RESEARCH ARTICLE | JULY 10 2017

One-prey two-predator model with prey harvesting in a food chain interaction FREE

I. M. Sayekti; M. Malik; D. Aldila

AIP Conf. Proc. 1862, 030124 (2017)
https://doi.org/10.1063/1.4991228

One-Prey Two-Predator Model with Prey Harvesting in a Food Chain Interaction

I. M. Sayekti, M. Malik, and D. Aldila[a)]

Department of Mathematics, Faculty of Mathematics and Natural Sciences (FMIPA), Universitas Indonesia, Depok 16424, Indonesia

[a)]Corresponding author: aldiladipo@sci.ui.ac.id

Abstract. The interaction between prey, secondary predator, and primary predator as a mathematical model of the one-prey and two-predator system with constant harvesting in prey population will be introduced in this article. Their interaction might describe as a food pyramid, with the preys is in the lowest level of the pyramid, secondary predators in the middle, and primary predators in the top of the pyramid. Human intervention to controlling prey population is needed and will be analyzed how this will effect on the existence of secondary predator and primary predator population. Equilibrium points and their existence criteria will be analyzed to find a threshold that will guarantee the coexistence of this system. Some numerical simulation will be given to illustrate the analytical results. We find that as long as harvesting rate in prey population is smaller than prey intrinsic growth rate, coexistence might achieve.

INTRODUCTION

The community is a collection of various populations in a given area that interact and influence each other. The interaction formed by the community and the environment is called ecosystems. In ecosystems, there is a series of interrelated organisms in their feeding habits called the food chain. The food chain has an important role in maintaining the balance of the ecosystem. If one of the food chains is missing, then the dynamic of the ecosystem will be disturbed and affect a population of prey or predators in the ecosystem.

An interaction example in the food chain relation is the interaction between the rats, snakes, and mongooses in the form of the food pyramid. Rats unable to build anti-predation to snakes and mongooses so that snakes and mongooses don't need to form a group to be able to prey on rats. In the other hands, snakes can build anti-predation to mongooses so that mongoose need to form a group to be able to prey on snakes.

To understand the ecology interaction between prey and predator, many ways can be used, for example with a mathematical model. Since Lotka [1] and Volterra [2] proposed the simple model of prey-predator interaction (Now it is known as Lotka-Volterra model), many mathematical models have been introducing to understand more complex ecology interaction, for example [3,-6] who discuss about the predator prey model in many circumstance that appears in the field, [7] who discussed about translocation strategy in black rhino population, [8] who discuss about effect of feeding saturation in a species interaction. Authors in [9] develop a predator prey model where there are two equal predators in ecosystem, while authors in [10] construct a model between two equal prey with single predator model. Differently with above authors, in this article, a mathematical model to describe the interaction between preys, secondary predators, and primary predators as a one-prey two-predator model will be introduced. The interaction is based on food pyramid model. Intervention from human such as harvesting in prey population will include into the model.

The paper will be organized as follow. In section 2 mathematical model with their assumption will be given in detail. Mathematical model analysis to find existence criteria of equilibrium points will be conducted in section 3. Some numerical simulation and conclusion will be given in section 4 and 5, respectively.

International Symposium on Current Progress in Mathematics and Sciences 2016 (ISCPMS 2016)
AIP Conf. Proc. 1862, 030124-1–030124-8; doi: 10.1063/1.4991228
Published by AIP Publishing. 978-0-7354-1536-2/$30.00

MATHEMATICS MODEL

In this chapter, the interaction between prey (t) , secondary predator $y(t)$, and primary predator $z(t)$ will be constructed as a deterministic model in three-dimensional system. We assumed that the populations are closed, and the interaction between these species followed the food pyramid in Fig. 1. Since when secondary predator and primary predator hunting prey individually, the predation term follow Holling type II [3, 8, 10]. In the other hands, primary predator needs to do a group hunting to hunt secondary predator. Therefore, we use a modified Holling type II [3, 8] to describe the predation term. Without any predation and harvesting rate, the dynamic of prey population described as a logistic equation with carrying capacity K.

