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Dynamics of Coupled Harmonic Oscillators in an Environment using White Noise Analysis

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Abstract. White noise analysis has gained considerable attention in the past years in the analysis of quantum systems, polymer entanglement, transport, and diffusion phenomena. However, there is dearth on the application of the analysis into open quantum systems specifically coupled harmonic oscillators in an environment. In this work, white noise analysis was used to study the dynamics of the propagator of an open quantum system consisting of coupled harmonic oscillators that are coupled to an environment consisting of N-2 multimode harmonic oscillators. The quantum propagators were obtained after solving for the normal modes of the system-environment interaction to decouple the coordinates in the Lagrangian describing the dynamics of the system, the environment and their interaction with each other. The decoupled Lagrangian was used in the path integral corresponding to the propagator of the system, with the path integral evaluated using white noise analysis. The resulting propagator was found to consist of the product of N simple harmonic oscillator propagators. N-3 of these propagators correspond to the degenerate normal mode frequencies of the system-environment interaction, while the other three correspond to the non-degenerate normal mode frequencies, two of these nondegeneracies correspond to complex normal modes. This approach of describing the dynamics of the open quantum system greatly simplifies the task mathematically.

INTRODUCTION

In the study of an open quantum system, the Caldeira-Leggett (CL) model is one of the fundamental models that has been used in the analysis of system-environment interactions. The CL model describes a quantum system with an arbitrary potential interacting with an environment modeled as an infinite number of harmonic oscillators [1, 2]. The system-bath interaction describes dissipation phenomena in solid state physics, quantum tunneling and quantum computing [3, 4]. Among others, the influence-functional method of Feynman and Vernon [5] was used to solve the propagator. Through the propagator, the description of the dissipative dynamics of the system was made. The CL model study can be extended to the analysis of a system composed of coupled harmonic oscillators in an environment modeled as an ensemble of harmonic oscillators. In the literature, this was called as double Caldeira-Leggett model and is important in the analysis of macroscopic quantum phenomena such as decoherence since any quantum system, and the environment with which it is interacting, can be decomposed into several components which are modeled as harmonic oscillators. The influence-functional method of Feynman and Vernon was utilized on its analysis, among others.

A review in the literature also reveals that the white noise analysis has not been applied to solve this model. Hence, this study aims to extend the application of white noise analysis to determine the dynamics of the propagator of the coupled harmonic oscillators in an environment. This extension is vital for understanding macroscopic quantum phenomena like decoherence.

In this paper, a system composed of coupled harmonic oscillators interacting with an environment composed of an ensemble of harmonic oscillators was considered. A method different from those considered in Refs. [1, 6, 7, 8] to derive the quantum propagator of the system was used. The method makes use of white noise analysis invented by Hida [9] and Streit [10]. As compared to the influence functional method which is considered to be mathematically ill-defined due to the presence of the Lebesgue measure, white noise analysis is a mathematically well-defined method, and is a powerful tool in evaluating the Feynman path integral [11]. It is the motivation here to show the promise of white noise analysis in evaluating propagators for open quantum systems.

This paper is organized as follows. The system and the bath considered in this study, together with their corresponding Hamiltonians, the discussion of some basics on white noise analysis, and the recasting of Feynman path integral in the context of white noise analysis following Ref. [11] are presented in the succeeding three sections. The last section presents the application of white noise analysis to derive the quantum propagator, before the short summary and the conclusion are presented.

