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Constructing Wave Functions in a Hyperspherical Basis Using Parentage Scheme of Symmetrization

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Abstract. Investigating few-body systems with identical particles in a hyperspherical basis yields the problem of obtaining symmetrized hyperspherical functions from functions with arbitrary quantum numbers. This article solves the problem of hyperspherical basis symmetrization for four-, five- , and six- body systems using Parentage Scheme of Symmetrization. Parentage coefficients and transformation coefficients corresponding to the [4], [31], [22], [211], representations of S_4 groups with specific quantum numbers are obtained, and relationship between parentage coefficients and transformation coefficients between different sets of Jacobi coordinates for six-body systems are derived. Young operators, acting on N = 4, 5, 6 body hyperspherical functions symmetrized with respect to (N-1) particles, are derived. The symmetrized N = 4, 5, 6 body hyperspherical functions are obtained with different values of quantum numbers. The connection between the transformation coefficients for the identical particle systems and the parentage coefficients is demonstrated and the corresponding formulas are introduced.

INTRODUCTION

The study of baryon-baryon interactions is one of the important problems of modern physics. As the experimental data of low energy hyperon-nucleon scattering are not available, the principal information about hyperon-nucleon interaction may be obtained by investigation of the structure and decays of light hypernuclei. The microscopic approach to the study of the structure and mesonic decays of four-body hypernucleus He_{λ}^{4} by the use of the Hyperspethical Function Method (HFM) in momentum space was developed and structure characteristics and decay rates were obtained [2-5]. The microscopic approach to the solution of five- body problem in hypernuclear physics within the framework of the HFM in momentum representation was developed and the dependence of the structure characteristics and wave functions on the types of nucleon-nucleon and hyperon-nucleon interaction potentials was studied [6]. Parentage Scheme of Symmetrization (PSS) to the N-body symmetrized basis construction necessary for the description of the structural characteristics and decay reactions of the hypernuclear and nuclear systems with arbitrary amount of particles was introduced [1,7].The PSS allows to construct N-particle symmetrized hyperspherical functions on the bases of N-particle hyperspherical functions symmetrized with respect to N–1-particles by the use of the transformation coefficients related with the (N-1)-th and N-th particle permutations. This article solves the problem of hyperspherical basis symmetrization for four-, five-, and six- body systems using Parentage Scheme of Symmetrization (PSS).

HYPERSPHERICAL BASIS TRANSFORMATIONS

Transformations of Hyperspherical functions become sufficiently complex when number of particles in the system increases. For four and more particle transformations include both particle permutations and transitions from one configuration to another. In four-body systems we have (3+1) and (2+2) configurations and transformations between different sets of Jacobi coordinates can be expressed as Kinematic Rotations (KR) in nine dimensional space [1]. Recurrent method [1-3] allows obtaining transformation coefficients for the systems with any number of particles. According to the recurrent method, initial coefficients with lowest quantum numbers are calculated by solving

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overlap integral of corresponding Hyperspherical Functions (HF) analytically, then wave functions with arbitrary quantum numbers are expanded in terms of basic Hyperspherical Functions (HF), and kinematic rotation of this expansion is performed with the use of already known coefficients with lowest quantum numbers [7]. This procedure allows calculating few-body transformation coefficients with any quantum numbers. Four-body transformation coefficients $\langle l_1'l_2'l_{12}'l_3'k_3' \,|\, l_1l_2l_{12}l_3k_3\rangle_{k_4L}^{1\leftrightarrow2}$ where $l_{1,}l_{2,}l_{12,}l_3$ are orbital momentums related with Jacobi coordinates, k_3 and $k_4$are generalized hyper momentums in six and nine dimensional space of Jacobi coordinates, and L is total orbital momentum were introduced in [1]. Table 1 represents four-body transformation coefficients describing transformations between (2+2) and (3+1) configurations for four identical particles with k_4=2 and L=0.

TABLE 1. Four-Body Transformation Coefficients $\langle l_1'l_2'l_{12}'l_3'k_3' \,|\, l_1l_2l_{12}l_3k_3\rangle_{20}^{1\leftrightarrow2}$ for Identical Particles

$l_1l_2l_{12}l_3k_3$	$\langle 11002 \vert$	$\langle 10111 \vert$	$\langle 01111 \vert$	$\langle 00000 \vert$	$\langle 00002 \vert$
$\vert 11002\rangle$	$-\sqrt{6}/3$	$-\sqrt{3}/3$	0	0	0
$\vert 10111\rangle$	$-\sqrt{3}/3$	$\sqrt{6}/3$	0	0	0
$\vert 01111\rangle$	0	0	$-1/3$	$\sqrt{6}/3$	$\sqrt{2}/3$
$\vert 00000\rangle$	0	0	$\sqrt{6}/3$	$1/2$	$-\sqrt{3}/6$
$\vert 00002\rangle$	0	0	$\sqrt{2}/3$	$-\sqrt{3}/6$	$5/6$