Therefore, based on above assumptions and Fig. 1, we proposed our three-dimensional one-prey and two-predator mathematical model with constant harvesting rate in prey population given by

$$\begin{aligned}\frac{dx}{dt} &= rx\left(1-\frac{x}{K}\right)-\frac{axy}{1+bx}-\frac{cxz}{1+bx}-hx\\ \frac{dy}{dt} &= \rho_1\frac{axy}{1+bx}-\frac{dyz}{1+ey+f(z-1)}-\delta_y y\\ \frac{dz}{dt} &= \rho_2\frac{axy}{1+bx}+\rho_3\frac{dyz}{1+ey+f(z-1)}-\delta_z y\end{aligned} \qquad (1)$$

with parameters, description is given in Table 1.

TABLE 1. Parameters Description for System (1)

Parameter	Description	Unit
r	Prey per capita intrinsic growth rate	$\frac{1}{time}$
K	Prey carrying capacity	$individu$
a, c, d	Predation intensity between x and y, x and z, y and z, respectively	$\frac{1}{individu\ x\ time}$
ρ_1, ρ_2, ρ_3	Food conversion coefficient for predation term between x and y, x and z, y and z, respectively	–
δ_y, δ_z	Natural death rate for secondary predator (y) and primer predator (z), respectively	$\frac{1}{time}$
h	Prey harvesting rate	$\frac{1}{time}$
b, e	Saturation rate of prey and Secondary predator, respectively	$\frac{1}{individu}$
f	Interference coefficient among Predator	$\frac{1}{individu}$

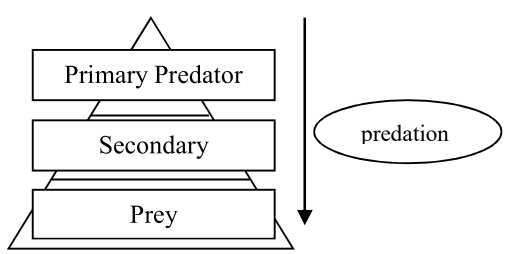

FIGURE 1. Food Pyramid

EQUILIBRIUM POINT ANALYSIS

We find 5 equilibrium points for System (1), namely $E_1 = (0,0,0)$, $E_2 = (x_2, 0,0)$, $E_3 = (x_3, y_3, 0)$, $E_4 = (x_4, 0, z_4)$, and $E_5 = (x_5, y_5, z_5)$. Here below we give an analysis of the existence of each equilibrium point.

1. Extinction of all population, $E_1 = (0,0,0)$
   This equilibrium is a trivial equilibrium of System (1) without positive criteria. To find it local stability criteria, firstly we find Jacobian matrix of System (1) evaluated in E_1 which given by

$$J_1 = \begin{bmatrix} r-h & 0 & 0 \\ 0 & -\delta_y & 0 \\ 0 & 0 & -\delta_z \end{bmatrix}. \tag{2}$$

   J_1 have three eigenvalues, i.e $r-h$, $-\delta_y$, and $-\delta_z$. Therefore, E_1 will locally asymptotic stable if only if all eigenvalues of J_1 are negative which lead us to the condition that should be satisfied, i.e $r-h$. This means that all population will be vanish from ecosystem if harvesting rate in prey population is larger than than prey intrinsic growth rate.

2. Extinction of all predators, $E_2 = (x_2, 0,0)$
   By it's name, E_2 describe a situation where only prey is the remaining population in the ecosystem. Explicitly, this equilibrium point is given by

$$E_2 = \left(\frac{K(r-h)}{r}, 0,0\right) \tag{3}$$

   which will exist if and only if $r > h$. This means that to guarantee the existence of prey population only in the ecosystem, intrinsic growth rate should be larger than the harvesting rate.
   In the other hand, as well as the local stability analysis from preview equilibrium, the Jacobian of System (1) evaluated in E_2 is given by

$$J_2 = \begin{bmatrix} h-r & \frac{-aK(h-r)}{Kb(h-r)-r} & \frac{-Kc(h-r)}{Kb(h-r)-r} \\ 0 & \frac{K(h-r)(a\rho_1-b\delta_y)+r\delta_y}{Kb(h-r)-r} & 0 \\ 0 & 0 & \frac{K(h-r)(c\rho_2-b\delta_z)+r\delta_z}{Kb(h-r)-r} \end{bmatrix} \tag{4}$$