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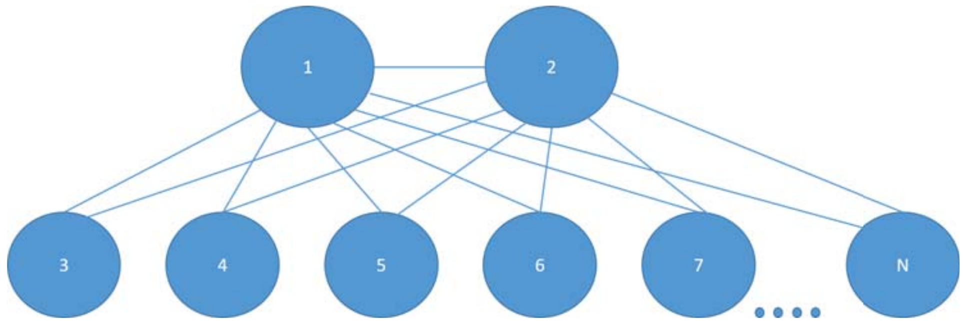

FIGURE 1. Schematic diagram of the coupled harmonic oscillators (system 1 and 2) in an environment of $N-2$ harmonic oscillators, with the coupling of the system to each individual oscillator in the environment represented in the diagram by solid blue lines.

COUPLED HARMONIC OSCILLATORS IN AN ENVIRONMENT

The Hamiltonians of the system H_S, which is coupled harmonic oscillators, the environment H_B, consisting of $N-2$ harmonic oscillators (see Fig. (1)); and the interaction between the system and the environment H_{SB}, are defined respectively as follows

$$H_S = \sum_{n=1}^{2}\left(\frac{p_i^2}{2m} + \frac{1}{2}m\omega_i^2 x_i^2\right) + Cx_1x_2, \tag{1}$$

$$H_B = \sum_{n=3}^{N}\left(\frac{p_{x_n}^2}{2m} + \frac{1}{2}m\omega_n^2 x_n^2\right), \tag{2}$$

$$H_{SB} = C(x_1+x_2)\sum_{n=3}^{N} x_n, \tag{3}$$

where $x's$, $p's$, and $\omega's$, are the corresponding positions, momenta, and frequencies of the system and bath oscillators, while C is the positive coupling constant of the system-environment interaction. In here, the environment coordinate is linearly coupled to the system, with the coupling described in terms of the Hamiltonian H_{SB}, given above. Thus, the total Hamiltonian can be written as

$$H = H_S + H_B + H_{SB}. \tag{4}$$

WHITE NOISE ANALYSIS FUNDAMENTALS

The Langevin equation for a stochastic process like Brownian motion is given by

$$\dot{X} = a(t,X) + b(t,X)\omega(t), \tag{5}$$

where X represents the stochastic variable, $a(t,X)$ and $[b(t,X)]^2$ are the drift and diffusion coefficients, respectively, $\omega(t) = \frac{dB(t)}{dt}$ is interpreted as the velocity of Brownian motion, and is called the Gaussian white noise with $B(t)$ as the Wiener process, $B(t) = \int_{t_o}^{t}\omega(\tau)d\tau = \left\langle \omega, 1_{[t_o,t)}\right\rangle$, where the notation for a contraction to a test function ξ can be written as $\langle\omega,\xi\rangle \equiv \int_{t_o}^{t}\omega(\tau)\xi(\tau)d\tau$.

The Hida and Streit's formulation treats the set $\omega(\tau)$ at different instants of time, $\{\omega(\tau); t\in\Re\}$ as a continuum coordinate system and operates in the Gelfand triple [10] $S\subset L^2\subset S^*$, linking the spaces of a Hida distribution S^* and test function S through a Hilbert space of square integrable functions L^2. Using Minlos' theorem a Hida white noise space (S^*,B,μ) where μ is the probability measure, B is the σ-algebra generated on S, and a characteristic functional

$$C(\xi) = \int_{S^*}\exp\left[i\langle\omega,\xi\rangle\right]d\mu(\omega) = \exp\left(-\frac{1}{2}\int\xi^2 d\tau\right), \tag{6}$$

are introduced with $\xi \in S$ and the white noise Gaussian measure $d\mu(\omega)$

$$d\mu(\omega) = N_\omega \exp\left(-\frac{1}{2}\int \omega^2(\tau)d\tau\right) d^\infty\omega, \tag{7}$$

with N_ω as a normalization constant.

