even-even nuclei appear composed of nucleon pairs treated as bosons with angular momentum J=0 (S-pairs or bosons) and J=2 (D-pairs) interacting through a suitable quantum Hamiltonian. The IBM-2 is an ideal framework to investigate the order-to-chaos transitions in nuclei and their relationships with the GOE eigenvalues' behavior, thus exploiting the potential of the Riemann Zeta function as a model for quantum chaos [24] and therefore its versatility of use in the analysis of complex physical systems.

We have taken into account all the possible shape transitions between the dynamical symmetries of the IBM-2 [25, 26] by applying to the calculated energy levels the following Brody distribution:

$$P(s,\omega) = \alpha(\omega+1)s^{\omega}\exp(-\alpha s^{\omega+1}), \tag{3}$$

where

$$\alpha = \left[\Gamma\left(\frac{\omega+2}{\omega+1}\right)\right]^{\omega+1}. \tag{4}$$

The Brody distribution interpolates between the Poisson ensemble distribution and the Wigner one, corresponding to a Gaussian orthogonal ensemble (GOE), depending on the ω parameter that ranges from 0 (Poisson distribution) to 1 (Wigner distribution). Fig. 2 shows an example of shape transition between a rigid rotor with axial symmetry and a vibrational nucleus where a maximum amount of quantum chaos was found. In general, we observe broad nearly regular regions for all the shape transitions (a behavior characteristic of a system at the edge of chaos) due to the rich variety of symmetries (partial dynamical symmetries, shape-phase transitions, critical-point transitions, etc.).

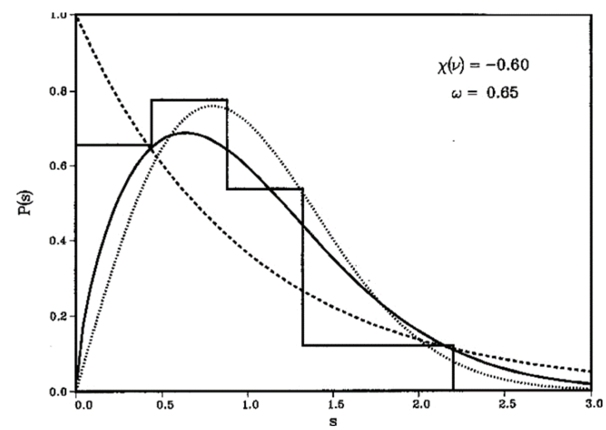


FIGURE 2. Level-spacing distribution, P(s), for the $SU_{\pi+\nu}(3) \rightarrow U_{\pi+\nu}(5)$ transition of the IBM-2 Hamiltonian, with $\chi_\pi = -1.25$ and $\chi_\nu = -0.60$ where π and ν suffixes correspond to proton and neutron degrees of freedom, respectively. The dashed, dotted and solid lines refer to Poisson, GOE and Brody distributions (eq.(3) with ω = 0.65).

To sum up, there is evidence that a suitable superconducting nuclear model such as IBM-2 and its extensions, which incorporates a rich variety of dynamic (super)symmetries due to both bosonic and fermionic degrees of freedom, can reproduce average statistical properties and quantum fluctuations of energy levels – in agreement with experimental data - in a wide range of ordered and quasi-chaotic patterns similar to those of the prime numbers' and Zeta function zeros' distributions.

SUPERCONDUCTIVITY IN CONDENSED AND NUCLEAR MATTER

Once ubiquitous and universal characteristics depending on general symmetry properties have been identified in complex physical systems, similar to those highlighted in number theory, according to the Hilbert–Pólya conjecture it is tempting to adopt a very general yet simple quantum Hamiltonian to deal with the RH problem.