In five-body systems with different particles we have (4+1), (3+2), (3+1+1), and (2+2+1) configurations. Five-body transformation coefficients for the systems with different particles are defined as the overlap integral of corresponding HF [6-7].

$$\langle l_1'l_2'l_{12}'l_3'l_{123}'l_4'k_3'k_4'|l_1l_2l_{12}l_3l_{123}l_4k_3k_4\rangle_{k_5L} = \int \Psi_{k_3'k_4'k_5'}^{l_1'l_2'l_{12}'l_3'l_{123}'l_4'L\,*}(\Omega')\Psi_{k_3k_4k_5}^{l_1l_2l_{12}l_3l_{123}l_4L}(\Omega)d\Omega \tag{1}$$

Where $l_{1,}l_{2,}l_{12,}l_{3,}l_{123,}l_{4,}$are orbital momentums corresponding to Jacobi coordinates, $k_{3,}k_4$, and k_5 are generalized hyper momenta in six, nine, and twelve dimensional space of Jacobi coordinates, and Ω is twelve dimensional volume element.

In six-body systems we have (5+1), (4+2), and (3+3) configurations. According to the mathematical formalism introduced in [2], transformation coefficients between HF defined on different sets of Jacobi coordinates for six-body systems with different particles can be introduced as:

$$\Phi_{k_3k_4k_5k_6}^{l_1l_2l_{12}l_3l_{123}l_4l_{1234}l_5L}(\omega) = \sum_{l_1'l_2'l_{12}'l_3'l_{123}'}\sum_{l_4'l_{1234}'l_5'}\sum_{k_3'k_4'k_5'}\Phi_{k_3'k_4'k_5'k_6}^{l_1'l_2'l_{12}'l_3'l_{123}'l_4'l_{1234}'l_5'L}(\omega')\cdot \\ \cdot\langle l_1'l_2'l_{12}'l_3'l_{123}'l_4'l_{1234}'l_5'k_3'k_4'k_5'|l_1l_2l_{12}l_3l_{123}l_4l_{1234}l_5k_3k_4k_5\rangle_{k_6L} \tag{2}$$

Where k_6 is a generalized hyper momentum in fifteen dimensional space of Jacobi coordinates, and different sets of Jacobi coordinates are connected by the matrix of kinematic rotations in fifteen dimensional space:

$$\begin{pmatrix}\vec{x}\\ \vec{y}\\ \vec{z}\\ \vec{w}\\ \vec{u}\end{pmatrix} = \begin{pmatrix}a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25}\\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35}\\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45}\\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55}\end{pmatrix}\begin{pmatrix}\vec{x}'\\ \vec{y}'\\ \vec{z}'\\ \vec{w}'\\ \vec{u}'\end{pmatrix} \tag{3}$$

When investigating few-body systems with identical particles problem of hyperspherical basis symmentrization arises. Four-body HF symmetrized with respect to three identical particles were introduced in [1]:

$$\Psi_{k_4L}^{[f](n)l_{12}l_3k_3} = \sum_{l_1l_2} C_{k_3l_{12}}^{[f]\nu}(l_1l_2)\Psi_{k_3L}^{l_1l_2}(\Omega) \tag{4}$$

Where $C_{k_3 l_{12}}^{[\bar{f}]\bar{\nu}}(l_1 l_2)$ are three-body symmetrization coefficients, $[\bar{f}]$ is three particle Young diagram obtained from four particle Young diagram $[f]$ by removing the cell corresponding to the fourth particle, $\bar{\nu}$ is the number of $\bar{f}$ with given K and L. Four-Body transformation coefficients with three identical particles $\left\langle [\bar{f}]' \bar{m}' \nu' l'_{12} l'_3 k'_3 \middle| [\bar{f}]\, \bar{m} \bar{\nu} l_{12} l_3 k_3 \right\rangle$ were introduced in [1]. Table 2 represents values of these coefficients with $K_4 = 2, L = 0$ in general form where a_{ik} matrix elements represent Jacobi coordinate transformations.

