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Min Liu; Rui Sun

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Min Liu\textsuperscript{a)} and Rui Sun\textsuperscript{b)}

AFFILIATIONS
School of Mathematics, Liaoning Normal University, Dalian 116029, People’s Republic of China

\textsuperscript{a)}Author to whom correspondence should be addressed: minliu@lnnu.edu.cn
\textsuperscript{b)}Email: ruisun99@163.com

ABSTRACT
In this paper, we are concerned with a Kirchhoff-Choquard type equation with $L^2$-prescribed mass. Under different cases of the potential, we prove the existence of normalized ground state solutions to this equation. To obtain the boundedness from below of the energy functional and the compactness of the minimizing sequence, we apply the Gagliardo-Nirenberg inequality with the Riesz potential and the relationship between the different minimal energies corresponding to different mass. We also extend the results to the fractional Kirchhoff-Choquard type equation.

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I. INTRODUCTION

We consider the following Kirchhoff-type equation

$$-\left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right) \Delta u + (V(x) + \lambda) u = f(x, u) \quad \text{in } \mathbb{R}^N$$

with prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 \, dx = c^2,$$

where $a, b, c > 0$ and $\lambda \in \mathbb{R}$ is unknown. The nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2 \, dx \Delta u$ implies that the above equation is not a pointwise identity and causes some mathematical difficulties which make the research of Eq. (1.1) particularly challenging. The research of Eq. (1.1) arises in the physical background. If we replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$, Eq. (1.1) is viewed as Kirchhoff’s model, which takes into account the changes in length of the string produced by transverse vibrations. It is associated with the stationary analog of equation

$$\begin{cases}
  u_{tt} - \left(a + b \int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u) & \text{in } \Omega \\
  u = 0 & \text{on } \partial \Omega
\end{cases}$$

introduced by Kirchhoff\textsuperscript{66} as an extension of the classical D’Alembert wave equations to describe free vibration of elastic strings. It is worth noting that such a nonlocal model also appears in biological systems, where $u$ describes a process depending on the average of itself, for instance, population density. In the past few decades, there has been increasing interest in studying Eq. (1.1). Li and Ye\textsuperscript{12} studied the existence and concentration of normalized solutions for Eq. (1.1) with prescribed mass (1.2) by the Gagliardo-Nirenberg inequality when $V(x) \geq 0$, $V(x) \in L^\infty_\text{loc}(\mathbb{R}^N)$ and $V(x) \to \infty$ as $|x| \to \infty$, where $f(x, u) = |u|^{p^*-2}u$ and $p \in (2, 2^*)$ ($2^* := \frac{2N}{N-2} \quad \text{if } N \geq 3$ and $2^* := +\infty \quad \text{if } N = 1, 2$) with $N = 1, 2, 3$. He et al.\textsuperscript{10} proved the existence of normalized ground state solutions for Eq. (1.1) in dimension $N = 3$ with (1.2) on a Nehari-Pohozaev manifold under suitable hypotheses on $V(x)$, where $f(x, u) = g(u) + u^r$, $g(u)$ is very general and mass supercritical. When $a \geq 0$, $b > 0$, $N = 1, 2, 3$. The following inequality is known as Gagliardo-Nirenberg inequality

$$\int_{\mathbb{R}^N} |u|^p \, dx \leq C \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{\frac{p-2}{2}}$$

for $p \in (2, 2^*)$ with $a, b > 0$. We denote by $\mathcal{E}$ the energy functional

$$\mathcal{E}(u) = \int_{\mathbb{R}^N} \frac{1}{2} \left|\nabla u\right|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} \frac{1}{2^*} \left|u\right|^{2^*} \, dx - \lambda \int_{\mathbb{R}^N} u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V(x) - V(y)}{|x-y|^{N+2}} |u(x)|^2 |u(y)|^2 \, dx \, dy,$$

where $\lambda \in \mathbb{R}$ is unknown. The critical exponent $2^* := \frac{2N}{N-2} \quad \text{if } N \geq 3$ and $2^* := +\infty \quad \text{if } N = 1, 2$. The following energy inequality is known as Sobolev inequality

$$\int_{\mathbb{R}^N} |u|^2 \, dx \leq C \left(a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u|^2 \, dx\right)^{\frac{1}{2}}$$

for $p \in (2, 2^*)$ with $a, b > 0$. We denote by $\mathcal{H}$ the Nehari-Pohozaev manifold

$$\mathcal{H} = \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V(x) - V(y)}{|x-y|^{N+2}} |u(x)|^2 |u(y)|^2 \, dx \, dy = 0 \}.$$
The existence of $L^2$-norm prescribed ground state solution and the concentration behavior of the ground state under suitable assumptions of $V(x)$ by using blow-up analysis and the optimal energy estimates of the constraint minimizer.