   We have three eigenvalues of J_2, i.e $\lambda_1 = h-r$, $\lambda_2 = \frac{K(h-r)(a\rho_1-b\delta_y)+r\delta_y}{Kb(h-r)-r}$, and $\lambda_3 = \frac{K(h-r)(c\rho_2-b\delta_z)+r\delta_z}{Kb(h-r)-r}$, where λ_1 will be negative if only if $r > h$, λ_2 will be negative if only if $\frac{r\delta_y}{K(r-h)(a\rho_1-b\delta_y)} > 1$, and λ_3 will be negative if only if $\frac{r\delta_z}{K(r-h)(c\rho_2-b\delta_z)} > 1$.

3. Survival of prey and secondary Predator, $E_3 = (x_3, y_3, 0)$
   E_3 describe a situation when only prey and secondary predator exist in ecosystem while the primary predator will be extinct. Explicitly, this equilibrium is given by

$$E_3 = (x_3, y_3, 0) = \left(\frac{\delta_y}{a\rho_1-b\delta_y}, \frac{\rho_1(K(r-h)(a\rho_1-b\delta_y)-r\delta_y)}{(a\rho_1-b\delta_y)^2 K}, 0\right), \tag{5}$$

   which will positive if and only if $a\rho_1 - b\delta_y > 0$ and $K(r-h)(a\rho_1 - b\delta_y) - r\delta_y > 0$. Since to ensure the existence of it should be, therefore we have the existence criteria of E_3 are

$$\frac{a\rho_1}{b\delta_y} > 1 \tag{6}$$

$$\frac{K(r-h)(a\rho_1-b\delta_y)}{r\delta_y} > 1. \tag{7}$$

In the other hand, as well as the local stability analysis from preview equilibrium, the Jacobian of System (1) evaluated in E_3 is given by

$$J_3 = \begin{bmatrix} p & q & s \\ u & 0 & v \\ 0 & 0 & w \end{bmatrix}, \tag{8}$$

with

$$p = -\frac{(-K\delta_y(h-r)b^2+(r\delta_y+Ka\rho_1(h-r))b+ar\rho_1)\delta_y}{(\rho_1 a(a\rho_1-b\delta_y)K)}, q = -\frac{\delta_y}{\rho_1}, s = -\frac{\delta_y c}{a\rho_1}, u = -\frac{K(h-r)(a\rho_1-b\delta_y)+r\delta_y}{Ka},$$

$$v = \frac{\rho_1 d\left((h-r)(-a\rho_1+b\delta_y)K-r\delta_y\right)}{(-a\rho_1+b\delta_y)\left(((-f+1)a-e(h-r))\rho_1+\delta_y b(f-1)\right)K+er\rho_1\delta_y},$$

$$w = \frac{\alpha+\beta+\gamma}{\theta}, \alpha = Ka^2\rho_1(-a\delta_z(f-1)+(h-r)(d\rho_3-e\delta_z))^3,$$

$$\beta = a\delta_y\rho_1^2\left(K\left(a(f-1)(2b\delta_z+c\rho_2)+\left((-d\rho_3+e\delta_z)b+\rho_2 ce\right)(h-r)\right)+r(d\rho_3-e\delta_z)\right),$$
$$\gamma = -2\delta_y^2\rho_1\left(Kb\left(a\left(\rho_2 c+\frac{1}{2}\delta_z b\right)(f-1)+\frac{1}{2}\rho_2 ce(h-r)\right)-\frac{1}{2}\rho_2 cer\right)+\rho_2\delta_y^3 b^2 c(f-1),$$
$$\theta = a\rho_1(Ka\rho_1^2((h-r)e+a(f-1))-2\delta_y\rho_1(Kb(a(f-1)+\frac{1}{2}(h-r)e)-\frac{1}{2}+K\delta_y^2 b^2(f-1)).$$