The evaluation of the Feynman integral in the context of white noise analysis is carried out by the evaluation of the Gaussian white noise measure $d\mu(\omega)$. This can be facilitated through the T-transform of a generalized white noise functional $\Phi(\omega)$ given by

$$T\Phi(\xi) = \int_{S^*} \exp\left(i\langle\omega,\xi\rangle\right)\Phi(\omega)d\mu(\omega), \tag{8}$$

and the S-transform

$$S\Phi(\xi) = C(\xi)T\Phi(-i\xi), \tag{9}$$

where $C(\xi)$ is the characteristic functional.

FEYNMAN PATH INTEGRAL IN THE CONTEXT OF WHITE NOISE ANALYSIS

The propagator for the quantum mechanical oscillator, say in x-dimension, has the form as presented by Feynman [12]

$$K(x,x_o;\tau) = \int \exp\left(\frac{i}{\hbar}S\right) D[x], \tag{10}$$

where $S = \int L\ dt$ is the classical action with $L = \frac{1}{2}m\dot{x}^2 - V(x)$ as the Lagrangian of the system, and $D[x]$ is the infinite-dimensional Lebesgue measure. Now, the parametrization of the path is introduced [10, 11]

$$x(t) = x_o + \sqrt{\frac{\hbar}{m}}\int_0^t \omega(\tau)d\tau, \tag{11}$$

which shows how the value $x(t)$ is affected by the white noise variable as τ ranges from 0 to t. Taking the derivative of Eq. (11), substituting it into the exponential expression of Eq. (10) yields

$$\exp\left(\frac{i}{\hbar}S\right) = \exp\left[\frac{i}{2}\int_0^t \omega(\tau)^2 d\tau\right]\exp\left[-\frac{i}{\hbar}\int_0^\tau V(x)d\tau\right]. \tag{12}$$

The evaluation of the Lebesgue measure $D[x]$ leads to an integration over the Gaussian white noise measure $d\mu(\omega)$ in the relation

$$D[x] = \lim_{N\to\infty}\prod^{N}(A_j)\prod^{N-1}(dx_j) = Nd^\infty x, \tag{13}$$

where

$$Nd^\infty x \to Nd^\infty\omega = \exp\left[\frac{1}{2}\int_0^t \omega(\tau)^2 d\tau\right] d\mu(\omega), \tag{14}$$

with N as the normalization constant. To fix the endpoint of the path parametrization in Eq. (11) the Donsker delta function,

$$\delta(x(t)-x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(i\lambda(x(t)-x)\right)d\lambda, \tag{15}$$

is multiplied such that at time t the particle is located at x. Thus, with Eqs. (12), (14) and (15) the Feynman propagator in the context of white noise analysis can be written as

$$K(x,x_o;\tau) = N\int \exp\left[\frac{i+1}{2}\int_0^t \omega(\tau)^2 d\tau\right]\exp\left[-\frac{i}{\hbar}\int_0^\tau V(x)d\tau\right]\delta(x(t)-x)d\mu(\omega). \tag{16}$$

RESULTS AND DISCUSSION

The presentation of the results is as follows. First, the eigenvalues which correspond to the normal modes of the entire system-environment interaction were solved. By determining the amplitudes of the eigenvectors, one can effectively decouple the system and environment coordinates. After the system and environment coordinates have been decoupled, the Lagrangian was rewritten and the path integral corresponding to the propagator for the system was evaluated using the white noise analysis.