TABLE 2. Four-Body Transformation Coefficients $\left\langle [\bar{f}]' \bar{m}' \nu' l'_{12} l'_3 k'_3 \middle| [\bar{f}]' \bar{m} \bar{\nu} l_{12} l_3 k_3 \right\rangle$ with $K_4 = 2, L = 0$

$[\bar{f}]\bar{m}, l_{12} l_3 k_3$	$\langle [3], 000 \vert$	$\langle [21]1,002 \vert$	$\langle [21]1,111 \vert$
$\vert [3],\ 000 \rangle\rangle$	$^1/_2\,(3a_{33}^2 - 1)$	$^{\sqrt{3}}/_2\,(a_{31}^2 - a_{32}^2)$	$-\sqrt{3} a_{32} a_{33}$
$\vert [21]1,002 \rangle$	$^{\sqrt{3}}/_2\,(a_{13}^2 - a_{23}^2)$	$^1/_2\,(a_{11}^2 - a_{12}^2 - a_{21}^2 + a_{22}^2)$	$a_{22} a_{23} - a_{12} a_{13}$
$\vert [21]1,111 \rangle$	$-\sqrt{3} a_{23} a_{33}$	$a_{22} a_{32} - a_{21} a_{31}$	$a_{22} a_{33} + a_{32} a_{23}$
$\vert [21]2,002 \rangle$	$\sqrt{3} a_{13} a_{23}$	$-a_{12} a_{22} + a_{11} a_{21}$	$-a_{22} a_{13} - a_{12} a_{23}$
$\vert [21]2,111 \rangle$	$-\sqrt{3} a_{13} a_{33}$	$-a_{11} a_{31} + a_{32} a_{12}$	$a_{12} a_{33} + a_{32} a_{13}$
	$\langle [21]2,002 \vert$	$\langle [21]1,111 \vert$	
$\vert [3], 000 \rangle$	$\sqrt{3} a_{31} a_{32}$	$-\sqrt{3} a_{31} a_{33}$	
$\vert [21]1,002 \rangle$	$a_{11} a_{12} - a_{21} a_{22}$	$a_{21} a_{23} - a_{11} a_{13}$	
$\vert [21]1, 111 \rangle$	$-a_{22} a_{31} - a_{21} a_{32}$	$a_{21} a_{33} + a_{31} a_{23}$	
$\vert [21]2,002 \rangle$	$a_{11} a_{22} + a_{12} a_{21}$	$-a_{11} a_{23} - a_{21} a_{13}$	
$\vert [21]2,111 \rangle$	$-a_{11} a_{32} - a_{12} a_{31}$	$a_{11} a_{33} + a_{31} a_{13}$	

Transformation matrix of Jacobi coordinates for four-body systems with three identical particles $M_i = M_j = M_k = M$ corresponding to the permutations of the third and fourth M_l particle is represented as:

$$\hat{\boldsymbol{a}}(P_{34}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{M_l}{3(2M+M_l)}} & \sqrt{\frac{2(3M+M_l)}{3(2M+M_l)}} \\ 0 & \sqrt{\frac{2(3M+M_l)}{3(2M+M_l)}} & -\sqrt{\frac{M_l}{3(2M+M_l)}} \end{pmatrix} \tag{5}$$

Using Table 2 and matrix (5) we can obtain four-body transformation coefficients for identical particles (equal masses) in (3+1) configuration when $K_4 = 2$ and L=0 (see Table 3).

TABLE 3. Four-Body Transformation Coefficients for Identical Particles when $K_4 = 2$ and L=0

$[\bar{f}]\bar{m}_1 l_{12} l_3 k_3$	$\langle [3], 000 \vert$	$\langle [21]1,002 \vert$	$\langle [21]1,111 \vert$	$\langle [21]2,002 \vert$	$\langle [21]2,111 \vert$
$\vert [3], 000 \rangle$	$-1/3$	$-4\sqrt{3}/9$	$2\sqrt{6}/9$	0	0
$\vert [21]1,002 \rangle$	$-4\sqrt{3}/9$	$5/9$	$2\sqrt{2}/9$	0	0
$\vert [21]1,111 \rangle$	$2\sqrt{6}/9$	$2\sqrt{2}/9$	$7/9$	0	0
$\vert [21]2,002 \rangle$	0	0	0	$1/3$	$-2\sqrt{2}/3$
$\vert [21]2,111 \rangle$	0	0	0	$-2\sqrt{2}/3$	$-1/3$

