Bayesian contour detection in a time series of ultrasound images through dynamic deformable template models

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SUMMARY
Deformable template models and Markov chain Monte Carlo methods are used for analysing a space–
time process of intracoronary ultrasound images in order to detect the artery contour and various other
characteristics as a function of time.

Keywords: Bayesian image analysis; Deformable templates; Intracoronary ultrasound images; Markov chain Monte
Carlo; Space–time modelling.

1. INTRODUCTION

In medical imaging, in vivo video films of dynamic behaviour such as blood oxygenation in the brain
under stimuli, beating hearts, and pulsating artery walls, object detection in both space and time are
required: see for example, Mulet-Parada and Noble (2000). In this paper we develop an automatic method
for contour detection in a sequence of high-frequency intracoronary ultrasound (ICUS) images, using a
fully Bayesian approach. To the best of our knowledge, the paper provides the first published attempt of a
fully Bayesian analysis of a space–time process based on deformable template models and Markov chain
Monte Carlo (MCMC) simulations.

The ICUS images show the dynamic behaviour of the coronary artery; the data set is further described
in Section 2. Extracting information from the ICUS images is important, for example, in assessing the
extent of arteriosclerosis and flow response to various stimuli, and one feature of interest is the dynamic
behaviour of the cross sectional area of the artery at a given place. Analysing the images for anomalies
by manually following the contour of the artery in a digitized version of each video sequence is both time
consuming and observer dependent. A quick and accurate method for automatic contour detection and
feature extraction would be useful to practitioners.

Many papers deal with automatic detection of the contour of an object in a single image. In the
engineering literature the active contour approach has been popular (Kass et al., 1987), where the outline

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Table 1. List of key notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tr>
<td>$S$, $V$</td>
<td>Pixel centers, true coronary artery</td>
</tr>
<tr>
<td>$y = (y_s)_{s \in S}$</td>
<td>Data</td>
</tr>
<tr>
<td>$\mu_i$, $i = 1, 2$</td>
<td>Mean greylevels</td>
</tr>
<tr>
<td>$\sigma_i^2$, $a_i$, $b_i$, $i = 1, 2$</td>
<td>Variance of greylevels and associated hyperparameters</td>
</tr>
<tr>
<td>$\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$</td>
<td>Vector of equipment dependent parameters</td>
</tr>
<tr>
<td>$v_1, \ldots, v_n; e_1, \ldots, e_n$</td>
<td>Vertices and edges of the contour</td>
</tr>
<tr>
<td>$v_0, \ldots, v_0; e_0, \ldots, e_0$</td>
<td>Vertices and edges of the template</td>
</tr>
<tr>
<td>$\kappa, \delta, \epsilon$</td>
<td>Variance of edge coordinates and associated hyperparameters</td>
</tr>
<tr>
<td>$r = (x, v_1, \kappa, \theta)$</td>
<td>Vector of all ‘parameters’</td>
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is normally modeled by a connected and sufficiently smooth curve. The best fit is found as the curve that minimizes a functional which balances the degree of misfit, some smoothness criterion, and a quantity describing the closeness to some important image features. Yet other methods have been developed for medical image analysis: see Mulet-Parada and Noble (2000) and references therein. Many of the methods produce a good ‘point estimate’ of the contour and various characteristics belonging to it, but say nothing about the uncertainty of the derived characteristics, a crucial requirement when it comes to diagnosis. This has led to a number of more statistical formulations of the contour detection problem, usually in a Bayesian setting.

Possibly the most popular class of prior models for contours is provided by the deformable polygonal template approach of Grenander and others: see for example, Grenander et al. (1991), Grenander and Miller (1994), and Kent et al. (1996). Another approach is based on marked point processes for object detection (Baddeley and van Lieshout, 1993; Pientaolo and Green, 1998). The two approaches of deformable templates and marked point processes are combined in Rue and Syversveen (1998), Rue and Husby (1998), and Hurn et al. (2001). Likelihood based methods for template matching have recently been studied in Hobolth and Jensen (2000) and Hobolth and Pedersen (1999).

There is an extensive engineering literature dealing with space–time images, including Bayesian approaches based on deformable templates: see Kervrann (1998, 1999) and references therein. Mulet-Parada and Noble (2000) extend ideas from deterministic spatial feature detection to the temporal domain, and apply this to boundary detection in echocardiography. Kervrann (1999), in particular, combines a spatial segmentation procedure based on deformed templates with a fixed number of vertices, with edge-based temporal tracking. The tracking procedure over long image sequences was supplemented with Kalman filtering.

We also use a deformable template as prior distribution, but do not find it necessary to include global deformations as in Kervrann (1999). In order to account for uncertainty of the parameters, we impose prior distributions on the parameters of the deformation model in contrast to Kervrann (1999) who estimates them by maximum marginal likelihood. In contrast to Kervrann’s approach based on maximum a posteriori estimates, we use the posterior means as estimates of the contour and to predict the templates. Furthermore, we supply the posterior mean calculations of the cross sectional area of the artery and other parameters with calculations of credibility intervals.