If $a = 1, b = 0$ and $f(x, u) = (I_0 * G(u))G'(u)$, Eq. (1.1) becomes the generalized Choquard equation

$$-\Delta u + (V(x) + \lambda)u = (I_0 * G(u))G'(u) \quad \text{in} \quad \mathbb{R}^N,$$

where $I_0 : \mathbb{R}^N \to \mathbb{R}$ is called the Riesz potential and defined as follows:

$$I_0(x) := \frac{\Gamma(\frac{N-\theta}{2})}{2\theta \pi^{\frac{N}{2}}} |x|^{N-\theta}, \quad x \in \mathbb{R}^N \setminus \{0\}.$$  

When $V(x) = 1$, $G'(u) = u$ with $\theta = 2$ and $N = 3$, Eq. (1.3) turns to be the well-known Pekar-Choquard equation, which appears in several physics contexts such as the description by Pekar in 1954 of the quantum mechanics of a polaron at rest,\textsuperscript{19} the model by Choquard in 1976 of a one-component plasma,\textsuperscript{13} and the coupling of the Schrödinger equation of quantum physics with nonrelativistic Newtonian gravity.\textsuperscript{15} Furthermore, it is also related to the Einstein–Klein–Gordon and Einstein–Dirac system. In a very recent paper, Ao et al.\textsuperscript{1} considered the existence of normalized solutions for Eq. (1.3) with (1.2) for $N \geq 3$, and they proved the compactness of every minimizing sequence with respect to the energy functional under some conditions imposed on $G$, where $V$ satisfies $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^N)$, $V_2 \in L^2(\mathbb{R}^N)$, $\frac{N}{2} < p_1, p_2 < \infty$ and $V(x) = \frac{V(x)}{|x|^}{\leq 0}$. If $V(x) = 0, N \geq 1$ and $G(u)$ is mass supercritical in Eq. (1.3), Xia and Zhang\textsuperscript{4} proved the existence of saddle type normalized solutions by concentration compactness principle with a minimax procedure in the saddle type symmetric space. For the existence of normalized solutions to the Choquard equation, we also refer to Ref. 6 and the references therein.

Recently, Lü\textsuperscript{17} considered the following Kirchhoff-Choquard type equation:

$$-\left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = (I_0 * |u|^p)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $a > 0, b \geq 0, N = 3$, $\theta \in (0, 3)$, $p \in (2, 6 - \theta)$, $V(x) = 1 + g(x)$ with $g > 0$ and $g(x)$ being a nonnegative continuous potential, and they obtained the existence and concentration of the ground state solution by the Nehari manifold and the concentration compactness principle for $N$ large enough. Chen and Liu\textsuperscript{1} proved the existence of the ground state solution for Eq. (1.4), when $N = 3, a, b > 0, p \in (\frac{6-\theta}{2}, 3 + \theta)$ with $\theta \in (0, 3)$, $V \in C^2(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, there is $k \in (0, a)$ such that $\sup_{x \in \mathbb{R}^3}|x|^2 (\nabla V, x) \leq k/2$, and $0 \leq V(x) \leq \lim \inf_{|x| \to \infty} V(x) := V_\infty < \infty$ for $x \in \mathbb{R}^3$. Lü\textsuperscript{17} studied the existence of normalized solutions for Eq. (1.4) with (1.2), where $a, b, c > 0, N \geq 3, V(x) = -v \in \mathbb{R}$, $\theta \in (0, N) \cap \mathbb{Z}$ and $p \in (\frac{4-N}{2}, \frac{6-N}{2})$. They also considered the behaviors of the Lagrange multiplier $v$ and the energies corresponding to the constrained critical points when $c \to 0$ and $c \to +\infty$, respectively.

Inspired by the above works, in this paper we study the existence of normalized ground state solutions for the following Kirchhoff-Choquard type equation with prescribed mass (1.2):

$$-\left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + (V(x) + \lambda)u = (I_0 * |u|^p)|u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$

where $a, b, c > 0, \theta \in (0, N), p \in \left( \frac{N+\theta}{N-\theta}, \frac{N+\theta+4}{N-\theta} \right)$ with $N = 1, 2, 3, \lambda \in \mathbb{R}$ is unknown appearing as a Lagrange multiplier. We consider four different types of the potential $V(x)$. Our idea mainly comes from Alves and Ji\textsuperscript{2} where they established the existence of normalized solutions for the following Schrödinger equation with (1.2) under different types of the potential when $N \geq 2$ and $p \in (2, \frac{2N+4}{N-2})$:

$$-\Delta u + (V(x) + \lambda)u = |u|^{p-2}u \quad \text{in} \quad \mathbb{R}^N,$$

and they also gave the corresponding results for the Schrödinger equations with magnetic field. Our results are proved by applying the constraint minimization technique and variational method. In fact, we use the Gagliardo-Nirenberg type inequality with the Riesz potential $I_0$ to obtain the boundedness from below of the energy functional and apply the relationship between the different minimal energies corresponding to different $c$ to get the compactness of the minimizing sequence. We weaken the condition of the third existence result compared with Ref. 2 and remove the nonemptiness of the interior of $W^{-1}(0)$ in the fourth case of $V$ supposed in Ref. 2. The fourth assumption of $V$ in Ref. 2 comes from Ref. 4, where it is imposed to obtain the uniform boundedness of Palais-Smale sequence for the energy functional. However, in the present paper, the minimal energy of the energy functional we consider on the constraint set is negative under some conditions, which jointly with Gagliardo-Nirenberg type inequality ensures the uniform boundedness of the minimizing sequence. By employing the fractional version of Gagliardo-Nirenberg inequality with the Riesz potential $I_0$, we find that the corresponding results for the fractional Kirchhoff-Choquard type equation are also true.

Now we assume $V(x)$ is nonnegative, continuous and satisfies the following four cases.