HYPERSPHERICAL BASIS SYMMETRIZATION

Parentage Scheme of Symmetrization introduced in [1] can be easily generalized for N-body systems by introducing Young operators acting on N-particle functions corresponding to the irreducible representation of the N-1 particle permutation group S_{N-1}. For four-body systems we have [4], [22], [211], and [1111] representations and Young operators acting on four-body hyperspherical functions symmetrized with respect to 3 particles can be expressed as:

$$
\begin{gathered}
\omega_{[3]}^{[4]}\Psi^{[3]} = \frac{1}{4}[1 + P_{14} + P_{24} + P_{34}]\Psi^{[3]} \\
\omega_{[1^3]}^{[1^4]}\Psi^{[1^3]} = \frac{1}{4}[1 - P_{14} - P_{24} - P_{34}]\Psi^{[1^3]} \\
\omega_{[21]_1}^{[22]}\Psi_1^{[21]} = \frac{1}{4}\left[\Psi_1^{[21]} + P_{34}\Psi_1^{[21]} + P_{14}\left(-\frac{1}{2}\Psi_1^{[21]} + \frac{\sqrt{3}}{2}\Psi_2^{[21]}\right) + P_{24}\left(-\frac{1}{2}\Psi_1^{[21]} - \frac{\sqrt{3}}{2}\Psi_2^{[21]}\right)\right] \\
\omega_{[21]_2}^{[22]}\Psi_2^{[21]} = \frac{1}{4}\left[\Psi_2^{[21]} - P_{34}\Psi_2^{[21]} + P_{14}\left(\frac{\sqrt{3}}{2}\Psi_1^{[21]} + \frac{1}{2}\Psi_2^{[21]}\right) + P_{24}\left(-\frac{\sqrt{3}}{2}\Psi_1^{[21]} + \frac{1}{2}\Psi_2^{[21]}\right)\right] \\
\omega_{[1^3]_1}^{[211]}\Psi_1^{[1^3]} = \frac{\sqrt{6}}{4}[P_{14} - P_{24}]\Psi^{[1^3]} \\
\omega_{[1^3]_2}^{[211]}\Psi^{[1^3]} = \frac{\sqrt{2}}{4}[2P_{34} - P_{14} - P_{24}]\Psi^{[1^3]} \\
\omega_{[1^3]_3}^{[211]}\Psi^{[1^3]} = \frac{1}{4}[3 + P_{14} + P_{24} + P_{34}]\Psi^{[1^3]}
\end{gathered}
\tag{6}
$$

Young operators corresponding to [5], [11111], and [2111] representations and acting on the five -body hyperspherical functions symmetrized with respect to four particles in the (4+1) configuration were derived in [7]. Using PSS [1], we can construct Young operators for [221] and [311] representations of S_5 group:

$$
\begin{gathered}
\omega_{[22]_1}^{[221]}\Psi^{[22]} = \frac{1}{2}\left[1 - \frac{1}{2}(P_{15} + P_{25} + P_{35} + P_{45})\right]\Psi_1^{[22]} \\
\omega_{[22]_1}^{[221]}\Psi^{[22]} = \frac{1}{2}\left[1 - \frac{1}{2}(P_{15} + P_{25} + P_{35} + P_{45})\right]\Psi_1^{[22]} \\
\omega_{[22]_1}^{[221]}\Psi^{[22]} = \frac{1}{2}\left[1 - \frac{1}{2}(P_{15} + P_{25} + P_{35} + P_{45})\right]\Psi_1^{[22]} \\
\omega_{[22]_2}^{[221]}\Psi^{[22]} = \frac{1}{4}\left[(P_{25} - P_{15})\Psi_2^{[22]} + \sqrt{3}(P_{45} - P_{35})\Psi_1^{[22]}\right] \\
\omega_{[22]_3}^{[221]}\Psi^{[22]} = \frac{1}{2}\left[1 - \frac{1}{2}(P_{15} + P_{25} + P_{35} + P_{45})\right]\Psi_2^{[22]} \\
\omega_{[22]_4}^{[221]}\Psi^{[22]} = \frac{1}{4}\left[(P_{25} - P_{15})\Psi_1^{[22]} + \frac{2}{\sqrt{3}}\left(\frac{1}{2}P_{35} - P_{15} - P_{25} + \frac{3}{2}P_{45}\right)\Psi_2^{[22]}\right] \\
\omega_{[22]_5}^{[221]}\Psi^{[22]} = \frac{\sqrt{2}}{4}\left[(P_{25} - P_{15})\Psi_1^{[22]} + \frac{1}{\sqrt{3}}(P_{15} + P_{25} - 2P_{35})\Psi_2^{[22]}\right] \\
\omega_{[211]_1}^{[311]}\Psi^{[211]} = \frac{1}{\sqrt{30}}\left[(P_{15} + P_{25} - 2P_{35})\Psi_1^{[211]} + \sqrt{3}(P_{15} - P_{25})\Psi_2^{[211]}\right] \\
\omega_{[211]_2}^{[311]}\Psi^{[211]} = \frac{1}{\sqrt{30}}\left[(3P_{45} - P_{15} - P_{25} - P_{35})\Psi_1^{[211]} + \sqrt{3}(P_{15} - P_{25})\Psi_3^{[211]}\right] \\
\omega_{[211]_3}^{[311]}\Psi^{[211]} = \frac{3}{10}\left[\left(\frac{4}{3} + P_{15} + P_{25} + \frac{1}{3}(P_{35} + P_{45})\right)\Psi_1^{[211]} - \frac{2}{3\sqrt{3}}(P_{15} - P_{25})\Psi_2^{[211]} - \frac{2}{3\sqrt{6}}(P_{25} - P_{15})\Psi_3^{[211]}\right] \\
\omega_{[211]_4}^{[311]}\Psi^{[211]} = \frac{1}{\sqrt{30}}\left[\frac{1}{\sqrt{2}}(3P_{45} - P_{15} - P_{25} - P_{35})\Psi_2^{[211]} + (2P_{35} - P_{15} - P_{25})\Psi_3^{[211]}\right] \\
\omega_{[211]_5}^{[311]}\Psi^{[211]} = \frac{1}{10}\left[\left(4 + \frac{5}{3}(P_{15} + P_{25}) + \frac{11}{3}P_{35} + P_{45}\right)\Psi_2^{[211]} + \frac{2}{\sqrt{3}}(P_{25} - P_{15})\Psi_1^{[211]} - \frac{\sqrt{2}}{3}(P_{15} + P_{25} - 2P_{35})\Psi_3^{[211]}\right] \\
\omega_{[211]_6}^{[311]}\Psi^{[211]} = \frac{1}{10}\left[\left(4 + \frac{4}{3}(P_{15} + P_{25} + P_{35} + 3P_{45})\right)\Psi_3^{[211]} + \sqrt{\frac{2}{3}}(P_{15} - P_{25})\Psi_1^{[211]} - \frac{\sqrt{2}}{3}(P_{15} + P_{25} - 2P_{35})\Psi_2^{[211]}\right]
\end{gathered}
\tag{7}
$$