A possible candidate is a superconducting-type Hamiltonian of the following form:

$$H = \sum_{k,l} \varepsilon_{kl} a_k^{+} a_l - \sum_{k,l,m,n} g_{klmn} a_k^{+} a_l^{+} a_m a_n, \tag{5}$$

where a^{+} and a are creation and annihilation operators, respectively, in the second quantization formalism and can be fermion or boson operators, like in the IBM or VM. Hamiltonian of eq. (5) acts in the L_2 infinite-dimensional

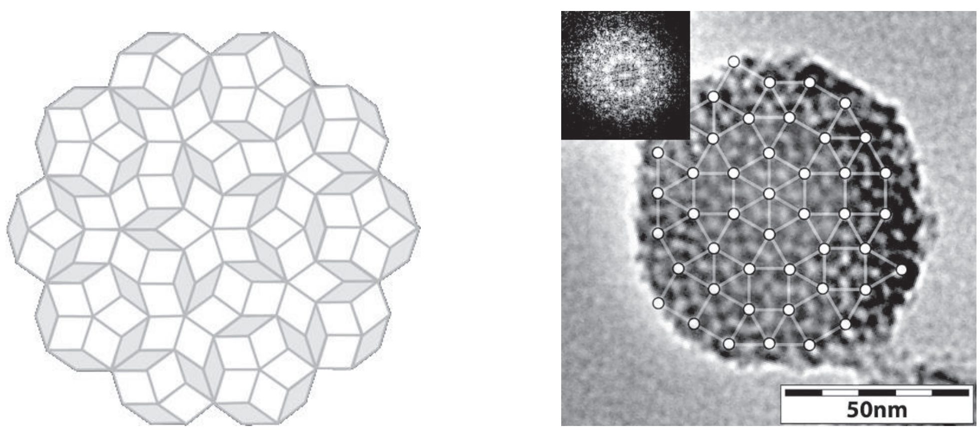
Hilbert-Fock space and, by suitable choice of the ε and g parameters reduces to the well-known BCS Hamiltonian describing traditional superconductors [27].The ground-state of this Hamiltonian consists of pair condensate including correlations between pairs of particles according to the Bardeen, Cooper and Schrieffer (BCS) approach, so determining the ground-state of a superconductor. Analogously, by means of the Bogoliubov-Valatin transformation [28] the ground state of a nucleus can be written in terms of interacting quasiparticle (qp) operators (see, for instance, ref. [27]). Moreover, a boson model of cuprate high-temperature superconductors based on the analogy with atomic nuclei has been proposed by Iachello [29] and Karl Alex Müller [30].

The vacuum expectation energy value of eq. (5), in suitable units, is ½ while other eigenvalues can be related to ½ plus the rest energies of qp and anti-qp pairs (the number of quasiparticles is not conserved and in many physical cases it can be determined by means of projection techniques or Lagrange multipliers). The complete mathematical treatment will be presented in a forthcoming paper.

Freeman Dyson suggested a different strategy to deal with the RH problem [31]. Quasicrystals were discovered in 1984 by Dan Shechtman (for this discovery he was awarded the 2011 Nobel Prize for Chemistry), they are aperiodic solids and exist in spaces of one, two, or three dimensions. From a mathematical point of view, "almost periodic" functions were studied by Harald Bohr, the mathematician brother of Niels, and Roger Penrose who, in 1974, conceived the famous Penrose tiling, a quasiperiodic pattern, since sliding an exact copy of the pattern around will never produce an exact match.

Dyson's idea is to obtain a complete enumeration and classification of all one-dimensional quasicrystals, the most prevalent type, with the aim of identifying one with a spectrum that corresponds to the Riemann Zeta function and one that corresponds to the L-functions that resemble the Riemann Zeta function. If it can be proved that a one-dimensional quasicrystal has properties that identify it with the zeros of the Riemann zeta function, then the RH will be proved.

Quite surprisingly, superconductivity was recently discovered in quasicrystals, too [33, 34].



(a) (b)

FIGURE 3. A schematic one-dimensional quasicrystal (a) and a transmission electron microscope image of a mesoporous silica nanoparticle, showing the tiling with triangles and squares, and the Fourier analysis (inset) showing 12-fold symmetry [32] (b).

CONCLUDING REMARKS

On November 18, 2009, the 150th anniversary of the RH was celebrated with lectures all over the world. Even it represents a yet unsolved problem, from 160 years, it continues to inspire new ideas and promote interesting and fruitful connections between physics and mathematics in many fields, such as quantum chaos, fractals and quasicrystals, string theory and duality, quantum fields and motives [35] and noncommutative geometry [36].