(V1) $V$ is one-periodic in $x_1, x_2, \ldots, x_N$. 

\[ \tag{V1} \]
(V2) $V$ is asymptotically periodic: there is a one-periodic function $V_p : \mathbb{R}^N \to \mathbb{R}$ such that $V(x) \leq V_p(x)$ for $x \in \mathbb{R}^N$ and

$$|V(x) - V_p(x)| \to 0 \text{ as } |x| \to \infty.$$  \hspace{1cm} (1.6)

(V3) $V(x) = K(\varepsilon x)$, where $\varepsilon > 0$ is a parameter, $K \in L_+^\infty(\mathbb{R}^N)$ and

$$K_\infty := \liminf_{|x| \to \infty} K(x) > K_0 := \inf_{x \in \mathbb{R}^N} K(x).$$ \hspace{1cm} (1.7)

(V4) $V(x) = \mu W(x)$, where $\mu > 0$ is a parameter, $W \in L_+^\infty(\mathbb{R}^N)$, and there is $W_0 > 0$ such that the measure of the set $\{x \in \mathbb{R}^N : W(x) < W_0\}$ is finite.

Throughout this paper, we make the following notations.

- $B_r(x) := \{y \in \mathbb{R}^N : |x - y| < r\}$ with $r > 0$, and $B_0(x) := \mathbb{R}^N \backslash B_r(x)$.
- $C$ denotes any positive constants possibly different in different places.
- $|u|_q$ is the norm of a function $u$ in $L^q(\mathbb{R}^N)$ with $1 \leq q \leq \infty$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to \infty$.
- $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$.

Before stating our results, we give the definition of the normalized ground state solution for Eq. (1.5) with (1.2).

Definition. The function $u \in H^1(\mathbb{R}^N)$ is called a normalized ground state solution to Eq. (1.5), provided that it solves Eq. (1.5) for some $\lambda \in \mathbb{R}$ and has minimal energy among all the functions satisfying the prescribed mass (1.2).

The main results of Eq. (1.5) are stated as follows.

Theorem 1.1. Assume that $V$ satisfies (V1). Then for each $c > 0$, there exist $\sigma = \sigma(c) > 0$ and $\hat{a} > 0$ such that for $|V|_\infty < \sigma$ and $a < \hat{a}$, Eq. (1.5) admits a positive normalized ground state solution for some $\lambda \in \mathbb{R}$.

Theorem 1.2. Assume that $V$ satisfies (V2). Then for each $c > 0$, there exist $\sigma = \sigma(c) > 0$ and $\hat{a} > 0$ such that for $|V|_\infty < \sigma$ and $a < \hat{a}$, Eq. (1.5) admits a positive normalized ground state solution for some $\lambda \in \mathbb{R}$.

Theorem 1.3. Assume that $V$ satisfies (V3). Then for each $c > 0$, there exist $\sigma = \sigma(c) > 0$, $\hat{a} > 0$ and $\hat{\epsilon} = \hat{\epsilon}(c) > 0$ such that for $K_\infty < \sigma$, $a < \hat{a}$ and all $\epsilon \in (0, \hat{\epsilon})$, Eq. (1.5) admits a positive normalized ground state solution for some $\lambda \in \mathbb{R}$.

Theorem 1.4. Assume that $V$ satisfies (V4). Then for each $c > 0$, there exist $\sigma = \sigma(c) > 0$, $\hat{a} > 0$ and $\hat{\mu} = \hat{\mu}(c) > 0$ such that for all $\mu \in [\hat{\mu}, +\infty)$, $|W|_\infty < \sigma$ and $a < \hat{a}$, Eq. (1.5) admits a positive normalized ground state solution for some $\lambda \in \mathbb{R}$.

As a matter of fact, the above results can also be extended to the following fractional Kirchhoff-Choquard type equation with prescribed mass (1.2):

$$\left(\epsilon^2 + \int_{\mathbb{R}^N} (-\Delta)^s u^2 \, dx\right) (-\Delta)^s u + (V(x) + \lambda) u = (I_\theta * |u|^\theta) |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N, \quad \text{for} \quad \epsilon > 0, \quad \epsilon \to 0,$$ \hspace{1cm} (1.8)

where $a, b, c > 0$, $\theta \in (0, N)$, $p \in \left(\frac{N+\theta}{N-\theta+4\theta}, \frac{N+\theta+4\theta}{N-\theta}\right)$ with $s \in \left(\frac{1}{2}, 1\right)$ and $N \in (2s, 4s)$, $\lambda \in \mathbb{R}$ is unknown and $V : \mathbb{R}^N \to [0, +\infty)$ is bounded and continuous. The fractional Laplacian $(-\Delta)^s$ is defined by

$$(-\Delta)^s u(x) = C_{N,s} \lim_{\epsilon \to 0} \int_{|y| < \epsilon} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy$$

with $C_{N,s} > 0$ a dimensional constant. We may work in the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{u \in L^2(\mathbb{R}^N) : (-\Delta)^s u \in L^2(\mathbb{R}^N)\right\}$$

with norm $|u|_{H^s(\mathbb{R}^N)}^2 = |u|^2_{L^2} + \int_{\mathbb{R}^N} (-\Delta)^s u^2 \, dx$. The definition of the normalized ground state solution of Eq. (1.8) is similar to that of Eq. (1.5). According to the fractional version of Gagliardo-Nirenberg inequality, there exists a constant $C_{N,s,\theta,q} > 0$ such that