Normal mode analysis

Consider again the system-environment interaction in Fig. (1) and by the method of small oscillations the coordinates can be defined as

$$\eta_i = x_i - x_{oi}, \tag{17}$$

where η_i is the small oscillation displacement and x_{oi} is the distance between the system and i^{th} harmonic oscillator environment. Then the potential and kinetic energy of the system and environment are given by

$$\begin{aligned} V = &\frac{1}{2}C(\eta_1-\eta_2)^2+\frac{1}{2}C(\eta_1-\eta_3)^2+\frac{1}{2}C(\eta_1-\eta_4)^2+\cdots+\frac{1}{2}C(\eta_1-\eta_N)^2 \\ &+\frac{1}{2}C(\eta_2-\eta_3)^2+\frac{1}{2}C(\eta_2-\eta_4)^2+\frac{1}{2}C(\eta_2-\eta_5)^2+\cdots+\frac{1}{2}C(\eta_2-\eta_N)^2, \end{aligned} \tag{18}$$

and

$$T = \frac{1}{2}m\dot{\eta_1}^2+\frac{1}{2}m\dot{\eta_2}^2+\frac{1}{2}m\dot{\eta_3}^2+\cdots+\frac{1}{2}m\dot{\eta_N}^2. \tag{19}$$

In the expression for the potential energy, the coordinates η_i are coupled with each other. In order to decouple them, the normal mode frequencies of the system were computed. The normal mode frequencies are the eigenvalues of the characteristic equation defined as $\left|V-\omega^2T\right| = |V-\lambda I| = 0$ or

$$0 = \begin{bmatrix} V_{11}-\lambda & V_{12} & \cdots V_{1N} \\ V_{21} & V_{22}-\lambda & \cdots V_{2N} \\ \vdots & \vdots & \vdots \\ V_{N1} & V_{N2} & \cdots V_{NN}-\lambda \end{bmatrix},$$

where I is an $N\times N$ identity matrix, $\lambda = \omega^2 m$ and

$$V_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}, \qquad T_{ij} = m\frac{\partial \vec{r}_k}{\partial q_{oi}}\frac{\partial \vec{r}_k}{\partial q_{oi}}, \tag{20}$$

with q's as the generalized coordinates. Inserting Eq. (20) into the characteristic equation yields,

$$0 = \begin{bmatrix} (N-1)C-\lambda & C & \cdots C \\ -C & (N-1)C-\lambda & \cdots 0 \\ \vdots & \vdots & \vdots \\ -C & 0 & \cdots -2C-\lambda \end{bmatrix},$$

where N has integer values greater than or equal to three (3). Solving for the eigenvalues of this matrix, one can find that for $N \geq 5$, there are $N-3$ degenerate eigenvalues, all of which are equal to $-2C$, and three nondegenerate (two complex and one real) eigenvalues.

By solving for the normal modes, one finds that not only he/she decouples the coordinates η_i, but also manage to decompose the dynamics of the system into two components. The first component of the dynamics, described by the $N-3$ degenerate eigenvalues with value $-2C$, correspond to only two oscillators in the environment moving in sync with each other, with the rest of the environment, as well as the coupled system, remaining at rest. The second component of the dynamics, described by the two nondegenerate eigenvalues, correspond to both the system and the environment moving in sync with each other. This result holds for any number $N \geq 5$ of oscillators in the system-environment interaction.

Evaluation of the Path Integral for $N = 5$ Using White Noise Analysis

To illustrate the concepts presented in the previous subsection, the dynamics of the coupled harmonic oscillator interacting with an environment composed of $N = 5$ system-bath harmonic oscillators was analyzed. The eigenvectors corresponding to the normal modes for $N = 5$ are given by

$$\vec{A} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \tag{21}$$

where $\alpha's$ are real numbers and $\beta's$ are complex numbers. Notice in general that one can deduce what particular elements are oscillating through the equation, $\eta_i = a_{ij}\exp(i\omega_i t)$, where a_{ij} corresponds to the amplitude of a certain element on the entire system-environment interaction and ω_i as the normal mode frequency.