The paper is organized as follows. Section 2 considers the data and the observation model corresponding to the likelihood term of our posterior. Deformable template priors related to our application are studied in Section 3. Section 4 is concerned with Bayesian inference and MCMC algorithms. The empirical results are discussed in Section 5.

Finally, since there is a fair amount of notation in the paper, Table 1 provides a list of key notation.
Contour detection by dynamic deformable templates

2. DATA AND OBSERVATION MODEL

The data originate from a study of patients with angina pectoris (pain in the chest) without angiographic evidence of arteriosclerosis. The study was conducted by the Department of Cardiology, Aarhus University Hospitals, Skejby, with the purpose of diagnosing whether or not chest pain is caused by inappropriate responses of the coronary artery to various stimuli. Notice that the present paper does not deal with the diagnosis of the patients but merely presents a way of doing the tracing and feature extraction automatically.

The data we consider arise from a 2 seconds long video film recorded by ultrasound scanning at a prespecified place in a coronary artery, approximately perpendicular to the blood flow. The frame rate is 25 images per second, yielding 49 ultrasound images which have been digitized into grey-level pictures. Figure 1 shows two of these grey-level images. Additional figures, which show the development over time, are given in Figures 6–9.

Consider the right image in Figure 1. The intracoronary scanning is performed by inserting into the artery (the black part in the middle of the image) a catheter (indicated by 3) with a rotating ultrasound transducer at the top. The brightest grey-level represents the greatest reflection of the ultrasound. The drop outs at 1 and 2 (compare with the left image in Figure 1) may be true image artifacts or offsprings of vessels.

It is customary to consider an image as a positive function defined on some planar subset $W$. However, in ‘computerized’ imaging one is forced to discretize the images to get a computer representation. The model for the discretization process is termed an observational model. In the present context we assume that the true underlying artery can be identified with some subset $V \subset W$.

Each discretized grey-level image is made up of $N^2 = 128^2$ square pixels, which we identify with unit squares. We denote the union of these pixels by $W = [0,N] \times [0,N]$. The set of observed grey-levels of a given ultrasound image is denoted by $y = (y_s)_{s \in S}$, where $S = \{0.5,1.5, \ldots, N-0.5\} \times \{0.5,1.5, \ldots, N-0.5\}$ is the set of centres of pixels. Although the range of grey-levels is $[0,1, \ldots, 255]$ (where 0 is black and 255 is white), $y_s$ is conceptually viewed as being real-valued.

Given the true artery $V \subset W$ we assume that the observed grey-levels $y_s$ are independent and Normally distributed with mean $\mu_1$ and variance $\sigma_1$ if $s \in V$ (the inside of the coronary artery) and mean $\mu_2$ and variance $\sigma_2$ if $s \in W \setminus V$ (the outside of the coronary artery). So if $S_1 = S \cap V$ and $S_2 = S \setminus V$ denote the pixels inside and outside of the coronary artery, the likelihood for the data $y$ given $\theta = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ and $V$ is

$$L(y|V, \theta) = (\sigma_1^2)^{-n_1/2}(\sigma_2^2)^{-n_2/2} \exp((-SS_1/(2\sigma_1^2) - SS_2/(2\sigma_2^2))) \times \exp(-n_1(\bar{y}_1 - \mu_1)^2/(2\sigma_1^2) - n_2(\bar{y}_2 - \mu_2)^2/(2\sigma_2^2))$$

(1)
where
\[ n_i = \#(S_i \cap V), \quad \bar{y}_i = \sum_{s \in S_i} y_s / n_i, \quad SS_i = \sum_{s \in S_i} (y_s - \bar{y}_i)^2, \quad i = 1, 2. \] (2)

As the grey-levels inside the coronary artery in general are supposed to be smaller than those outside, the domain of variation for the parameter \( \theta \) is assumed to be
\[ 0 < \mu_1 < \mu_2 < 255, \quad \sigma_1^2 > 0, \quad \sigma_2^2 > 0. \] (3)

Note that owing to the discretization into pixels, there is for each fixed value of \((y, \theta)\) an upper bound and a strictly positive lower bound on the likelihood,
\[ 0 < c_1(y, \theta) \leq L(y|V, \theta) \leq c_2(y, \theta) < \infty. \] (4)

In the sequel, we assume that \( V \) is a polygon specified by \( n \) vertices \( v_1, \ldots, v_n \). Its boundary
\[ \partial V = \bigcup_{j=1}^{n} [v_{j-1}, v_j] \]
is a closed self-avoiding planar contour as illustrated in Figure 2, where \( v_0 = v_n \) and \([v_{j-1}, v_j]\) is the closed line segment with endpoints \( v_{j-1} \) and \( v_j \) (‘self-avoiding’ means that for all \( 1 \leq j < k \leq n, [v_{j-1}, v_j] \cap [v_{k-1}, v_k] \) equals \( \{v_k\} \) for \( k = j + 1 \) and is empty otherwise). In some applications of object restoration it is relevant to let \( n \) be random (see for example, Pievatolo and Green (1998) and Rue and Hurn (1999)). In our case, where \( \partial V \) is expected to be close to a circular \( n \)-gon, we found it sufficient and much simpler to fix \( n = 16 \).