We have three eigenvalues of J_3 that will be negative if only if

$$\frac{\alpha+\beta+\gamma}{\theta} < 0 \tag{9}$$

$$\frac{r\delta_y}{K(r-h)(a\rho_1-b\delta_y)} > 1 \tag{10}$$

$$\frac{b(r\delta_y+Ka\rho_1(h-r))+ar\rho_1}{k\delta_y(h-r)b^2} > 1. \tag{11}$$

4. Survival of prey and secondary Predator $E_4 = (x_4, 0, z_4)$
E_4 describe a situation when only prey and primary predator exist in the ecosystem while the secondary predator will be extinct. Explicitly, this equilibrium is given by

$$E_4 = \left(\frac{\delta_z}{c\rho_2-b\delta_z}, 0, \frac{\rho_2(K(r-h)(c\rho_2-b\delta_z)-r\delta_z)}{(c\rho_1-b\delta_z)^2 K}\right) \tag{12}$$

which will positive if and only if if $\frac{\delta_z}{c\rho_2-b\delta_z} > 0$ and $\frac{\rho_2(K(r-h)(c\rho_2-b\delta_z)-r\delta_z)}{(c\rho_2-b\delta_z)^2 K} > 0$. Since to ensure the existence of it should be, therefore we have the existence criteria of $E_4$are

$$\frac{c\rho_2}{b\delta_z} > 1 \tag{13}$$

$$\frac{K(r-h)(c\rho_2-b\delta_z)}{r\delta_z} > 1. \tag{14}$$

In the other hand, as well as the local stability analysis from previous equilibrium, the Jacobian of System (1) evaluated in E_4 is given by

$$J_4 = \begin{bmatrix} k & l & m \\ 0 & n & 0 \\ o & p & 0 \end{bmatrix}, \tag{15}$$

with

$$k = -\frac{\delta_z(k\delta_z(h-r)b^2+(-r\delta_z-Kc\rho_2(h-r))b-cr\rho_2)}{Kc\rho_2(b\delta_z-c\rho_2}, l = -\frac{a\delta_z}{c\rho_2}, m = -\frac{\delta_z}{\rho_2}, n = \frac{\phi+\psi+\tau}{\sigma}, o = \frac{K(h-r)(b\delta_z-c\rho_2)-r\delta_z}{Kc}$$

$$p = -\frac{\rho_2 d(K(h-r)(b\delta_z-c\rho_2)-r\delta_z)\rho_3}{K(b\delta_z-c\rho_2)(((-f+1)c-f(h-r))\rho_2+\delta_z b(f-1))+fr\rho_2\delta_z}, \phi = -Kc^2\rho_2^3\left(\delta_y(f-1)c+(h-r)\left(f\delta_y+d\right)\right)$$

$$\psi = 2c\rho_2^2\delta_z\left((f-1)\left(\delta_y b+\frac{1}{2}a\rho_1\right)c+\frac{1}{2}\left(\left(a\rho_1+b\delta_y\right)f+bd\right)(h-r)\right)K-\frac{1}{2}r(f\delta_y+d))$$

$$\tau = -\left(\left((f-1)\left(2a\rho_1+b\delta_y\right)c+af\rho_1(h-r)\right)bK-afr\rho_1\right)\delta_z^2\rho_2+K\delta_z^3ab^2\rho_1(f-1)$$

$$\sigma = c\rho_2\left(Kc\rho_2^2\left((f-1)c+f(h-r)\right)-2\delta_z\rho_2\left(Kb\left((f-1)c+\frac{1}{2}f(h-r)\right)-\frac{1}{2}fr\right)+K\delta_z^2b^2(f-1)\right).$$

