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# Quantum Particle Model of Free Particle Solution in One-Dimensional Klein-Gordon Equation

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**Abstract.** We have discussed the quantum particle model for the case of a free particle solution in the one-dimensional Klein-Gordon nonlinear equations. This model was obtained through the two equations from the conservation laws of the classical physics, namely, the Hamilton-Jacobi equation for the relativistic motion and continuity equations. In this case, the Hamilton-Jacobi equation describes a part of the particle while the continuity equation describes the wave side. The derivation of this equation did not use the two postulates in the quantum mechanics, namely, the Einstein and de Broglie's postulates regarding the quantization of energy and momentum. According to this derivation, the particle side has almost most of the energy of quantum particle that accumulate at a point while the wave part has only a small portion of the energy of the quantum particle that surrounds the part of the particle. In addition, this paper also shows the form of mathematical functions that represent the particle and wave parts for a free particle solution. This form is obtained through a special solution of the free particle which is a plane-wave solution.



## INTRODUCTION

Quantum mechanics is a theory that describes the equations of microscopic particle motion, most of which are examined through the Schrödinger equation. The Schrödinger's equation itself is formulated through the two fundamental postulates, namely, the Einstein and de Broglie's postulates, regarding the quantization of energy and momentum. Then, the relationship between particle and wave that can appear in the microscopic area can be proven by the experiments, such as a Stern-Gerlach experiment, Young interference, and photoelectric phenomenon.

At the same time, the special relativity theory proposed by Einstein has also been verified by experiment. So, physicists attempted to unify the quantum mechanics and the special relativity. The first equation obtained from the unification is the Klein-Gordon equation, which represents the dynamics of a spinless particle. This equation predicts two different energies, a positive energy and a negative energy, which can be occupied by a particle. A negative energy, in this case, does not give the physical interpretation. Later, the Klein-Gordon equation was extended to the so-called the Dirac equation, which give a reliable interpretation of the negative energy. In this case, the negative energy can be interpreted as the anti-particle having a different charge.

Although the quantum mechanics can give a satisfying explanation for several experiments, it was believed that the fundamental of the quantum mechanics should be formed in the nonlinear equation. Long time ago, some physicists tried to extend the Schrodinger equation in the nonlinear form [1-4]. They proposed a nonlinear Schrodinger by combining the Hamilton-Jacobi equation and the continuity equation, a good review can be found in Ref. [5]. In this case, they did not consider the Einstein and de Broglie's postulates. They introduced a quantum particle, which contains two terms, a particle term represented by the Hamilton-Jacobi equation and a wave term represented by the continuity equation. In addition, the argued that the energy of a particle term has a large portion of energy of quantum particle than that of the wave term.

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Our purpose is to discuss a mathematical model solution of the one-dimensional nonlinear Klein-Gordon equation for a free particle based on the above approach. We choose the nonlinear Klein-Gordon equation because it has an interesting property, such as the solitary wave solution [6-8]. In addition, the mathematical and numerical aspects of the nonlinear Klein-Gordon had also become a lot interest in physics [9-12]. In this paper, we give an analytical solution by allowing the linear superposition of the wave functions for a particle term and a wave term simultaneously.

## METHOD

In this section, we will review the Klein-Gordon nonlinear equation to see the particle and wave sides. Let's consider the three-dimensional nonlinear Klein-Gordon equation which has a form [13]

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}-\nabla^2\psi+\frac{m^2c^2}{\hbar^2}\psi-\frac{1}{c^2 a}\frac{\partial^2 a}{\partial t^2}\psi+\frac{\nabla^2 a}{a}\psi=0, \tag{1}$$

where $m$ is the rest mass, $c$ is the velocity of light, and $(\nabla^2 a)/a$ is called the quantum potential. From Eq. (1), it is clear that the nonlinear term comes from the quantum potential. Here, the wave function of a particle is given by

$$\psi(\vec{r},t)=a(\vec{r},t)\,e^{\frac{i}{\hbar}\varphi(\vec{r},t)}, \tag{2}$$

where $a$ and $\varphi$ are the real amplitude and phase, respectively.

If the solution in Eq. (2) is substituted into Eq. (1), we obtain the two classical equations of motion of relativistic particles, namely, the Hamilton-Jacobi equation

$$\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2-(\nabla\varphi)^2-m^2c^2=0, \tag{3}$$

and the continuity equation

$$\nabla\cdot\left(a^2\nabla\varphi\right)=\frac{1}{c^2}\frac{\partial}{\partial t}\left(a^2\frac{\partial\varphi}{\partial t}\right). \tag{4}$$

For the complete derivation of the nonlinear Klein-Gordon equation using the Hamilton-Jacobi and continuity equation can be found in Ref. [13].

To find the solution of the nonlinear Klein-Gordon, we solve those two equations in Eqs. (3) and (4). For the convenience, we first solve the Hamilton-Jacobi equation in Eq. (3) by writing the ansatz solution of $\varphi(\vec{r},t)$

$$\varphi(\vec{r},t)=\varphi_1(t)+\varphi_2(x)+\varphi_3(y)+\varphi_4(z). \tag{5}$$

Substituting Eq. (5) into Eq. (3), we obtain the solution of the phase

$$\varphi(\vec{r},t)=-Et+\vec{p}\cdot\vec{r}, \tag{6}$$

where $E$ and $\vec{p}$ are the total energy and relativistic momentum of a free particle. Next, the solution of the amplitude $a(\vec{r},t)$ can be found by solving the continuity equation in Eq. (4). Note that we are also allowed to set $a(\vec{r},t)=A$ (constant), which is a plane-wave solution in the ordinary Klein-Gordon equation. However, since we want to describe the particle and wave sides at the same time, we expect the general solution for $a(\vec{r},t)\neq$ constant.