$$\int_{\mathbb{R}^N} (I_\theta * |u|^\theta) |u|^q \, dx \leq C_{N,s,\theta,q} \left(\int_{\mathbb{R}^N} (-\Delta)^s u^2 \, dx\right)^{\frac{N-\theta}{2s}} |u|_{L^2}^{\frac{N+\theta}{2s}} \left(\int_{\mathbb{R}^N} \frac{|u|^q}{\left|x \right|^{N-q \theta \xi}} \, dx\right)^{\frac{2s}{q-N}} \quad \forall u \in H^s(\mathbb{R}^N),$$

where $q \in \left(\frac{N\theta}{N-\theta}, \frac{N\theta+4\theta}{N-\theta}\right)$ with $s \in (0, 1)$, $N > 2s$ and $\theta \in (0, N)$, we can obtain the corresponding four results for Eq. (1.8) by making minor modifications in the proof of Sec. III.

This paper is organized as follows: In Sec. II, we present some preliminary results. In Sec. III, we offer the proofs of Theorems 1.1–1.4.
II. PRELIMINARIES

In this section, we introduce some useful facts about the Riesz potential and prepare some preliminary results for proving the main theorems.

Lemma 2.1 (see Ref. 21). Let \( R > 0 \) and \( q \in [2, 2^*). \) If \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^N) \) and
\[
\sup_{y \in \mathbb{R}^n} \int_{B_y(\epsilon)} |u_n|^q dx \to 0 \quad \text{as} \quad n \to \infty,
\]
then \( u_n \to 0 \) in \( L^r(\mathbb{R}^N) \) for any \( r \in (2, 2^*). \)

Lemma 2.2 (Hardy-Littlewood-Sobolev inequality'). Let \( \theta \in (0, N) \) and \( t \in \left(1, \frac{N}{\theta} \right). \) Then for any \( f \in L^t(\mathbb{R}^N), \) there holds \( I_\theta \ast f \in L^{\frac{N}{t-\theta}}(\mathbb{R}^N) \) and there exists a constant \( C_{N, \theta, t} > 0 \) such that
\[
\left( \int_{\mathbb{R}^N} |I_\theta \ast f|^\frac{N}{t-\theta} dx \right)^{\frac{t-\theta}{N}} \leq C_{N, \theta, t} \left( \int_{\mathbb{R}^N} |f|^t dx \right)^{\frac{t-\theta}{N}}.
\]

By Lemma 2.2, we have the nonlocal Brézis-Lieb type lemma.

Lemma 2.3 (see Ref. 17). Let \( \theta \in (0, N) \), \( q \in \left[1, \frac{N}{N+\theta} \right) \) and \( \{u_n\} \) be a bounded sequence in \( L^{\frac{2N}{N+\theta}}(\mathbb{R}^N). \) If \( u_n \to u \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty, \) then
\[
\int_{\mathbb{R}^N} (I_\theta \ast |u_n|^q)|u_n|^q dx - \int_{\mathbb{R}^N} (I_\theta \ast |u-u|^q)|u-u|^q dx \to \int_{\mathbb{R}^N} (I_\theta \ast |u|^q)|u|^q dx \quad \text{as} \quad n \to \infty.
\]

If \( f = |u|^q \in L^{\frac{2N}{N+\theta}}(\mathbb{R}^N) \) in Lemma 2.2, the following result is true by the Hölder inequality.

Lemma 2.4 (See Ref. 18). Let \( \theta \in (0, N). \) Then for any \( u \in L^{\frac{2N}{N+\theta}}(\mathbb{R}^N), \) there exists a constant \( C_{N, \theta} > 0 \) such that
\[
\int_{\mathbb{R}^N} (I_\theta \ast |u|^q)|u|^q dx \leq C_{N, \theta} \left( \int_{\mathbb{R}^N} |u|^{\frac{2N}{N+\theta}} dx \right)^{\frac{N+\theta}{N}}.
\]

By Lemma 2.4 and the Gagliardo-Nirenberg inequality, we get the following Gagliardo-Nirenberg inequality with the Riesz potential \( I_\theta \) which is important to show the boundedness from below of the energy functional.

Lemma 2.5. For \( \theta \in (0, N), \) \( q \in \left(\frac{N+\theta}{N}, \frac{N+\theta}{N-q} \right) \) with \( N \geq 3 \) and \( q \in \left(\frac{N+\theta}{N}, \infty \right) \) with \( N = 1, 2, \) there exists a constant \( C_{N, \theta, q} > 0 \) such that
\[
\int_{\mathbb{R}^N} (I_\theta \ast |u|^q)|u|^q dx \leq C_{N, \theta, q} \left( \int_{\mathbb{R}^N} |u|^{\frac{N+\theta}{N-q}} dx \right)^{\frac{N-q}{N}} \left( \int_{\mathbb{R}^N} |u|^q dx \right)^{\frac{N-q}{N}}.
\]