Another unexpected property of prime numbers is that in certain large intervals they possess order across length scales and represent the first example of a new class of many-particle systems with pure point diffraction patterns [37, 38]. In particular, the primes in this regime are hyperuniform and their behavior resembles quasicrystals in some respects. A quasi-regular ordered pattern that we already encountered in nuclear shape transitions.

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I would like to thank Franco Iachello for many useful discussions and suggestions. For four decades I have been bound by friendship and scientific collaboration with Franco, and for this reason I am very indebted to him and it is with great pleasure that I dedicate this essay to him, since it is placed at the borders between mathematics, nuclear physics and condensed matter physics, all subjects in which he is a master.

REFERENCES

1. T. M. Apostol, "Zeta and Related Functions", chapt. 25 in *NIST Handbook of Mathematical Functions*, edited by F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (NIST and Cambridge University Press, Cambridge, 2010), pp. 601-616.
2. G. A. Hiary and A. M. Odlyzko, Math. Comp. **81**, 1723-1752 (2012).
3. M. Odlyzko, "Correspondence about the origins of the Hilbert-Pólya Conjecture", http://www.dtc.umn.edu/~odlyzko/polya/index.html.
4. D. Hilbert, Bull. of the American Math. Soc. **8** (10), 437–479 (1902). Earlier publications in German appeared in D. Hilbert, Göttinger Nachrichten, 253–297 (1900), and Archiv der Mathematik und Physik. **3**, 44–63, 213–237 (1901).
5. H. L. Montgomery, "The pair correlation of zeros of the zeta function", in *Analytic Number Theory*, Proc. Sympos. Pure Math., XXIV, (American Mathematical Society, Providence, R.I., 1973) pp. 181–193.
6. A. Selberg, J. Indian Math. Soc. (N.S.) **20**, 47–87 (1956).
7. V. R. Manfredi and L. Salasnich, Int. J. Mod. Phys. E **9**, 279-297 (2000).
8. O. Bohigas and M. J. Giannoni, "Chaotic motion and random matrix theories", in *Mathematical and Computational Methods in Nuclear Physics*, edited by I. M. Gomez et al. (Springer-Verlag, New York, 1984), Lecture Notes in Physics **209**, pp. 1-99.
9. J. V. Armitage, "The Riemann Hypothesis and the Hamiltonian of a Quantum Mechanical System", chapt. 10 in *Number Theory and Dynamical Systems*, edited by M. M. Dodson and J. A. G. Vickers (Cambridge University Press, New York, 1989), London Math. Soc. Lecture Note Ser. **134**, pp. 153–172.
10. Z. Rudnick and P. Sarnak, Duke J. Math. **81** (2), 269–322 (1996).
11. A. Connes, Selecta Mathematica, New Series **5** (1), 29-106 (1999), arXiv:math/9811068.
12. M. V. Berry and J. P. Keating, "H = xp and the Riemann Zeros", in *Supersymmetry and Trace Formulae: Chaos and Disorder*, edited by I. V. Lerner et al. (Plenum, New York, 1999) vol. **370**, NATO ASI Series, pp. 355-367; SIAM Review **41** (2), 236–266 (1999).
13. C. M. Bender, D. C. Brody and M. P. Müller, Phys. Rev. Lett. **118**, 130201, (2017).
14. J. V. Bellissard, "Comment on "Hamiltonian for the Zeros of the Riemann Zeta Function"", arXiv:1704.02644v1 (2017).
15. C. M. Bender, D. C. Brody and M. P. Müller, "Comment on 'Comment on "Hamiltonian for the Zeros of the Riemann Zeta Function"'", arXiv:1705.06767 (2017); M. P. Müller, "A note on "Hamiltonian for the zeros of the Riemann zeta function"", arXiv:1704.04705v2 (2017).
16. F. I. Moxley III, "A Schrödinger equation for solving the Bender-Brody-Müller conjecture", in AIP Conference Proceedings 1905, 030024 (2017).
17. D. Schumayer and D.A. Hutchinson, Reviews of Modern Physics **83** (2), 307-330 (2011).
18. E. Elizalde, *Ten physical applications of spectral zeta functions* (Springer Science & Business Media, New York, 2008) vol. 35; E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, *Zeta Regularization Techniques with Applications* (World Scientific, Singapore, 1994).
19. M. F. Atiyah, "The Riemann Hypothesis", https://aperiodical.com/wp-content/uploads/2018/09/2018-The_Riemann_Hypothesis.pdf (2018).
20. M. F. Atiyah, "Arithmetic Physics", in Proceedings of ICM Rio de Janeiro (2018); "The Fine Structure Constant", submitted to Proc. Roy. Soc. A (2018).
21. A. Arima and F. Iachello, Ann. Phys. (N.Y.) **99**, 253 (1976); ibid. **111**, 201 (1978); ibid., **123**, 468 (1979).
22. F. Iachello, Phys. Rev. Lett. **85**, 3580 (2000); ibid. **87**, 052502 (2001); ibid. **91**, 132502 (2003).
23. F. Iachello, J. Phys.: Conf. Ser. **1071,** 012013 (2018).