Perantage scheme of symmetrization [1] generalized in [7], allows to construct N-particle symmetrized hyperspherical functions on the bases of N-particle hyperspherical functions symmetrized with respect to N−1-particles by the use of the Kinematic Rotation Coefficients (KRC) related with the (N-1)-th and N-th particle permutations:

$$\Psi_{k_iL}^{[f]m\nu} = \sum_\alpha B_{k_iL}^{[f]\nu}\left([\bar{f}]_m\alpha\right)\Psi_{k_iL}^{[\bar{f}]_m\bar{n}_m\alpha} \tag{8}$$

Where $B_{k_iL}^{[f]\nu}\left([\bar{f}]_m\alpha\right)$ are parentage coefficients acting on N-particle hyperspherical functions symmetrized with respect to N−1-particles, i=4,5,6 and α represents set of parameters defining hyperspherical functions. Connection between parentage coefficients (8) and transformation coefficients between different sets of Jacobi coordinates for N-body hyperspherical basis symmetrized with respect to N-1 identical particles (N=4,5) was found in [6-7] and the corresponding formulas were introduced. Table 4 presents calculated values of parentage coefficients acting on four-body HF Symmetrized with respect to three identical particles with $K_4 = 4$ and L=0.

TABLE 4: Four-Body Parentage Coefficients $B_{K_4L}^{[f]\nu}\left([\bar{f}]_m\alpha\right)$ with $K_4 = 4$ and L-0

$[f]_\nu[\bar{f}]$ \ $l_{12}l_3k_3$	000	002	222	111	113	004
$[4]_1[3]$	$\sqrt{26}/9$	0	$\frac{1}{9}\sqrt{11/2}$	0	$-\frac{1}{3}\sqrt{11/3}$	$\frac{1}{3}\sqrt{11/6}$
$[4]_2[3]$	0	0	$1/\sqrt{2}$	0	$1/\sqrt{3}$	$1/\sqrt{6}$
$[31]_1[3]$	$\sqrt{55}/9$	$-\frac{1}{9}\sqrt{13/5}$	0	0	$\frac{1}{3}\sqrt{26/15}$	$-\frac{1}{3}\sqrt{13/15}$
$[31]_1[\widetilde{21}]$	0	$\frac{7}{9\sqrt{5}}$	$\frac{1}{9}\sqrt{26/5}$	$-\frac{4}{9}\sqrt{6/5}$	$-\frac{1}{3}\sqrt{26/15}$	$\frac{2}{3}\sqrt{13/15}$
$[31]_2[3]$	0	0	$\sqrt{2/5}$	0	$-1/\sqrt{15}$	$-4/\sqrt{30}$
$[31]_2[\widetilde{21}]$	0	$-\frac{1}{3}\sqrt{13/10}$	$\frac{4}{3\sqrt{5}}$	$\frac{1}{6}\sqrt{39/5}$	$-\frac{1}{2}\sqrt{3/5}$	$\sqrt{2/15}$
$[211][1^3]$	0	0	0	0	1	0
$[211][21]$	0	$\frac{1}{3}\sqrt{13/6}$	$-\frac{2}{3\sqrt{3}}$	$\sqrt{13}/6$	$\frac{1}{6}$	$-\sqrt{2}/3$
$[22]_1[21]$	0	$\frac{2\sqrt{10}}{9}$	$-\frac{1}{9}\sqrt{13/5}$	$-\frac{1}{3}\sqrt{5/3}$	$\frac{1}{3}\sqrt{13/5}$	$-\frac{1}{9}\sqrt{78/5}$
$[22]_2[21]$	0	0	$\sqrt{2/5}$	0	$4/\sqrt{30}$	$1/\sqrt{15}$