Normalized ground state solutions of Eq. (1.5) with \( |u|_2 = c > 0 \) can be searched as critical points of the following energy functional
\[
\mathcal{J}(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^2
\]
\[
+ \frac{1}{2} \int_{\mathbb{R}^N} V(x)|u|^2 dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\theta \ast |u|^q)|u|^q dx - \frac{C_{N, \theta, q} N^{\frac{N+\theta}{N-q}}}{2} |u|_2^{N-\theta} - \frac{C_{N, \theta, q} N^{\frac{N+\theta}{N-q}}}{2} \cdot \frac{N^{\frac{N+\theta}{N-q}}}{2} |u|_2^{N-\theta}.
\]

Hence we will study the existence of minimizers to the following minimization problem
\[
\Phi_c := \inf_{u \in S(c)} \mathcal{J}(u).
\]

Lemma 2.6. The energy functional \( \mathcal{J}(u) \) is bounded from below on \( S(c) \) for any \( c > 0. \)

Proof. For any \( u \in S(c), \) it follows from Lemma 2.5 that
\[
\mathcal{J}(u) \geq \frac{b}{4} |\nabla u|_2^4 + \frac{a}{2} |\nabla u|_2^2 - \frac{C_{N, \theta, q} N^{\frac{N+\theta}{N-q}}}{2} |\nabla u|_2^{N-\theta}.
\]
Noticing $p \in (\frac{N\theta}{N-\theta}, \frac{N+\theta+4}{N})$, we have $N\theta - N - \theta < 4$, then $\mathcal{J}(u)$ is bounded from below on $S(c)$ for any $c > 0$.

Lemma 2.7. Let $V \in L^\infty(\mathbb{R}^N)$. Then there exist $\sigma = \sigma(c) > 0$ and $\hat{a} > 0$ such that $\Phi_c < 0$ when $|V|_\infty < \sigma$ and $a < \hat{a}$.

Proof. Choose $v_0 \in S(c)$. Let $t \in \mathbb{R}$, we set $K(t_0, t)(x) = e^{\frac{Nt}{t_0}}v_0(\cdot e^t x)$ for $x \in \mathbb{R}^N$. By direct computation, we have

$$|K(t_0, t)(x)|_2^2 = \frac{e^{2t}}{|\nabla v_0|_2^2}, \quad |\nabla K(t_0, t)(x)|_2^2 = e^{2t}|\nabla v_0|_2^2, \quad \int_{\mathbb{R}^N} (I_0 * |v_0|^p) K(t_0, t)(x)) dx = e^{N\theta-\theta-t_0} \int_{\mathbb{R}^N} (I_0 * |v_0|^p) |v_0|^p dx.$$

Therefore,

$$\mathcal{J}(K(t_0, t)) \leq \frac{ae^{2t}}{2} \int_{\mathbb{R}^N} |\nabla v_0|_2^2 dx + \frac{be^{4t}}{4} \left( \int_{\mathbb{R}^N} |\nabla v_0|_2^2 dx \right)^2 + \frac{|V|_\infty c^2}{2} - \frac{\varepsilon^{Np-N-\theta t}}{2p} \int_{\mathbb{R}^N} (I_0 * |v_0|^p) |v_0|^p dx.$$

Due to $p \in (\frac{N+\theta}{N}, \frac{N+\theta+4}{N})$, there is $t_0 > 0$ such that

$$\mathcal{H}_{n_0} := \frac{b\varepsilon^{4t_0}}{4} \left( \int_{\mathbb{R}^N} |\nabla v_0|_2^2 dx \right)^2 - \frac{\varepsilon^{Np-N-\theta t_0}}{2p} \int_{\mathbb{R}^N} (I_0 * |v_0|^p) |v_0|^p dx < 0.$$

We choose $\sigma := \frac{-2\varepsilon^{4t_0}}{3c^2}$ and $\hat{a} := \frac{-2\varepsilon^{4t_0}}{3c^2 |\nabla v_0|_2^2}$, then for $|V|_\infty < \sigma$ and $a < \hat{a}$,

$$\mathcal{J}(K(t_0, t_0)) < \mathcal{H}_{n_0} = \frac{2 \mathcal{H}_{n_0}}{3} = \frac{\mathcal{H}_{n_0}}{3} < 0.$$

Hence $\Phi_c < 0$.

Lemma 2.8. If $0 < c_1 < c_2$ and $\Phi_{c_1} < 0$, then $\Phi_{c_2} < 0$ and $c_1^2 \Phi_{c_2} < c_2^2 \Phi_{c_1}$.

Proof. Let $\eta = \frac{c_2}{c_1} > 1$. Choose $(u_n) \subset S(c_1)$ as a minimizing sequence of $\Phi_{c_1}$. Now we prove there are $C > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} (I_0 * |u_n|^p) |u_n|^p dx \geq C, \quad \forall n \geq n_0. \quad (2.2)$$