3. Deformable Template Priors

For each single ultrasound image, we assume that the contour \( \partial V \) and the parameters \( \sigma_1^2, \sigma_2^2, \) and \( (\mu_1, \mu_2) \) are mutually independent, where \( \sigma_j^2 \) follows a flat \( IG(\alpha_j, \beta_j) \) distribution, \( j = 1, 2 \) (the inverse gamma distribution with shape parameter \( \alpha_j \) and scale parameter \( \beta_j \)), while \( (\mu_1, \mu_2) \) is uniformly distributed on the set where \( 0 < \mu_1 < \mu_2 < 255 \), cf. (3). The hyperparameters \( \alpha_j \) and \( \beta_j \) are specified in Section 5.1. In Sections 3.1 and 3.2 we specify two kinds of deformable template priors for \( \partial V \). As discussed in Section 4, this can be combined with the likelihood (1) to obtain a joint distribution for the entire sequence of ultrasound images. Note that by (4) and the independence between \( \partial V \) and \( \theta \), posterior propriety is equivalent to prior propriety for \( \partial V \).
3.1 A simple deformable template prior

The contour in the $i$th image is modeled as a deformation of a template

$$\partial V^0 = \bigcup_{j=1}^n [v_{i,j-1}^0, v_{i,j}^0]$$

which in our case is expected to be close to a circular $n$-gon. If $i = 1$, the vertices of the template are found by manual tracing of the artery wall. This is done by means of a computer screen and mouse where the operator clicks in the vertices on the displayed image such that the produced $n$-gon best captures what is believed to be the artery wall, cf. the figures in Section 5.3. For $i > 1$, the template is given by the (estimated) posterior mean of the vertices of the deformed template in the $(i-1)$th image, cf. Section 4.2.

An alternative to the manual detection would be an automated version. However, we found it most natural to base the initial template on an expert opinion. Various experiments with initial templates showed that the method is quite robust, and when a large misspecification is introduced in the first image, it will decreasingly influence successive reconstructions.

Prior models for deformable templates are usually obtained by specifying a conditional cyclic autoregression (CAR) model on the vertices or edges before conditioning on closeness of the contour. Grenander et al. (1991), Grenander (1993), and Grenander and Miller (1994) propose a multiplicative model for the edges, while additive models for vertices and edges are considered in Kent et al. (1996). The CAR model we suggest is for the edges, and is obtained as a certain limit of proper first-order cyclic CAR models so that closeness is automatically satisfied. The advantage of using this template model compared with those considered in the abovementioned references is explained below.

Specifically, let

$$e_{sl} = (e_{1l}^0, \ldots, e_{ml}^0)^T, \quad e_{s0}^0 = (e_{01}^0, \ldots, e_{ml}^0)^T, \quad l = 1, 2;$$

be the vectors of first ($l = 1$) and second ($l = 2$) coordinates of the edges in $\partial V$ and $\partial V^0$ as defined by

$$e_j = v_j - v_{j-1}, \quad e_j^0 = v_j^0 - v_{j-1}^0, \quad j = 1, \ldots, n.$$

The CAR model is now specified by assuming that $e_{s1}$ and $e_{s2}$ are independent and Normally distributed with means $\mu_1$ and $\mu_2$, and a common precision matrix $\kappa^{-1}R(\gamma)$, where $\kappa > 0$ is the variance. Setting $r_1, 0 = r_{1,n}$ and $r_{n,n+1} = r_{n,1}$, the non-zero entries of $R(\gamma)$ are

$$r_{jj} = 1, \quad r_{j,j+1} = r_{j,j-1} = -\gamma/2, \quad j = 1, \ldots, n.$$ 

Additionally, we condition upon $\partial V$ being closed, i.e.

$$\sum_{j=1}^n e_{jl} = \sum_{j=1}^n e_{0jl} = 0, \quad l = 1, 2, \quad (5)$$

and self-avoiding. Notice that $R(\gamma)$ is positive definite for $|\gamma| < 1$, but only positive semi-definite if $\gamma = 1$. The latter case corresponds to an intrinsic CAR (see, for example, Besag and Kooperberg (1995)) and (5) is automatically satisfied. For our application, simulations from the posterior distribution, using a uniform prior on $(-1, 1)$ for $\gamma$ and remaining priors as specified below, showed $\gamma$ to be concentrated on values very close to 1.