Propagator dynamics described by the first degenerate normal mode frequency

Consider the first degenerate eigenvalue $-2C$ whose eigenvector is given by the fourth column of Eq. (21). From the eigenvector, $x_1 = x_2 = x_4 = 0$ and $x_3 = -x_5$, hence the total Hamiltonian in Eq. (4) becomes,

$$H = \frac{p_3^2}{2m} + \frac{1}{2}m\omega^2 x_3^2 + \frac{p_5^2}{2m} + \frac{1}{2}m\omega^2 x_5^2, \tag{22}$$

where $\omega_3 = \omega_5 = \omega = \sqrt{\frac{-2C}{m}}$. Note that the coordinate notation was changed from η or q to x for simplicity, since they represent position coordinates. Clearly, it is evident that Eq. (22) is separable into propagators for two independent harmonic oscillators written as $K(x_3, x_5; x_{30}, x_{50}; \tau) = K_{x_3 x_5} = K(x_3, x_{30}; \tau)K(x_5, x_{50}; \tau)$ where $K(x_3, x_{30}; \tau) = K_{x_3} = \int \exp\left[\frac{i}{\hbar}S_3\right] D[x_3]$ and $K(x_5, x_{5o}; \tau) = K_{x_5} = \int \exp\left[\frac{i}{\hbar}S_5\right] D[x_5]$. Now, following the same procedure in subsection 4.1 of Ref. [13] to obtain the explicit form of the propagator for the first degenerate normal mode frequency $K_{x_3 x_5} = K(x_3, x_5; x_{30} = 0, x_{50} = 0; t, 0)$ as

$$K(x_3, x_5; 0, 0; t, 0) = \left[\frac{m\omega}{2\pi i\hbar t \sin\omega t}\right] \exp\left[\frac{im\omega}{2\hbar}(x_3^2 + x_5^2)\cot\omega t\right]. \tag{23}$$

Propagator dynamics described by the second degenerate normal mode frequency

Observe that the eigenvalue is similar to the previous subsection and that $x_1 = x_2 = x_5 = 0$ and $x_3 = -x_4$, hence the total Hamiltonian in Eq. (4) becomes,

$$H = \frac{p_3^2}{2m} + \frac{1}{2}m\omega^2 x_3^2 + \frac{p_4^2}{2m} + \frac{1}{2}m\omega^2 x_4^2, \tag{24}$$

where $\omega_3 = \omega_4 = \omega = \sqrt{\frac{-2C}{m}}$. Following the same procedure in the previous subsection, the full propagator yielded

$$K(x_3, x_4; 0, 0; t, 0) = \left[\frac{m\omega}{2\pi i\hbar t \sin\omega t}\right] \exp\left[\frac{im\omega}{2\hbar}(x_3^2 + x_4^2)\cot\omega t\right]. \tag{25}$$

Propagator dynamics described by the first nondegenerate normal mode frequency

The third eigenvalue (real) describes the case where all elements in the system and environment are oscillating. Referring to the third column of the eigenvector above, it is observed that $x_3 = x_4 = x_5$ oscillate at the same amplitude while x_1 and x_2 are different. So the Hamiltonian becomes,

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_\alpha^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_\alpha^2 x_2^2 + 3\left[\frac{p_3^2}{2m} + \frac{1}{2}m\omega_\alpha^2 x_3^2\right] + Cx_1x_2 + \kappa(x_1+x_2)x_3, \tag{26}$$

where $m_1 = m_2 = m$, $m_3 = 3m$ and $\kappa = 3C$. Eq. (26) has the resemblance that of a tricoupled harmonic oscillator. Following the works of Ref. ([13]), the propagator can be expressed as

$$K_{x_1x_2x_3} = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}} \left[\frac{\Omega_{1\alpha}\Phi_\alpha\Phi_{2\alpha}}{\sin\Omega_{1\alpha}t\sin\Phi_\alpha t\sin\Phi_{2\alpha}t}\right]^{\frac{1}{2}} \exp\left[\frac{im\Omega_{1\alpha}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\alpha}t\right]$$
$$\times \exp\left[\frac{im\Phi_{2\alpha}}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3\right]^2\cot\Phi_{2\alpha}t\right]\exp\left[\frac{im\Phi_\alpha}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3\right]^2\cot\Phi_\alpha t\right], \tag{27}$$