Assume by contradiction that $\int_{\mathbb{R}^N} (I_0 * |u_n|^p) |u_n|^p dx \to 0$ as $n \to \infty$, up to a subsequence if necessary. Since

$$\Phi_{c_2} + o_n(1) = \mathcal{J}(u_n) \geq -\frac{1}{2p} \int_{\mathbb{R}^N} (I_0 * |u_n|^p) |u_n|^p dx,$$

we have $\Phi_{c_2} \geq 0$ by letting $n \to \infty$, which contradicts with the assumption. By Lemma 2.5, there is $\theta > 0$ independent of $n$ such that

$$|\nabla u_n|_2^2 \geq \theta, \quad \forall n \geq n_0. \quad (2.3)$$

Setting $w_n = u_n(\eta^{-\frac{1}{\theta}} x)$, then $w_n \in S(c_2)$. By (2.2) and (2.3), we have

$$\mathcal{J}(w_n) = \eta^2 \mathcal{J}(u_n) + \left( \eta^{2 - \frac{\theta}{2}} - \eta^2 \right) \frac{\theta}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \left( \eta^{4 - \frac{\theta}{2}} - \eta^2 \right) \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^2$$

$$- \left( \eta^{2 + \frac{\theta}{2}} - \eta^2 \right) \frac{1}{2p} \int_{\mathbb{R}^N} (I_0 * |u_n|^p) |u_n|^p dx \leq \eta^2 \mathcal{J}(u_n) - \mathcal{J}(\eta), \quad \forall n \geq n_0, \quad (2.4)$$

where

$$\mathcal{J}(\eta) := \eta^2 \left( (1 - \eta^{-\frac{1}{\theta}}) \frac{\theta}{2} + (1 - \eta^{4 - \frac{\theta}{2}}) \frac{b}{4} + (\eta^{2 + \frac{\theta}{2}} - 1) \frac{C}{2p} \right)$$

is strictly positive and independent of $n$. Thus $\Phi_{c_2} < 0$ and $c_1^2 \Phi_{c_2} < c_2^2 \Phi_{c_1}$ by letting $n \to \infty$ in (2.4) and noticing $\mathcal{J}(w_n) \geq \Phi_{c_2}$. □

Lemma 2.9. Assume that $V \in L^\infty(\mathbb{R}^N)$ and $\Phi_c < 0$. Let $(u_n) \subset S(c)$ be a minimizing sequence of $\Phi_c$ with $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$, $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and $u \neq 0$. Then $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, $u \in S(c)$ and $\mathcal{J}(u) = \Phi_c$. □
III. PROOF OF MAIN RESULTS

In this section, we give the proofs of Theorems 1.1–1.4.

A. Proof of Theorem 1.1

We assume (V1) holds in this subsection.

Lemma 3.1. If $\Phi_0 < 0$, then the minimizing sequence $\{u_n\} \subset S(c)$ of $\Phi_c$ can be chosen such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $u \neq 0$.

Proof. Let $\{v_n\} \subset S(c)$ be a minimizing sequence of $\Phi_c$. It follows from $p \in \left(\frac{N \theta}{N - \theta}, \frac{N + \theta}{N - \theta}\right)$ and (2.1) that $|\nabla v_n|^2$ is bounded. Hence $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Similar to the proof of (2.2) and noticing $\Phi_c < 0$, we have $\int_{\mathbb{R}^N} (I_0 + |v_n|^p) |v_n|^p \, dx \geq C_0 > 0$ for $n$ large enough. Additionally, there exist $R, C_2 > 0$ and $t_0 \in \mathbb{R}^N$ such that

$$
\int_{B_R(t_0)} |v_n|^2 \, dx \geq C_2.
$$

(3.1)

Otherwise, by Lemma 2.1, we get $v_n \to 0$ in $L^1(\mathbb{R}^N)$ for any $t \in (2, 2^*)$ as $n \to \infty$, which together with Lemma 2.4, implies that $\int_{\mathbb{R}^N} (I_0 + |v_n|^p) |v_n|^p \, dx \to 0$ as $n \to \infty$, which is impossible. We may choose $t_0 \in \mathbb{Z}^N$ and increase $R$ if necessary in (3.1). Setting $u_n(x) = v_n(x + t_0)$, we get that $\{u_n\} \subset S(c)$ and it is also a bounded minimizing sequence of $\Phi_c$ in view of (V1). Hence there is $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ along a subsequence. By (3.1), we have $\int_{B_R(t_0)} |u_n|^2 \, dx \geq C_2$. Thus $u \neq 0$.

Proposition 3.2. For each $c > 0$, there exist $\sigma = \sigma(c) > 0$ and $\tilde{a} > 0$ such that if $|V|_{\infty} < \sigma$ and $a < \tilde{a}$, then $\Phi_c < 0$ and $\Phi_c$ is attained by a positive function.

Proof. By Lemma 2.7, for any $c > 0$, there are $\sigma = \sigma(c) > 0$ and $\tilde{a} > 0$ such that $\Phi_c < 0$ when $|V|_{\infty} < \sigma$ and $a < \tilde{a}$. By Lemma 3.1, there is a minimizing sequence $\{u_n\} \subset S(c)$ of $\Phi_c$ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $u \neq 0$. By Lemma 2.9, $u \in S(c)$, $J(u) = \Phi_c$ and $u_0 \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. Hence by the Lagrange multiplier method, there exists $\lambda \in \mathbb{R}$ such that

$$
J'(u) + \lambda \Psi'(u) = 0 \quad \text{in} \quad H^1(\mathbb{R}^N),
$$

(3.2)
where $\Psi : H^1(\mathbb{R}^N) \to \mathbb{R}$ is given by $\Psi(u) = |u|^2$ for $u \in H^1(\mathbb{R}^N)$. By (3.2), we get that $(u, \lambda_i)$ is a couple of solution to the following equation

$$-\left( a + b \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right) \Delta u + (V(x) + \lambda_i) u = \left( I_0 + |u|^p \right) |u|^{p-2} u \quad \text{in} \quad \mathbb{R}^N.$$ 

Now we show that $u$ can be chosen to be positive. Observe that if $u \in H^1(\mathbb{R}^N)$, then $|u| \in H^1(\mathbb{R}^N)$ with $|\nabla |u||^2 = |\nabla u|^2$. Furthermore, $u \in S(c)$ implies that $|u| \notin S(c)$. Then we have $J(|u|) = J(u) = \Phi_c$. Hence we can replace $u$ by $|u|$. By the standard regularity theory, $u \in C^2(\mathbb{R}^N)$.