We have three eigenvalues of J_4 that will be negative if only if

$$\frac{\phi+\psi+\tau}{\sigma} < 0 \tag{16}$$

$$\frac{k(r-h)(c\rho_2-b\delta_z)}{r\delta_z} > 1 \tag{17}$$

$$\frac{b(r\delta_z+Kc\rho_2(h-r))+cr\rho_2}{K\delta_z b^2(h-r)} > 1. \tag{18}$$

5. Coexistence of Prey and All Predators, $E_5 = (x_5, y_5, z_5)$
E_5 represent a situation when all population living alongside one and another. Eliminate x variable from System (1), give us x as a function of all parameters and y and z which given by

$$x = \frac{ey\delta_y+fz\delta_y+dz-f\delta_y+\delta_y}{ae\rho_1 y+af\rho_1 z-bey\delta_y-bfz\delta_y-af\rho_1-bdz+bf\delta_y+a\rho_1-b\delta_y}. \tag{19}$$

Substituting (19) in to $\frac{dy}{dt} = 0$ dan $\frac{dz}{dt} = 0$, lead us in to a couple of polynomial which will guarantee the existence of (x_5, y_5, z_5), let call it as $G_1(y,z,\Omega) = 0$ and $G_2(y,z,\Omega) = 0$, with Ω is the set of all parameters. Since it is not possible to show $G_1(y,z,\Omega)$ and $G_2(y,z,\Omega)$ explicitely, the existence of this equilibrium will be shown numerically.

To perform the existense E_5 numerically, we use parameters value of $r = 0.8$, $K = 60$, $a = 0.9$, $b = 0.0025$, $c = 0.1$, $h = 0.01$, $\delta_y = 0.0001$, $\delta_z = 0.01$, $\rho_1 = 0.125$, $\rho_2 = 0.125$, $\rho_3 = 0.004$, $d = 0.001$, $e = 0.003$, and $f = 0.01$. Intersection of $G_1(y,z,\Omega) = 0$ and $G_2(y,z,\Omega) = 0$ in the first quadran will guarantee the existence of the equilibrium E_5 which is shown in Fig. 2. With this parameters value, we have E_5 is (0.8, 5.27, 5.17).

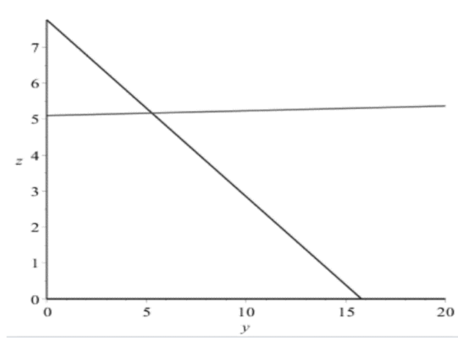

FIGURE 2. Intersection between y and z when $G_1(y,z,\Omega) = 0$ and $G_2(y,z,\Omega) = 0$.

NUMERICAL EXPERIMENT

In this section, we will perform some numerical simulation of System (1) with various scenario. The first scenario is given with varying the harvesting rate (h), the second scenario with varying the saturation rate of prey (b) and the last scenario with varying the prey carrying capacity (K).

a) First Scenario
As a first scenario, we use various value of harvesting rate $(h = 0, h = 0.25, h = 0.35)$ to understand the effects of harvesting scenario for the persistence of all three species in a same ecosystem. The parameters that we use are $r = 0.3$, $K = 60$, $a = 1$, $b = 0.04$, $c = 1$, $\delta_y = 0.25$, $\delta_z = 0.25$, $\rho_1 = 0.125$, $\rho_2 = 0.125$, $\rho_3 = 0.25$, $d = 0.01$, $e = 0.04$, dan $f = 0.5$. As it can be seen in Fig. 3, when harvesting rate of prey population is larger than intrinsic growth rate, then population of prey will tends to zero. As a consequence, the dynamic of all predators also will tends to zero. In the other hands, harvesting of prey population is still allowed as long as it is still smaller than intrinsic growth rate of prey population.