In the sequel we assume that $\gamma = 1$, or more precisely that $e_{sl}$ has covariance matrix $\kappa R(1)^{-1}$, where $R(1)^{-1}$ is the generalized inverse of $R(1)$ of rank $n - 1$. Exploiting the circulant structure of $R(\gamma)$, we can identify computationally tractable diagonal and symmetric matrices $\Lambda$ and $F$, respectively, such that

$$R(1)^{-1} = (n^{1/2}FA^{1/2})(n^{1/2}FA^{1/2})^*, \quad (6)$$
where \( * \) denotes conjugate transpose; for details, see the Appendix. By (6), we can set
\[
(e - e^0) = (n\kappa)^{1/2} F A^{1/2} \begin{bmatrix} 0 \\ z \end{bmatrix},
\]
where \( e = e_{s1} + i e_{s2}, \) \( e^0 = e_{s1}^0 + i e_{s2}^0, \) and \( z = z_{s1} + i z_{s2} \) with \( z_{s1} \) and \( z_{s2} \) independent \((n - 1)\)-dimensional standard Normally distributed. For the calculation of the right hand side in (7), we use a radix-2 FFT-algorithm (van Loan, 1992).

Note that \( \partial V \) is determined by \((x, v_1, e^0)\) where \( x = \sqrt{\kappa} z. \) We assume that \( z, v_1, \kappa, \theta \) are mutually independent, \( v_1 \) is uniformly distributed on \( W, \) and \( \kappa \) follows a flat IG(\( \delta, \epsilon \)) distribution. Note that we do not require any knowledge of the location \( v^{0}_1 \) of the template. The hyperparameters \( \delta, \epsilon, \omega \) are specified in Section 5.1.

Incidentally, consider for the moment the case where \(|\gamma| < 1.\) Then the full conditional distribution of \( e_{jl} \) given the rest is Normal with mean \( e_{jl}^0 + \gamma(e_{j-1,l} - e_{j-l}^0 + e_{j+1,l} - e_{j+1,l}^0)/2 \) and variance \( 1/\kappa. \) However, when we condition on the linear constraint (5), the CAR model loses its simple dependence on nearest neighbours. In this case note that in many Bayesian applications (including but not restricted to deformable template modelling) an improper prior density is used instead, provided the posterior becomes proper: see for example, Besag and Kooperberg (1995), Besag and Higdon (1999), and Kent et al. (1996). When \( \kappa \) is fixed, the improper prior density for \((e_{s1}, e_{s2})\) is proportional to
\[
\exp \left( -\sum_{j=1}^{n} \sum_{l=1}^{n} (e_{jl} - e_{jl}^0 - (e_{j-1,l} - e_{j-l}^0) - (e_{j+1,l} - e_{j+1,l}^0))/2 \right) / (4\kappa) \right).
\]
As noted, in our case this causes impropriety of the posterior; see also the discussion in Besag and Kooperberg (1995).

### 3.2 A refined deformable template prior

As mentioned in Section 2, in some of the images part of the contour of the artery wall is missing. It turns out that if the simple prior of Section 3.1 is used, deformations of the template with sharp interior angles are likely because of artifacts, in particular lost parts of contours such as the drop outs shown in Figure 1. In order better to incorporate the prior knowledge that the coronary artery is close to a circular \( n \)-gon, we consider (before conditioning on \( \partial V \) being self-avoiding) a refined prior density for \( x \) defined as follows.

Given \((x, \kappa, \omega), \) where \( \omega \) is a hyperparameter, the density of the refined prior is
\[
\pi(x|\kappa, \omega) \propto \exp \left( -\left( ||x_{s1}||^2 + ||x_{s2}||^2 \right)/(2\kappa) - \omega \sum_{j=1}^{n} (\phi_j - (n - 2)\pi/n)^2 \right),
\]
where \( \phi_j \) is the interior angle of \( V \) at the vertex \( v_j \) (measured in radians) and \((n - 2)\pi/n\) is the interior angle in a regular \( n \)-gon. Setting \( \omega = 0 \) we obtain the prior in Section 3.1, but in accordance with our prior belief we shall assume that \( \omega > 0. \)

We compare in Section 5.3 the results of the Bayesian analysis using either the simple or refined prior.

Pievatolo and Green (1998) consider a somewhat similar model, but without incorporating a deformable template and the circular structure. Another approach was suggested by Grenander (1993, Remark 16.2.3). There the lost part of the boundary is modeled by a stochastic destructive deformation field acting on the template. Rue and Husby (1998) show that such a field is well suited to model partly destroyed objects. However, this approach complicates the statistical modelling and is heavily computer intensive. We therefore prefer the simpler approach described above.
4. BAYESIAN INFERENCE AND SIMULATION

At this stage it seems appropriate to summarize the aim of the inference, especially which parameters are to be inferred, cf. Section 1. The dynamic behaviour of the contours, in particular the area of $V$, is clearly of diagnostic importance, while that of $\kappa$ may show how well the contours are approximated by the templates, and that of $\theta$ the equipment-dependent circumstances under which the images were taken. In Section 5 we have therefore chosen to concentrate on estimating these parameters by their posterior means and to measure the uncertainty by corresponding credibility intervals. However, in principle the entire posterior distribution is of interest, and below we describe how simulations can be performed so that Monte Carlo methods apply. The discussion falls naturally in two parts: posterior analysis for a single ultrasound image and for the time series of ultrasound images.