Assume by contradiction that there exists $x_1 \in \mathbb{R}^N$ such that $u(x_1) = 0$. Since $u \neq 0$, there exists $x_2 \in \mathbb{R}^N$ such that $u(x_2) > 0$. Then fix $R > 0$ sufficiently large such that $x_1, x_2 \in B_R(0)$. By Theorem 8.20 in Ref. 8, there exists $C > 0$ such that

$$\sup_{z \in B_R(0)} u(z) \leq C \inf_{z \in B_R(0)} u(z),$$

which is impossible, since $\sup_{z \in B_R(0)} u(z) > 0$ and $\inf_{z \in B_R(0)} u(z) = 0$. $\square$

**B. Proof of Theorem 1.2**

In this subsection, we assume (V2) holds and $V \neq V_p$. Hence there exists a measurable set $C \subset \mathbb{R}^N$ with $|C| > 0$ such that $V(x) < V_p(x)$ for all $x \in C$. Let us denote by $J_p : H^1(\mathbb{R}^N) \to \mathbb{R}$ the following energy functional

$$J_p(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} V_p(x) |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} \left( I_0 + |u|^p \right) |u|^p \, dx,$$

and consider the constrained minimization problem

$$\Phi_{c,p} := \inf_{u \in S(c)} J_p(u).$$

Due to Theorem 1.1, for each $c > 0$, there exists a positive function $u_p \in S(c)$ such that $J_p(u_p) = \Phi_{c,p} < 0$ when $|V_p|_{\infty} < \sigma(c)$ and $a < \tilde{a}$, where $\sigma(c)$ and $\tilde{a}$ are the existing constants in Theorem 1.1. Since $V(x) < V_p(x)$ for $x \in C$ and $|C| > 0$, we get that for $|V_p|_{\infty} < \sigma(c)$ and $a < \tilde{a}$,

$$\Phi_c = \inf_{u \in S(c)} J(u) \leq J(u_p) < J_p(u_p) = \Phi_{c,p} < 0. \quad (3.3)$$

By Lemma 2.6, $J$ is bounded from below on $S(c)$ for any $c > 0$.

**Lemma 3.3.** For any $c > 0$, there exist $\sigma = \sigma(c) > 0$ and $\tilde{a} > 0$ such that if $|V_p|_{\infty} < \sigma$ and $a < \tilde{a}$, then the minimizing sequence $\{u_n\} \subset S(c)$ of $\Phi_c$ can be chosen such that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $u \neq 0$.

**Proof.** Note that $p \in \left( \frac{N a}{2} \frac{N \tilde{a}}{4}, \frac{N a}{2} \frac{N a}{4} \right)$ and (2.1), $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Hence there are a subsequence of $\{u_n\}$, still denoted by itself, and $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. Suppose $u = 0$. It follows from $u_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$ as $n \to \infty$ and (1.6) that $\int_{\mathbb{R}^N} \left( V(x) - V_p(x) \right) |u_n|^2 \, dx \to 0$ as $n \to \infty$. Clearly,

$$\Phi_c + o_n(1) = J(u_n) \geq J(u_p) + \frac{1}{2} \int_{\mathbb{R}^N} \left( V(x) - V_p(x) \right) |u_n|^2 \, dx.$$

Letting $n \to \infty$, we get $\Phi_c \geq \Phi_{c,p}$, which contradicts with (3.3). Thus $u \neq 0$. $\square$

**Proposition 3.4.** For each $c > 0$, there exist $\sigma = \sigma(c) > 0$ and $\tilde{a} > 0$ such that if $|V_p|_{\infty} < \sigma$ and $a < \tilde{a}$, then $\Phi_c < 0$ and $\Phi_c$ is attained by a positive function.

**Proof.** By Lemma 2.7, for any $c > 0$, there are $\sigma = \sigma(c) > 0$ and $\tilde{a} > 0$ such that $\Phi_{c,p} < 0$ when $|V_p|_{\infty} < \sigma$ and $a < \tilde{a}$, which together with (3.3) implies $\Phi_c < 0$. By Lemma 3.3, there exists a bounded minimizing sequence $\{u_n\} \subset S(c)$ of $\Phi_c$ such that $u_n \to u$ in $H^1(\mathbb{R}^N)$ and $u_n(x) \to u(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $u \neq 0$. By Lemma 2.9, $u \in S(c)$, $J(u) = \Phi_c$ and $u_n \to u$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The following is similar to the Proof of Proposition 3.2, so the details are omitted. $\square$
C. Proof of Theorem 1.3

We assume (V3) holds in this subsection. The third result is related to the energy functional \( J_{\varepsilon} : H^1(\mathbb{R}^N) \to \mathbb{R} \) defined as below,

\[
J_{\varepsilon}(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} K(\varepsilon x) |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\theta} * |u|^p) |u|^p \, dx,
\]

and the \( L^2 \)-constraint minimization problem

\[
\Phi_{\varepsilon} := \inf_{u \in S(c)} J_{\varepsilon}(u).
\]

By Lemma 2.6, \( J_{\varepsilon} \) is bounded from below on \( S(c) \) for any \( c > 0 \).