## RESULT AND DISCUSSION

To obtain the general solution, we first write the general solution of the amplitude from Eq. (4)

$$a(\vec{r},t)=a\left[\left(mc^2\right)\vec{p}\cdot\vec{r}-\left(E^2-m^2c^4\right)t\right]. \tag{7}$$

Thus, the general solution of the wavefunction can be written as

$$\psi(\vec{r},t)=a\left[\left(mc^2\right)\vec{p}\cdot\vec{r}-\left(E^2-m^2c^4\right)t\right]e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)}. \tag{8}$$

Since the wave function should describe a particle term at one side and a wave at another side, we write the linear combination of two functions

$$\psi(\vec{r},t)=\xi(\vec{r},t)+\theta(\vec{r},t), \tag{9}$$

where $\xi(\vec{r},t)$ and $\theta(\vec{r},t)$ are the functions describing a particle term and a wave, respectively.

The next step is to determine the appropriate mathematical function, where the particle part has a large portion of the energy of the quantum particle, whereas the wave part has only a small portion of the energy of the quantum particle. In this case, we use the Gaussian function for the two functions above

$$\xi(\vec{r},t)=A\exp\left(-\frac{\left[\left(mc^2\right)\vec{p}\cdot\vec{r}-\left(E^2-m^2c^4\right)t\right]^2}{2\sigma_\xi\hbar^2}\right)e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)}, \tag{10}$$

$$\theta(\vec{r},t)=B\exp\left(-\frac{\left[\left(mc^2\right)\vec{p}\cdot\vec{r}-\left(E^2-m^2c^4\right)t\right]^2}{2\sigma_\theta\hbar^2}\right)e^{\frac{i}{\hbar}(\vec{p}\cdot\vec{r}-Et)}, \tag{11}$$

where $\sigma_\xi$ and $\sigma_\theta$ describe the widths of the Gaussian function for the particle and wave, respectively.

The description of the low or high energy possessed by a particle or a wave, for the first step, can be given by means of the postulates of Einstein and de Broglie

$$E=\hbar\omega,\ \vec{p}=\hbar\vec{k}\,. \tag{12}$$

Substituting Eq. (12) into Eqs. (10) and (11), we obtain

$$\xi(\vec{r},t)=A\exp\left(-\frac{\left[\left(mc^2\right)\vec{k}\cdot\vec{r}-\left(\hbar\omega^2-\frac{m^2c^4}{\hbar}\right)t\right]^2}{2\sigma_\xi}\right)e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}, \tag{13}$$

$$\theta(\vec{r},t)=B\exp\left(-\frac{\left[\left(mc^2\right)\vec{k}\cdot\vec{r}-\left(\hbar\omega^2-\frac{m^2c^4}{\hbar}\right)t\right]^2}{2\sigma_\theta}\right)e^{i\left(\vec{k}\cdot\vec{r}-\omega t\right)}, \tag{14}$$

The last step is to determine the relationship between A and B in Eqs. (13) and (14) by imposing the condition that the energy of the particle term has the most portion than that of the wave term. To actualize it, we use the postulate of Einstein to describe the quantum particle energy

$$\hbar\omega=\int_{-\infty}^{\infty}|\psi|^2d^3x\cong\int_{-\infty}^{\infty}|\xi|^2d^3x. \tag{15}$$

However, we get the problem in Eq. (15) because the solution will be divergent in the three-dimensional case. The solution can only be obtained in the one-dimensional case. If we only choose the *x* variable and use the following integral

$$\int_{-\infty}^{\infty}\exp\left(-\frac{\left[\left(mc^2\right)kx-\left(\hbar\omega^2-\frac{m^2c^4}{\hbar}\right)t\right]^2}{\sigma_\xi}\right)dx=\frac{\sqrt{\pi\sigma_\xi}}{mc^2k}, \tag{16}$$

we obtain the constant *A*

$$A=\sqrt{\frac{mc^2 k\hbar\omega}{\sqrt{\pi\sigma_\xi}}}\,. \tag{17}$$

Regarding only the small portion energy possessed by the wave term, we can make a relationship between $A$ and $B$

$$B=\lambda A,\ \lambda \ll 1. \tag{18}$$

Therefore, the general solution of the one-dimensional Klein-Gordon equation for a free particle is given by

$$\psi(x,t)=\sqrt{\frac{mc^2 k\hbar\omega}{\sqrt{\pi\sigma_\xi}}}\;e^{i(kx-\omega t)}$$

$$\left\{\lambda\exp\left(-\frac{\left[\left(mc^2\right)kx-\left(\hbar\omega^2-\frac{m^2c^4}{\hbar}\right)t\right]^2}{2\sigma_\theta}\right)+\exp\left(-\frac{\left[\left(mc^2\right)kx-\left(\hbar\omega^2-\frac{m^2c^4}{\hbar}\right)t\right]^2}{2\sigma_\xi}\right)\right\}. \tag{19}$$

From the general solution in Eq. (19), we deduce that the general solution of the one-dimensional Klein-Gordon equation still hold the linear superposition of the two particular solutions. These two particular solutions describe the solution of a particle part and a wave at the same time. Indeed, this is not surprising because we take a plane-wave solution in the phase part.

## CONCLUSIONS

We present a mathematical model solution of the one-dimensional Klein-Gordon equation, which describes the quantum particle. In this model, a quantum particle is composed of two parts, a particle part and a wave part. The particle part gets the most energy of the quantum particle while the small portion of the energy of the quantum particle is given to the wave part. We also show that the linear combination of the two particular solutions still maintains to give the general solution.


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