4.1 Analysing a single ultrasound image

For each image we have a hierarchical model given the edges $e_0$ of the template and the hyperparameters $\alpha_1, \alpha_2, \beta_1, \beta_2, \delta, \epsilon, \omega$. At the first level, the conditional distribution of $y$ given $(x, v_1, \kappa, \theta)$ depends only on $(\partial V, \theta)$ as described in Section 2. At the next level the conditional distribution of $(x, v_1, \theta)$ given $\kappa$ is specified and at the third level the distribution of $\kappa$ is specified, where $x/\sqrt{\kappa}, v_1, (\mu_1, \mu_2), \sigma_1, \sigma_2, \kappa$ are mutually independent with marginal distributions as described in Section 3. Thereby, using either the simple or refined prior, we obtain the posterior distribution of $r = (x, v_1, \kappa, \theta)$ given $(y, e_0)$. Its density is

$$
\pi(r|y, e_0)
$$

$\propto I[\partial V \text{ is self-avoiding}]\pi(x|\kappa, \omega)\pi(\kappa|\delta, \epsilon)\pi(v_1)\pi(\mu_1, \mu_2)\pi(\sigma_1|\alpha_1, \beta_1)\pi(\sigma_2|\alpha_2, \beta_2)L(y|V, \theta)

= I[\partial V \text{ is self-avoiding}]\kappa^{-\delta-\epsilon}(\sigma_1^2)^{-\alpha_1-\alpha_2-2\epsilon-1}(\sigma_2^2)^{-\alpha_2-2\delta-1} \times \exp\left(-\sum_j (\|x_{1j}\|^2 + \|x_{2j}\|^2)/(2\kappa) - \epsilon/\kappa - \beta_1/\sigma_1^2 - \beta_2/\sigma_2^2 - SS_1/(2\sigma_1^2) - SS_2/(2\sigma_2^2)\right)

\times \exp\left(-n_1(\bar{y}_1 - \mu_1)^2/(2\sigma_1^2) - n_2(\bar{y}_2 - \mu_2)^2/(2\sigma_2^2) - \omega \sum_j (\phi_j - (n - 2)\pi/n)^2\right)

$\text{with } x \in \mathbb{C}^{n-1}, v_1 \in W, \kappa > 0, 0 < \mu_1 < \mu_2 < 255, \sigma_j^2 > 0, j = 1, 2$. Here it is only $V$ and the sufficient statistics in (2) which depend on $e_0$. The posterior (9) is clearly analytically intractable, but MCMC methods provide a feasible way of performing Bayesian inference (see for example Gilks et al. (1996) for background material on MCMC).

We use a Metropolis–Hastings algorithm with proposals corresponding to updates of either

1. the means $(\mu_1, \mu_2),$
2. the variances $(\sigma_1^2, \sigma_2^2),$
3. the position $v_1$ of the deformed template,
4. the $x$ associated with the edges of the deformed template, or
5. the hyperparameter $\kappa$.

Each of these five types of updates are chosen with equal probability $1/5$. For types 1, 3, and 4, we use Metropolis random walk algorithms with the following kind of proposals: in 1, a uniform distribution defined on a square of side length $a > 0$ centred around the current value of $(\mu_1, \mu_2)$; similarly in 3, where $b > 0$ denotes the side length of the square centred at the current value of $v_1$; and in 4, a multivariate Normal distribution with mean equal to the current $x$ value, independent coordinates (for the
real and imaginary parts), and a common variance \( c > 0 \). The proposals are then accepted or rejected in accordance with the usual Metropolis–Hastings criterion. The user-specified parameters \( a, b, c \) are adjusted in order to obtain reasonable acceptance probabilities, see Section 5.2. Finally, for types 2 and 5, we simply use Gibbs sampling updates corresponding to the following full conditionals obtained from (9): in 2, \( \sigma_i^2 | \cdots \sim \text{IG}(\alpha_i, n_i/2, \beta_i + n_i(j_i - \mu_i)^2/2 + SS_i/2) \) for \( i = 1, 2 \), and independent; in 5, \( \kappa | \cdots \sim \text{IG}(\delta + n - 1, (||x_{11}||^2 + ||x_{22}||^2)/2) \).

The proposals are accepted or rejected in accordance with the usual Metropolis–Hastings criterion based on the Metropolis–Hastings ratio, and the algorithm can be shown to be ergodic with equilibrium distribution \( \pi(r|y, e^0) \). Note that many of the terms in (9) cancel when the Metropolis–Hastings ratio is calculated. The updates in 1 and 2 are particularly simple as \( L(y|V, \theta^*)/L(y|V, \theta) \) depends on \( V \) only through the sufficient statistics in (2) which are all known from the previous update. The updates in 5 are likewise rather simple. The most time consuming steps are 3 and 4 because of the need for calculating a new likelihood term, although this calculation can be restricted to pixels where the suggested update yields a change in the grey-level.