b) Second Scenario
The second scenario is given to see how saturation rate affect the dynamic of ecosystem. The parameters that we use are $r = 0.3$, $K = 60$, $a = 1$, $K = 60$, $c = 1$, $\delta_y = 0.25$, $\delta_z = 0.25$, $\rho_1 = 0.125$, $\rho_2 = 0.125$, $\rho_3 = 0.25$, $d = 0.01$, $e = 0.04$, dan $f = 0.5$. When the value of the parameter $b = 0.2$, the time that required to reach the maximum number of populations in each population being slower than the system with $b = 0.04$ and $b = 0.1$. It can be seen in Fig. 4 that varying b will increase the number of prey population. Enlarging b, although it could delay the time window to reach the maximum number of prey population, but it also could increase a maximum number of the prey. In the other hands, it can be seen that the number of secondary predator population will decrease as a consequence of the increase of it predation handling time. This result might be interpreted that when secondary predator needs longer time to handling prey, the number of the prey will increase significantly.

c) Third Scenario
Last simulation is given to see how number of prey carrying capacity might affect the dynamic of each population in the ecosystem (see Fig. 5). The parameters that we use are $r = 0.3$, $h = 0.4$, $a = 1$, $b = 0.04$, $c = 1$, $\delta_y = 0.25$, $\delta_z = 0.25$, $\rho_1 = 0.125$, $\rho_2 = 0.125$, $\rho_3 = 0.25$, $d = 0.01$, $e = 0.04$, dan $f = 0.5$. It can be seen in Fig. 5 that increasing the carrying capacity of the environment for the growth of prey, in addition, to increasing the maximum number of prey populations (trivial), a number of secondary and primary predator also increased. However, the time to achieve the maximum value of predators becomes slower than before.

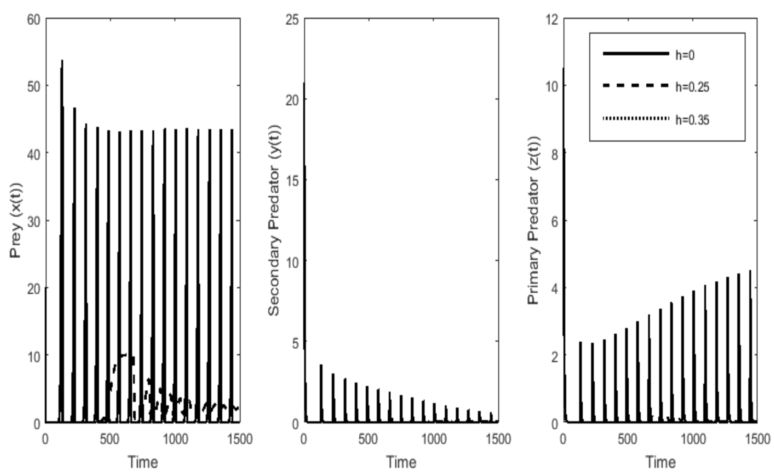

FIGURE 3. Dynamic of prey, secondary predator and primary predator (from left to right, respectively) respect to change of harvesting rate h. The time unit is month.

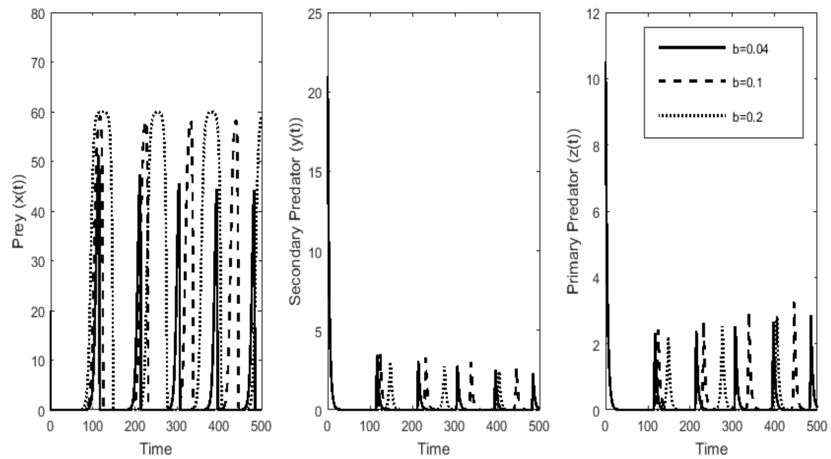

FIGURE 4. Dynamic of prey, secondary predator and primary predator (from left to right, respectively) respect to change of saturation rate b at $t = 100$ until $t = 200$. The time unit is month.