with the frequencies are given by $\Omega_{1\alpha} = \sqrt{\omega_\alpha^2 + \frac{C}{m}}$, $\Phi_\alpha = \sqrt{\omega_\alpha^2 - \frac{C}{m} - \frac{3\sqrt{2}C}{m}}$, and $\Phi_{2\alpha} = \sqrt{\omega_\alpha^2 - \frac{C}{m} + \frac{3\sqrt{2}C}{m}}$.

Propagator dynamics described by the second nondegenerate normal mode frequency

Observe that the second nondegenerate (complex β) eigenvalue corresponds to the movements where all elements in the system and environment are moving at the same frequency. The complex part of the eigenvalue corresponds to the dissipation of the energy from the environment into the coupled system. Similar to the first nondegenerate case, the total Hamiltonian in Eq. (4) becomes

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_\beta^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_\beta^2 x_2^2 + 3\left[\frac{p_3^2}{2m} + \frac{1}{2}m\omega_\beta^2 x_3^2\right] + Cx_1x_2 + \kappa(x_1+x_2)x_3, \tag{28}$$

where $\omega_3 = \omega_4 = \omega_5 = \omega_\beta$. Using the same result in the previous subsection, the propagator is obtained as

$$K_{x_1x_2x_3} = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}} \left[\frac{\Omega_{1\beta}\Phi_\beta\Phi_{2\beta}}{\sin\Omega_{1\beta}t\sin\Phi_\beta t\sin\Phi_{2\beta}t}\right]^{\frac{1}{2}} \exp\left[\frac{im\Omega_{1\beta}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta}t\right]$$
$$\times \exp\left[\frac{im\Phi_{2\beta}}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3\right]^2\cot\Phi_{2\beta}t\right]\exp\left[\frac{im\Phi_\beta}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3\right]^2\cot\Phi_\beta t\right], \tag{29}$$

with the frequencies are given by $\Omega_{1\beta} = \sqrt{\omega_\beta^2 + \frac{C}{m}}$, $\Phi_\beta = \sqrt{\omega_\beta^2 - \frac{C}{m} - \frac{3\sqrt{2}C}{m}}$, and $\Phi_{2\beta} = \sqrt{\omega_\beta^2 - \frac{C}{m} + \frac{3\sqrt{2}C}{m}}$.

Propagator dynamics described by the third nondegenerate normal mode frequency

Finally, notice also that the third nondegenerate (β conjugate) eigenvalue also corresponds to the movements where all elements in the system and environment are moving at the same frequency, hence the total Hamiltonian (Eq. (4)) becomes

$$H = \frac{p_1^2}{2m} + \frac{1}{2}m\omega_{\beta^*}^2 x_1^2 + \frac{p_2^2}{2m} + \frac{1}{2}m\omega_{\beta^*}^2 x_2^2 + 3\left[\frac{p_3^2}{2m} + \frac{1}{2}m\omega_{\beta^*}^2 x_3^2\right] + Cx_1x_2 + \kappa(x_1+x_2)x_3, \tag{30}$$

where $\omega_3 = \omega_4 = \omega_5 = \omega_{\beta^*}$. The propagator can be evaluated as

$$K_{x_1x_2x_3} = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{3}{2}} \left[\frac{\Omega_{1\beta^*}\Phi_{\beta^*}\Phi_{2\beta^*}}{\sin\Omega_{1\beta^*}t\sin\Phi_{\beta^*}t\sin\Phi_{2\beta^*}t}\right]^{\frac{1}{2}} \exp\left[\frac{im\Omega_{1\beta^*}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta^*}t\right]$$
$$\times \exp\left[\frac{im\Phi_{2\beta^*}}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3\right]^2\cot\Phi_{2\beta^*}t\right]\exp\left[\frac{im\Phi_{\beta^*}}{4\hbar}\left[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3\right]^2\cot\Phi_{\beta^*}t\right], \tag{31}$$