Preliminary runs of the algorithm showed that our ‘parametrization’ leads to better mixing properties than if we use the more immediate parametrization where \( x \) is replaced by \( z \). In the latter case, we are modifying type 4 updates in the algorithm by updating \( z \) instead of \( x \), using a proposal \( z' \sim \mathcal{N}(z, cI) \). Our experiments showed that the proposal \( z' \) is likely to be rejected for modest and large values of \( \kappa \), since \( x' = \sqrt{c}z' \sim \mathcal{N}(z, c^2I) \) is frequently not sufficiently small.

### 4.2 Analysing the time series of ultrasound images

It is convenient to analyse the images of the sequence of ultrasound images \( y(i), i = 1, \ldots, 49 \), in turn using the Metropolis–Hastings algorithm in Section 4.1 with equilibrium distribution \( \pi(r|y, e^0) \) given by (9), where \( r(i) = (x(i), v_1(i), \kappa(i), \theta(i)) \) are the ‘parameters’ and \( e^0(i) \) is the template of edges associated to the \( i \)th image. Under the model assumptions imposed below, we show in (10) that this is equivalent to analysing the posterior distribution based on all images.

Assume that the sequence \( u(i) = (y(i), r(i), e^0(i)), i = 1, \ldots, 49 \), forms a Markov chain, which can be viewed as a directed graphical model (Lauritzen, 1996) with a conditional dependence structure as shown in Figure 3. Note that for each \( i > 1 \), \( (y(i), r(i)) \) is conditionally independent of \( (u(1), \ldots, u(i - 1)) \) given \( e^0(i) \). Further, assume that \( e^0(1) \) is given and define recursively

\[
e^0(i) = \mathbb{E}[e(i - 1)|y(i - 1), e^0(i - 1)], \quad i = 2, \ldots, 49,
\]

i.e. the edges of the template for each image \( i > 1 \) are given by the posterior mean of deformed edges from the previous image. Since \( e^0(2), \ldots, e^0(49) \) is determined by \( y(1), \ldots, y(49), e^0(1) \), it follows from Figure 3 that \( r(1), \ldots, r(49) \) are conditionally independent given \( y(1), \ldots, y(49), e^0(1) \). Finally, assume that for each \( i \geq 1 \), \( y(i)|(r(i), e^0(i)) \) is distributed as \( y(i)|(V(i), \theta(i)) \) (see Section 2) and \( r(i) \) follows the same distribution as \( r = (x, v_1, \kappa, \theta) \) (see Section 3). Then the posterior distribution given the data \( y(1), \ldots, y(49) \) and the first template of edges \( e^0(1) \) has density

\[
\pi(r(1), \ldots, r(49)|y(1), \ldots, y(49), e^0(1)) \propto \prod_{i=1}^{49} \pi(r(i)|y(i), e^0(i)),
\]

that is the product of posterior densities (9) associated to each image \( i = 1, \ldots, 49 \). Below we consider estimates of posterior means like \( \mathbb{E}[g(r(i))|y(i), e^0(i)] \) based on simulated samples from \( \pi(r(1)|y(1), e^0(1)), \ldots, \pi(r(49)|y(49), e^0(49)) \), respectively, using in each case the Metropolis–Hastings algorithm in Section 4.1. Ignoring the fact that in practice we need successively to estimate the
templates as explained below, this is equivalent to simulating samples from the posterior distribution (10) for all images.

One may ask for an extension of the model for \((u(1), \ldots, u(49))\) so that posterior information from the \((i – 1)\)th image is taken more into account when modelling the \(i\)th image. For example, in Figure 3 the hyperparameters \(\alpha_1, \alpha_2, \beta_1, \beta_2, \delta, \epsilon, \omega\) are left unchanged for all images, but as reported in Section 5.1 the effect of incorporating posterior information about, for example, \((\beta_1, \beta_2)\) was minor in our application. In fact, as explained below we do incorporate posterior information when starting our MCMC runs.

We describe now in more detail how simulations from the posterior (10) are performed.

For the first image the template is initially determined manually using a computer mouse as described in the beginning of Section 3.1. As this estimate of the template is observer dependent, we generate first a chain \(r_1^{(0)}(1), \ldots, r_J^{(0)}(1)\) of length \(J > 0\) by the Metropolis–Hastings algorithm with equilibrium distribution \(\pi(r|y(1), e^{0}(1))\) and initial values specified as follows: \(\theta\) is the moment estimate \(\theta_0 = (\mu_0, \mu_0, \sigma_{01}^2, \sigma_{02}^2)\) obtained from the likelihood (1) with \((y, V) = (y(1), V^{(0)}(1))\); \(x = 0\) and \(v_1\) is the first vertex 'clicked in'; \(\kappa\) is set equal to its prior mean. Then we update the template by the Monte Carlo estimate based on \(r_1^{(0)}(1), \ldots, r_J^{(0)}(1)\) using an appropriate burn-in \(L < J\). This estimate is used throughout when generating further chains for the first image.