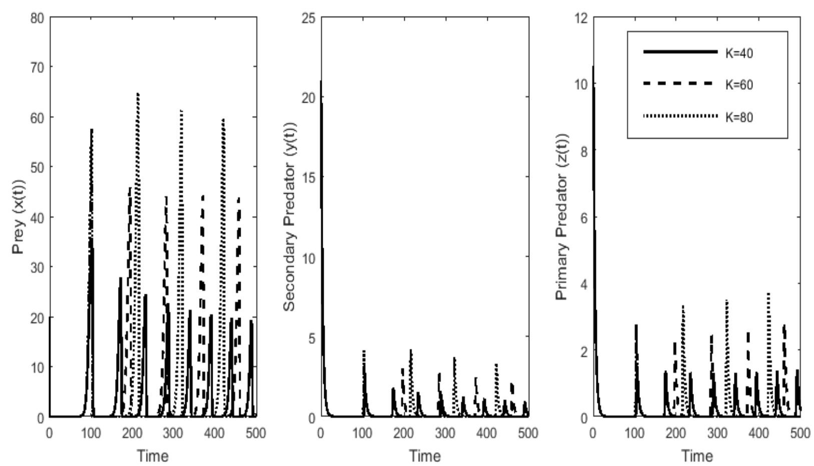

FIGURE 5. Dynamic of prey, secondary predator and primary predator (from left to right, respectively) respect to change of prey carrying capacity K. at $t = 100$ until $t = 200$. The time unit is month.

CONCLUSIONS

A mathematical model describing the dynamic of prey, secondary predator and primary predator in the same ecosystem as a prey-predator model have been introduced in this article. The system of Equation 1 has five equilibrium points. Four equilibrium points appears explicitly and the other equilibrium point appears implicitly. The threshold condition to guarantee the non-negative value of all populations is given analytically as a function of parameters. We find that if harvesting of prey population is larger than intrinsic growth rate of prey, then all population will be extinct. Modification of parameters in prey population (carrying capacity, saturation rate and harvesting rate) will change the dynamic in ecosystem globally. For further model development, readers may consider the dynamic of the model not only depend on time but also spatially. Considering the harvesting rate as a parameter which depending on time also could be considered to made the model become more realistic.

REFERENCES

1. A. J. Lotka, *Elements of Physical Biology* (Williams and Walkins Company, Baltimore, Maryland, 1925).
2. V. Volterra, in *Lecture Notes in Biomathematics: Variation and Fluctuations in the Number of Coexisting Animal Species* (Springer, Berlin, 1978), Vol. 22, pp. 65–236.
3. J. R. Beddington, J. Anim. Ecol, **44**, 331–340 (1975).
4. T. K. Kar, J. Biol. Sys. **12**, 61–71 (2003).
5. T. G. Skalski and J. F. Gillian, Ecology **82**, 3083–3092 (2001).
6. D. Xiao and S. Ruan, J. Math. Biol. **43**, 68–290 (2001).
7. D. Aldila, A. J. Hutchinson, M. Woolway, N. O. Smith, and E. Soewono, J. Math. Fund. Sci. **47**, 104–115 (2015).
8. D. L. DeAngelis, R. A. Goldstein, and R. V. O'Neill, Ecology **56**, 881–892 (1975).
9. B. Dubey and R. K. Upadhyay, Nonlinear Anal. Model. Control **9**, 307–329 (2004).
10. A. Edwin, *Modeling and Analysis of Two-prey Two-predator System with Harvesting, Holling Type II and Ratio Dependent System*, Ph.D. thesis, Makerere University, Kampala, 2010.