where the frequencies are given by $\Omega_{1\beta^*} = \sqrt{\omega_{\beta^*}^2 + \frac{C}{m}}$, $\Phi_{\beta^*} = \sqrt{\omega_{\beta^*}^2 - \frac{C}{m} - \frac{3\sqrt{2}C}{m}}$, and $\Phi_{2\beta^*} = \sqrt{\omega_{\beta^*}^2 - \frac{C}{m} + \frac{3\sqrt{2}C}{m}}$.

The full propagator for the dynamics of the system

Finally, the entire system-environment propagator for $N=5$ can be obtained by taking the product of all propagators in Eqs. (23), (25), (27), (29) and (31). The resulting propagator, $K_F = K(x_1,x_2,x_3,x_4,x_5;0,0,0,0,0;t,0)$, has the form

$$K_F = \left(\frac{m}{2\pi i\hbar t}\right)^{\frac{13}{2}} \left(\frac{\omega}{\sin\omega t}\right)^2 \left[\frac{\Omega_{1\alpha}\Phi_\alpha\Phi_{2\alpha}\Omega_{1\beta}\Phi_\beta\Phi_{2\beta}\Omega_{1\beta^*}\Phi_{\beta^*}\Phi_{2\beta^*}}{\sin\Omega_{1\alpha}t\sin\Phi_\alpha t\sin\Phi_{2\alpha}t\sin\Omega_{1\beta}t\sin\Phi_\beta t\sin\Phi_{2\beta}t\sin\Omega_{1\beta^*}t\sin\Phi_{\beta^*}t\sin\Phi_{2\beta^*}t}\right]^{\frac{1}{2}}$$
$$\times \exp\left[\frac{im\Omega_{1\alpha}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\alpha}t\right]\exp\left[\frac{im\Phi_{2\alpha}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\alpha}t\right]$$
$$\times \exp\left[\frac{im\Phi_{\alpha}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\alpha}t\right]\exp\left[\frac{im\Omega_{1\beta}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta}t\right]$$
$$\times \exp\left[\frac{im\Phi_{2\beta}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\beta}t\right]\exp\left[\frac{im\Phi_{\beta}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\beta}t\right]$$
$$\times \exp\left[\frac{im\Omega_{1\beta^*}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta^*}t\right]\exp\left[\frac{im\Phi_{2\beta^*}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\beta^*}t\right]$$
$$\times \exp\left[\frac{im\Phi_{\beta^*}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\beta^*}t\right]\exp\left[\frac{im\omega}{2\hbar}\left([2x_3^2+x_4^2+x_5^2]\cot\omega t\right)\right]. \quad (32)$$

GENERALIZATION TO THE CASE WHERE THERE ARE $N-2$ OSCILLATORS IN THE ENVIRONMENT

The generalization to the case where there are $N-2$ oscillators in the environment can be done by following the same procedures in the previous section. Noting that there will always be three nondegenerate and $N-3$ degenerate normal mode frequencies, one can show that for $N \geq 5$ the full propagator becomes