Next we run through the images \(i = 1, \ldots, 49\), generating a chain of length \(K > L\) for each image \(i\). Let \(r_1(i), \ldots, r_K(i)\) denote the chain for the \(i\)th image with equilibrium distribution \(\pi(r|y(i), e^{0}(i))\). As the initial value of \(x\) we use again \(x = 0\) for all images. If \(i = 1\), as the initial value of \(v_1\) we use the Monte Carlo estimate based on \(r_1^{(0)}(1), \ldots, r_J^{(0)}(1)\), while the initial values of \(\theta\) and \(\kappa\) are specified as before. If \(i > 1\) then we estimate \(e^{0}(i)\) and use as the initial values of \(\theta, v_1, \kappa\) the estimated Monte Carlo posterior means obtained from \(r_L(i – 1), \ldots, r_K(i – 1)\).

In order to obtain more accurate results we simulate \(M\) independent realizations

\[
(r_1^{(u)}(1), \ldots, r_K^{(u)}(1), \ldots, r_1^{(u)}(49), \ldots, r_K^{(u)}(49)), \ u = 1, \ldots, M,
\]

in the same way as we generate \((r_1(1), \ldots, r_K(1), \ldots, r_1(49), \ldots, r_K(49))\). Then, for any statistic \(g(r(i))\) associated to the \(i\)th image, we obtain a Monte Carlo estimate of its posterior mean,

\[
\mathbb{E}[g(r(i))|y(i), e^{0}(i)] \approx \sum_{u=1}^{M} \sum_{t=L}^{K} g(r_t^{(u)}(i))/(M(K – L + 1)). \tag{11}
\]

The values of \(J, K, L, M\) are specified in Section 5.2, and (11) is used in Figures 6–9 when estimating posterior means and percentiles.
5. EMPIRICAL RESULTS OF THE BAYESIAN ANALYSIS

In this section we discuss some empirical results using the simulation-based methods described in Section 4. Unless otherwise stated we use the refined prior of Section 3.2.

5.1 Specification of hyperparameters

For our application we found it appropriate to use the same value of the hyperparameters $\alpha_1, \beta_1, \alpha_2, \beta_2, \epsilon, \delta, \omega$ for all images. For the $\text{IG}(\delta, \epsilon)$ prior of $\kappa$, preliminary runs showed that the posterior (9) is robust against different choices of $\delta \leq 2$ and $\epsilon > 0$ corresponding to a flat prior. Note that $\delta \leq 2$ is equivalent to assuming an infinite variance of $\kappa \sim \text{IG}(\delta, \epsilon)$. We have chosen $\delta = 2$ and $\epsilon = 10$. Similarly, for $\sigma_j^2 \sim \text{IG}(\alpha_j, \beta_j)$, we set $\alpha_j = 2, \beta_j = 1, 2$. Then $\beta_j$ is the prior mean of $\sigma_j^2$, so we set $\beta_j = \sigma_{0j}^2, j = 1, 2$, where $\sigma_{0j}^2$ is the moment estimate based on the first image as described in Section 4.2. Finally, we found $\omega = 1000$ to be an appropriate value for the refined prior. The results presented in the following seemed rather robust for choices of $500 \leq \omega \leq 3000$.

We also investigated the effect of changing the values of the hyperparameters for each image $i = 1, \ldots, 49$. For example, if $i > 1$, each $\sigma_{0j}^2(i)$ may be estimated by the MCMC estimate of the posterior mean $E[\sigma_{0j}^2(i-1) | y(i-1)]$ from the $(i-1)$th image. However, the results were not seriously affected by such a change.

5.2 Output analysis of MCMC runs

The parameters $a, b, c$ in the random walk Metropolis algorithms are chosen to obtain reasonable acceptance rates. This is exemplified in Figure 4 for the special case of $c$ when the chain $r_1^{(0)}(1), \ldots, r_K^{(0)}(1)$ with $K = 50000$ is generated.

Fig. 4. Acceptance rates of updates of template deformations for images 1, 7, ..., 49. The horizontal axis refers to each one hundred of type 4 updates in the Metropolis–Hastings algorithm when the total number of type 1–5 updates is 50 000.
The performance of the Metropolis–Hastings algorithm was monitored with respect to various statistics for the different images. Time series plots (not shown here) indicate fast convergence with respect to $\theta$ and $\kappa$. For these parameters except the mean $(\mu_1, \mu_2)$, estimated autocorrelations (not shown here) are effectively 0, while for the mean they are nearly 0 at lags larger than 20. Time series for the area of the deformed template (not shown here) are heavily correlated and show a much slower convergence, but reliable results are obtained by repeated runs (i.e. when $M \gg 1$). Since the template scales and moves from image to image, a value of $K \gg 1$ is needed. In conclusion, we found it appropriate to use $J = 50000$, $K = 10000$, $L = 1000$, and $M = 20$ (these values and (11) are used in Figures 6–9 when estimating posterior means and percentiles). For example, Figure 5 shows time series for the area of the deformed template when $L = 1000$ and $M = 20$, but the chain length is either $K = 5000$, $10000$, $15000$, or $20000$. The posterior means and 95% credibility intervals shown in Figure 5 are rather stable for all the time series with values of $K \gg 10000$, while the result for $K = 5000$ is clearly different from the other cases.