Using formula (8) for six body systems, we can derive relationship between parentage coefficients and transformation coefficients between different sets of Jacobi coordinates for six-body systems:

$$\sum_\nu B_{k_6L}^{[42]\nu}([41]\alpha)\cdot B_{k_6L}^{[42]}([41]\alpha') = \frac{3}{8}\delta'_{\alpha\alpha} - \frac{15}{32}\langle[41](1)\alpha|[41](1)\alpha'\rangle^{56} + \frac{15}{32}\langle[41](4)\alpha|[41](4)\alpha'\rangle^{56}$$
$$\sum_\nu B_{k_6L}^{[42]\nu}([41]\alpha)\cdot B_{k_6L}^{[42]}([32]\alpha') = -\frac{3\sqrt{2}}{4}\langle[41](1)\alpha|[32](2)\alpha'\rangle^{56}$$
$$\sum_\nu B_{k_6L}^{[51]\nu}([5]\alpha)\cdot B_{k_6L}^{[51]\nu}([5]\alpha') = \frac{5}{6}\delta'_{\alpha\alpha} - \frac{5}{6}\langle[5]\alpha|[5]\alpha'\rangle^{56} \tag{9}$$

$$\sum_{\nu} B^{[51]\nu}_{k_6 L}([5]\alpha)\cdot B^{[51]\nu}_{k_6 L}([41]\alpha') = -\frac{5\sqrt{6}}{12}\langle [5]\alpha|[41]\alpha'\rangle^{56} \ \ (\alpha = \bar{\nu}, l_{1234}, l_5, k_5)$$

Symmetrization coefficients for the construction of N-body HF with identical particles can be found by the use of parentage coefficients (8). Four-body symmetrization coefficients $C^{[f](m)\nu}_{k_4 L}(l_1 l_2 l_{12} l_3 k_3)$ in terms of parentage coefficients are obtained by substituting definition (4) into formula (8) for four-body systems:

$$\Psi^{[f](m)\nu}_{k_4 L} = \sum_{l_1 l_2 l_{12} l_3 k_3} B^{[f]_m\nu}_{k_4 L}\left([\bar{f}]_m \bar{\nu} l_{12} l_3 k_3\right) C^{[\bar{f}]_m}_{k_3 l_{12}}(l_1 l_2)\Psi^{l_1 l_2 l_{12} l_3 L}_{k_3 k_4}(\Omega) = \sum_{l_1 l_2 l_{12} l_3 k_3} C^{[f]m\nu}_{k_4 L}(l_1 l_2 l_{12} l_3 k_3)\Psi^{l_1 l_2 l_{12} l_3 L}_{k_4 k_3}(\Omega)$$

Where $$\mathrm{C}^{[f](m)\nu}_{k_4 L}(l_1 l_2 l_{12} l_3 k_3) = \sum_{\bar{\nu}} \mathrm{B}^{[f]m\nu}_{k_4 L}\left([\bar{f}]_m \bar{\nu} l_{12} l_3 k_3\right) \mathrm{C}^{[\bar{f}]_m \bar{\nu}}_{k_4 L}(l_1 l_2) \quad (10)$$

Values of four-body symmetrization coefficients (10) in (2+2) configuration can be obtained using Table 4 and three-body symmetrization coefficients (4) calculated in [1] (see Table 5)

TABLE 5. $C^{[f]m\nu}_{20}(l_1 l_2 l_{12} l_3 k_3)$ Symmetrization Coefficients in $(2+2)$-configuration for Four Identical Bodies

$[f]_m$ \ $l_1 l_2 l_{12} l_3 k_3$	00000	00002	11002	10111	01111
[31]1	$-1/2$	$\sqrt{3}/6$	0	0	$-\sqrt{6}/3$
[31]2	$-\sqrt{2}/2$	$\sqrt{6}/6$	0	0	$\sqrt{3}/3$
[31]3	0	0	1	0	0
[22]1	1/2	$\sqrt{3}/2$	0	0	0
[22]2	0	0	0	1	0