24. M. V. Berry, “Riemann's Zeta function: A model for quantum chaos?”, in *Quantum Chaos and Statistical Nuclear Physics*, edited by T. H. Seligman and H. Nishioka, Lecture Notes in Physics, vol. 263 (Springer, Berlin - Heidelberg, 1986) pp. 1-17.
25. E. Canetta and G. Maino, Phys. Lett. B **483** (2000) 55-59.
26. G. Maino, “Dynamical symmetries and regular vs. chaotic quantum motion in realistic models of nuclear structure”, in *Nuclear Structure far from stability: new physics and new technology,* Proceedings of the International School of Physics ‘Enrico Fermi’, course CLXIX (IOS Press, Amsterdam, 2008) pp. 447-475.
27. P. Ring and P. Schuck*, The Nuclear Many-Body Problem* (Springer-Verlag, New York, 1980).
28. N. N. Bogoliubov, J. Phys. (USSR) **11**, 23 (1947).
29. F. Iachello, Phil. Mag. Lett. **82** (5) 289-295 (2002).
30. K. A. Müller, Nature **377**, 133-135 (1995); J. Supercond. **13**, 863 (2001) 863; Phil. Mag. Lett. **82** (5) 279-288 (2002).
31. F. J. Dyson, *Birds and Frogs. Selected papers, 1990-2014* (World Scientific, Singapore, 2015) pp. 40-43; originally prepared as Einstein Lecture for the American Mathematical Society meeting, Vancouver, Canada, November 4, 2008, Not. Am. Math. Soc. **56**, 212–223 (2009), and revised for Russian translation, June 2009.
32. Y. Sun, K. Ma, T. Kao, K. A. Spoth, H. Sai, D. Zhang, L. F. Kourkoutis, V. Elser and U. Wiesner, Nature Comm. **8** (1), article no. 252 (2017).
33. K. Kamiya, T. Takeuchi, N. Kabeya, N. Wada, T. Ishimasa, A. Ochiai, K. Deguchi, K. Imura and N. K. Sato, Nature Comm. **9**, article no. 154 (2018).
34. N. Takemori, "Towards understanding of the superconductivity in quasicrystals: superconductivity in quasiperiodic system", in Proceedings of *Aperiodic 2018 - 9th Conference on Aperiodic Crystals*, Iowa State University in Ames, July 8-13, 2018.
35. M. Marcolli, *Feynman Motives* (World Scientific, Singapore, 2010).
36. A. Connes, “An Essay on the Riemann Hypothesis”, chapt. 5 in *Open Problems in Mathematics*, edited by J. F. Nash, Jr. and M. Th. Rassias (Springer, Switzerland, 2016) pp. 225-257.
37. G. Zhang, F. Martelli and S. Torquato, J. Phys. A: Math. Theor. **51**, 115001 (2018).
38. S. Torquato, G. Zhang and M. de Courcy-Ireland, J. Stat. Mech. (2018) 093401, stacks.iop.org/JSTAT/2018/093401.