5.3 Comparison of results for the simple and refined priors

We have compared runs using the simple prior ($\omega = 0$) with those for the refined prior ($\omega = 1000$). For the simple prior, Figure 6 shows every sixth image with the initial template (lines with circles) and posterior means of the deformed templates (lines with crosses) superimposed. The deformed templates show clearly a departure from a circular $n$-gon, and especially where the contour is broken, the deformed templates seem to include areas outside the missing contour. Using the refined prior, much better results are obtained, see Figure 7. Now the deformed templates resembles circular $n$-gons and they do not tend to include areas outside the missing contour.
5.4 Conclusions

We have outlined how a fully Bayesian analysis can be performed for the time sequence of ultrasound images. Owing to the results reported in Section 5.3 we shall base our inference on the refined prior. As discussed in Section 5.2, the results of the Bayesian analysis are rather stable to the specification of the hyperparameters. Figure 8 shows time series of the posterior means and 95% credibility intervals for the area of $V$ and the parameters $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \kappa$. The size of the credibility interval is rather stable in each of the six plots. The posterior distribution of $\kappa$ is effectively the same for all images. The curves for $\mu_1, \mu_2, \sigma_1^2, \sigma_2^2$ show a somewhat parallel behaviour, where $\sigma_2^2$ is at least twice as large as $\sigma_1^2$, and $\mu_2$ is clearly larger than $\mu_1$ also. There is no clear relationship between the plot of the area and the other plots.

Figures 7–9 provide useful posterior information about the dynamic behaviour of the artery and the quality of the results. The area of the coronary artery on images 6–9 seems to be overestimated in Figure 8. Figure 9 shows a plausible reason for this: owing to the drop outs, the estimated posterior means of $\partial V$ for images 6–9 are less circular than those for images 5 and 10, and the estimated posterior means of $\partial V$ for images 6–9 seem to include area belonging to the region outside the contour of the artery. Using a larger value of $\omega$ would partly correct for this, but would also necessitate larger runs, when the random
Contour detection by dynamic deformable templates

Fig. 7. Images 1, 7, ..., 49 with superimposed template (lines with circles) and posterior means of deformed templates (lines with crosses). The refined prior model was used.

The walk scale parameter $c$ has to be adjusted to a smaller value in order to obtain acceptance probabilities of the same value as before. A more realistic model that takes into account broken contours may be needed to fully remedy the problem.

We have not included a critical examination of the performance of the methodology in a situation where the truth is known. For a somewhat simplified situation, this has been done for synthetic data in Segel et al. (1998), showing that the method works well if the contour does not have too sharp edges.

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Fig. 8. Posterior means and 2.5 and 97.5\% percentiles for each image and different statistics. From top left to bottom right: the area of $V$ and the parameters $\sigma_1^2$, $\mu_1$, $\kappa$, $\mu_2$, $\sigma_2^2$.

Fig. 9. Images 5–10 with superimposed template (lines with circles) and posterior means of deformed templates (lines with crosses).

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APPENDIX

We now give the details for the decomposition in (6). As $R(\gamma)$ is circulant (see Davis (1979)),

$$R(\gamma) = \frac{1}{n} F \text{diag}\{d_1(\gamma), \ldots, d_n(\gamma)\} F,$$

(A.1)

where $d_j(\gamma) = 1 - \gamma \cos(2(j - 1)\pi/n)$, $j = 1, 2, \ldots, n$, are the eigenvalues of $R(\gamma)$, $F = \{\exp(2\pi i (j - 1)(k - 1)/n)\}_{j,k=1}^n$ is the discrete Fourier transform matrix, and an overbar denotes complex conjugate.

Further, the conditional distribution of $e^*_l$ is Normal with mean $e_0^* l$ and covariance matrix

$$\frac{\kappa}{\Sigma(\kappa, \gamma)} = \kappa(I - E) R(\gamma)^{-1}(I - E),$$

where $I$ is the identity matrix and $E$ has entries equal to $1/n$. By (A.1), $\Sigma(\kappa, \gamma) \to \kappa R(1)^{-1}$ as $\gamma \to 1$, where $R(1)^{-1}$ is the generalized inverse of $R(1)$ of rank $n - 1$. Finally, we can decompose $R(1)^{-1}$ as in (6) where $\Lambda^{1/2} = \text{diag}(0, 1/d_2^2(1)^{1/2}, \ldots, 1/d_n(1)^{1/2})$.

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