Four-body symmetrized HF with given quantum numbers are derived using Table 5 and formula (10):

$$\Psi^{[31](1)}_{k_4=1} = \Psi^{[\widetilde{21}](1) l_{12}=1\, l_3=0}_{k_3=k_4=1} = \Psi^{l_1=1}_{k_3=k_4=1}$$

$$\Psi^{[31](2)}_{k_4=1} = \Psi^{[\widetilde{21}](2) l_{12}=1\, l_3=0}_{k_3=k_4=1} = -\Psi^{l_2=1}_{k_3=k_4=1}$$

$$\Psi^{[31](3)}_{k_4=1} = \Psi^{[3](2) l_{12}=0\, l_3=1}_{k_3=0\ k_4=1} = \Psi^{l_3=1}_{k_3=0\ k_4=1}$$

$$\Psi^{[22](i)}_{k_4=2} = \sqrt{1/3}\,\Psi^{[21](i) l_{12}=l_3=0}_{k_3=k_4=2} + \sqrt{2/3}\,\Psi^{[21](i) l_{12}=l_3=1}_{k_3=1\ k_4=2} \ \ (i=1,2)$$

$$\Psi^{[31](i)}_{k_4=2} = -\sqrt{2/3}\,\Psi^{[\widetilde{21}](i)}_{k_3=k_4=2} + \sqrt{1/3}\,\Psi^{[\widetilde{21}](i)}_{k_3=1\ k_4=2} \ (i=1,2)$$

$$\Psi^{[31](3)}_{k_4=2} = \sqrt{2/3}\,\Psi^{[3] l_{12}=l_3=0}_{k_3=0\ k_4=2} = \Psi^{l_i=0}_{k_3=0\ k_4=2} \quad (11)$$

Five-Body symmetrized basis for [41] representation with $K_5 = 1$ and L=0 was constructed in [7]. Using connection between parentage coefficients (8) and transformation coefficients between different sets of Jacobi coordinates for five-body hyperspherical basis symmetrized with respect to four identical particles [6-7] we can construct symmetrized HF corresponding to [41] representation with $\mathrm{K}_5 = 2$ and L=0:

$$\Phi^{[41](1)}_{20} = -\frac{\sqrt{5}}{3}\Phi^{l_1=l_2=1}_{k_2=2} - \frac{\sqrt{10}}{6}\Phi^{l_1=l_3=1}_{k_5=2} - \frac{\sqrt{6}}{6}\Phi^{l_1=l_4=1}_{k_5=2}$$

$$;\Phi^{[41](2)}_{20} = -\frac{\sqrt{5}}{3}\Phi^{l_i=0}_{k_3=k_4=k_5=2} + \frac{\sqrt{10}}{6}\Phi^{l_2=l_3=1}_{k_5=2} + \frac{\sqrt{6}}{6}\Phi^{l_2=l_4=1}_{k_5=2}$$

$$\Phi^{[41](3)}_{20} = -\sqrt{\frac{5}{6}}\Phi^{l_i=0}_{k_4=k_5=2} - \sqrt{\frac{1}{6}}\Phi^{l_3=l_4=1}_{k_5=2} \quad (12)$$

$$\Phi^{[41](4)}_{20} = \Phi^{l_i=0}_{k_5=2}$$

Using formulas (8) and (9) we can construct symmetrized basis for [51] and [42] representations of six-body systems with identical particles when $\mathrm{K}_6 = 2, \mathrm{L} = 0$:

$$\Phi^{[51](5)}_{20} = \Phi^{l_i=0}_{k_6=2}$$

$$\Phi^{[51](1)}_{20} = -\frac{1}{\sqrt{2}}\Phi^{l_1=l_2=1}_{k_6=2} - \frac{1}{2}\Phi^{l_1=l_3=1}_{k_6=2} - \frac{3}{2\sqrt{15}}\Phi^{l_1=l_4=1}_{k_6=2} - \frac{1}{\sqrt{10}}\Phi^{l_1=l_5=1}_{k_6=2}$$