$$K_F = \left[\frac{m}{2\pi i\hbar t}\right]^{[(N-3)+\frac{2}{3}(N-2)]} \left[\frac{\omega}{\sin\omega t}\right]^{N-3}$$
$$\times \left[\frac{\Omega_{1\alpha}\Phi_\alpha\Phi_{2\alpha}\Omega_{1\beta}\Phi_\beta\Phi_{2\beta}\Omega_{1\beta^*}\Phi_{\beta^*}\Phi_{2\beta^*}}{\sin\Omega_{1\alpha}t\sin\Phi_\alpha t\sin\Phi_{2\alpha}t\sin\Omega_{1\beta}t\sin\Phi_\beta t\sin\Phi_{2\beta}t\sin\Omega_{1\beta^*}t\sin\Phi_{\beta^*}t\sin\Phi_{2\beta^*}t}\right]^{\frac{1}{2}}$$
$$\times \exp\left[\frac{im\Omega_{1\alpha}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\alpha}t\right]\exp\left[\frac{im\Phi_{2\alpha}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\alpha}t\right]$$
$$\times \exp\left[\frac{im\Phi_{\alpha}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\alpha}t\right]\exp\left[\frac{im\Omega_{1\beta}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta}t\right]$$
$$\times \exp\left[\frac{im\Phi_{2\beta}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\beta}t\right]\exp\left[\frac{im\Phi_{\beta}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\beta}t\right]$$
$$\times \exp\left[\frac{im\Omega_{1\beta^*}}{4\hbar}(x_1-x_2)^2\cot\Omega_{1\beta^*}t\right]\exp\left[\frac{im\Phi_{2\beta^*}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)-x_3]^2\cot\Phi_{2\beta^*}t\right]$$
$$\times \exp\left[\frac{im\Phi_{\beta^*}}{4\hbar}[\frac{\sqrt{2}}{2}(x_1+x_2)+x_3]^2\cot\Phi_{\beta^*}t\right]\exp\left[\frac{im\omega}{2\hbar}\left(\left[(N-3)x_3^2+\sum_{j=4}^{N}x_j^2\right]\cot\omega t\right)\right], \quad (33)$$

where $\Omega_{1\alpha}=\sqrt{\omega_\alpha^2+\frac{C}{m}}$, $\Phi_\alpha=\sqrt{\omega_\alpha^2-\frac{C}{m}-\frac{(N-2)\sqrt{2}C}{m}}$, $\Phi_{2\alpha}=\sqrt{\omega_\alpha^2-\frac{C}{m}+\frac{(N-2)\sqrt{2}C}{m}}$, $\Omega_{1\beta}=\sqrt{\omega_\beta^2+\frac{C}{m}}$, $\Phi_\beta=\sqrt{\omega_\beta^2-\frac{C}{m}-\frac{(N-2)\sqrt{2}C}{m}}$, $\Phi_{2\beta}=\sqrt{\omega_\beta^2-\frac{C}{m}+\frac{(N-2)\sqrt{2}C}{m}}$, $\Omega_{1\beta^*}=\sqrt{\omega_{\beta^*}^2+\frac{C}{m}}$, $\Phi_{\beta^*}=\sqrt{\omega_{\beta^*}^2-\frac{C}{m}-\frac{(N-2)\sqrt{2}C}{m}}$, and $\Phi_{2\beta^*}=$

$\sqrt{\omega_{\beta^*}^2 - \frac{C}{m} + \frac{(N-2)\sqrt{2C}}{m}}$. In Eq. (33) the the first nine exponential expressions correspond to the nondegenerate normal mode frequencies, while the last exponential expression corresponds to the degenerate normal mode frequencies.

CONCLUSION

In this work, the dynamics of the quantum propagator of an open quantum system consisting of coupled harmonic oscillators in an environment was successfully presented. This was done through computing the normal mode frequencies of the system, obtaining three non-degenerate and $N-3$ degenerate normal modes for $N \geq 5$ oscillators in the environment in order to decouple the coordinates of the system and the environment. The dynamics of the degenerate eigenvalues always reduce into a product of $N-3$ individual propagators, while the nondegenerate case reduces into a tricoupled harmonic oscillators.

This work demonstrates that the white noise analysis posits promise in evaluating the propagators for open quantum systems, due to its mathematical rigor and ease of use.

The method can be applied to systems with N coupled oscillators which are all coupled to an environment. This system is useful in modelling quantum transport of energy excitation in solid state and biological systems. The author will explore these areas in further detail in future work.

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