$$\Phi_{20}^{[51](2)}=-\frac{1}{\sqrt{2}}\Phi_{k_3=k_4=k_5=k_6=2}^{l_i=0}+\frac{1}{2}\Phi_{k_6=2}^{l_2=l_3=1}+\frac{3}{2\sqrt{15}}\Phi_{k_6=2}^{l_2=l_4=1}+\frac{1}{\sqrt{10}}\Phi_{k_6=2}^{l_2=l_5=1}$$
$$\Phi_{20}^{[51](3)}=-\frac{\sqrt{3}}{2}\Phi_{k_4=k_5=k_6=2}^{l_i=0}-\frac{3}{2\sqrt{15}}\Phi_{k_6=2}^{l_3=l_4=1}-\frac{1}{\sqrt{10}}\Phi_{k_6=2}^{l_3=l_5=1}$$
$$\Phi_{20}^{[51](4)}=\frac{3}{\sqrt{10}}\Phi_{k_5=k_6=2}^{l_i=0}+\frac{1}{\sqrt{10}}\Phi_{k_6=2}^{l_4=l_5=1}$$
$$\Phi_{20}^{[51](5)}=\Phi_{k_6=2}^{l_i=0}$$
$$\Phi_{20}^{[42](1)}=-\frac{1}{\sqrt{3}}\Phi_{k_6=2}^{l_1=l_2=1}+\sqrt{\frac{2}{3}}\Phi_{k_6=2}^{l_1=l_3=1}$$
$$\Phi_{20}^{[42](2)}=\frac{1}{3}\Phi_{k_6=2}^{l_1=l_2=1}+\frac{\sqrt{2}}{6}\Phi_{k_6=2}^{l_1=l_3=1}-\sqrt{\frac{5}{6}}\Phi_{k_6=2}^{l_1=l_4=1} \tag{13}$$
$$\Phi_{20}^{[42](3)}=\frac{\sqrt{2}}{6}\Phi_{k_6=2}^{l_1=l_2=1}+\frac{1}{6}\Phi_{k_6=2}^{l_1=l_3=1}-\frac{1}{2\sqrt{15}}\Phi_{k_6=2}^{l_1=l_4=1}+\frac{3}{\sqrt{10}}\Phi_{k_6=2}^{l_1=l_5=1}$$
$$\Phi_{20}^{[42](4)}=-\frac{1}{\sqrt{3}}\Phi_{k_3=k_4=k_5=k_6=2}^{l_i=0}-\sqrt{\frac{2}{3}}\Phi_{k_6=2}^{l_2=l_3=1}$$
$$\Phi_{20}^{[42](5)}=\frac{1}{3}\Phi_{k_3=k_4=k_5=k_6=2}^{l_i=0}-\frac{\sqrt{2}}{6}\Phi_{k_6=2}^{l_2=l_3=1}+\sqrt{\frac{5}{6}}\Phi_{k_6=2}^{l_2=l_4=1}$$
$$\Phi_{20}^{[42](6)}=-\frac{\sqrt{2}}{6}\Phi_{k_3=k_4=k_5=k_6=2}^{l_i=0}+\frac{1}{6}\Phi_{k_6=2}^{l_2=l_3=1}+\frac{1}{2\sqrt{15}}\Phi_{k_6=2}^{l_2=l_4=1}-\frac{3}{\sqrt{10}}\Phi_{k_6=2}^{l_2=l_5=1}$$
$$\Phi_{20}^{[42](7)}=\frac{1}{\sqrt{6}}\Phi_{k_4=k_5=k_6=2}^{l_i=0}-\sqrt{\frac{5}{6}}\Phi_{k_6=2}^{l_3=l_4=1}$$
$$\Phi_{20}^{[42](8)}=\frac{\sqrt{3}}{6}\Phi_{k_4=k_5=k_6=2}^{l_i=0}+\frac{1}{2\sqrt{15}}\Phi_{k_6=2}^{l_3=l_4=1}+\frac{3}{\sqrt{10}}\Phi_{k_6=2}^{l_3=l_5=1}$$
$$\Phi_{20}^{[42](9)}=\frac{1}{\sqrt{10}}\Phi_{k_5=k_6=2}^{l_i=0}-\frac{3}{\sqrt{10}}\Phi_{k_6=2}^{l_4=l_5=1}$$

Fully symmetrized N-body (N=4, 5, 6) HF with higher quantum numbers can be easily constructed using expressions (11-13) and transformation coefficients of N-body HF symmetrized with respect to N-1 particles introduced in [7].

CONCLUSION

Parentage scheme of symmetrization (PSS) allows obtaining symmetrized hyperspherical functions from functions with arbitrary quantum numbers by the use of the transformation coefficients related with the permutations of the last two particles. Problem of hyperspherical basis transformation and construction is solved for few-body systems with identical particles and systems containing one distinguishable particle along with identical particles using the PSS. Young operators, acting on N-body (N=4,5,6) hyperspherical functions symmetrized with respect to (N-1) particles, are obtained and fully symmetrized Hyperspherical functions with different quantum numbers for the few-body systems containing identical particles are constructed. It is demonstrated when using the PSS no principal difficulties arise as number of particles increases. Therefore, proposed mathematical formalism can easily be generalized for the systems with any number of particles.

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