Let us denote by \( J_0, J_{\infty} : H^1(\mathbb{R}^N) \to \mathbb{R} \) the following energy functionals:

\[
J_0(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} K_0 |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\theta} * |u|^p) |u|^p \, dx,
\]

\[
J_{\infty}(u) := \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} K_{\infty} |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\theta} * |u|^p) |u|^p \, dx.
\]

Define

\[
\Phi_{0,0} := \inf_{u \in S(c)} J_0(u), \quad \Phi_{c,\infty} := \inf_{u \in S(c)} J_{\infty}(u).
\]

According to Theorem 1.1 and (1.7), for each \( c > 0 \), there exist \( \sigma = \sigma(c) > 0 \) and \( \tilde{a} > 0 \) such that if \( K_{\infty} < \sigma \) and \( a < \tilde{a} \), then there exist positive functions \( u_0, u_{\infty} \in S(c) \) satisfying \( J_0(u_0) = \Phi_{0,0} \) and \( J_{\infty}(u_{\infty}) = \Phi_{c,\infty} \). Therefore, if \( K_{\infty} < \sigma \) and \( a < \tilde{a} \), we have

\[
\Phi_{0,0} < \Phi_{c,\infty} < 0. \tag{3.4}
\]

**Lemma 3.5.** \( \text{lim sup}_{\varepsilon \to 0^+} \Phi_{\varepsilon,0} \leq \Phi_{0,0} \).

**Proof.** Let \( \{x_k\} \subset \mathbb{R}^N \) satisfy \( K(x_k) \to K_0 \) as \( k \to \infty \). If \( |x_k| \to \infty \) as \( k \to \infty \), then \( K_0 \geq K_{\infty} \), which contradicts with (1.7). Therefore, there is \( x_0 \in \mathbb{R}^N \) such that \( K(x_0) = K_0 \). Set \( u_n(x) = u_0(x - \frac{n}{N}) \). Then \( u_n(x) \in S(c) \) and

\[
\Phi_{0,0} \leq J_{\varepsilon}(u_n) = \frac{a}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^N} K(x+\epsilon)x_0 |u_n|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_{\theta} * |u_n|^p) |u_n|^p \, dx.
\]

Since \( K \in L^\infty(\mathbb{R}^N) \), we obtain \( \text{lim sup}_{\varepsilon \to 0^+} \Phi_{\varepsilon,0} \leq \text{lim sup}_{\varepsilon \to 0^+} J_{\varepsilon}(u_n) = \Phi_{0,0} \).

**Remark 3.6.** If \( K_{\infty} < \sigma \) and \( a < \tilde{a} \), then it follows from Lemma 3.5 and (3.4) that there exists \( \tilde{\varepsilon} = \tilde{\varepsilon}(c) > 0 \) such that \( \Phi_{\varepsilon,0} < \Phi_{c,\infty} < 0 \) for all \( \varepsilon \in (0, \tilde{\varepsilon}) \).

**Lemma 3.7.** If \( K_{\infty} < \sigma \) and \( a < \tilde{a} \), then for all \( \varepsilon \in (0, \tilde{\varepsilon}) \), any minimizing sequence \( \{u_{n,\varepsilon}\} \subset S(c) \) of \( \Phi_{\varepsilon,\varepsilon} \) is bounded in \( H^1(\mathbb{R}^N) \) uniformly in \( \varepsilon \) and it can be chosen such that \( u_{n,\varepsilon} \to u_\varepsilon \) in \( H^1(\mathbb{R}^N) \) and \( u_{n,\varepsilon}(x) \to u_\varepsilon(x) \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \) with \( u_\varepsilon \neq 0 \), where \( \tilde{\varepsilon} \) is given in Remark 3.6.

**Proof.** By Remark 3.6, \( \Phi_{\varepsilon,\varepsilon} < 0 \), which together with \( p \in \left( \frac{N+4}{N} \right) \) and (2.1) implies that \( |\nabla u_{n,\varepsilon}|^2 \) is bounded uniformly in \( \varepsilon \), from which it follows that \( \{u_{n,\varepsilon}\} \) is bounded in \( H^1(\mathbb{R}^N) \) uniformly in \( \varepsilon \). Thus there exist a subsequence of \( \{u_{n,\varepsilon}\} \), still denoted by itself, and
$u_\epsilon \in H^1(\mathbb{R}^N)$ such that $u_{n_\epsilon}, u_\epsilon \in H^1(\mathbb{R}^N)$ and $u_{n_\epsilon}(x) \to u_\epsilon(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$. By (V3), for any given $\zeta > 0$, there exists $R > 0$ such that $K(x) \geq K_\infty - \zeta$ for all $|x| \geq R$. Assume by contradiction that $u_\epsilon = 0$, then $u_{n_\epsilon} \to 0$ in $L^2(\mathbb{R}^N)$ as $n \to \infty$. Since

$$\Phi_{c,\epsilon} + o_\epsilon(1) = J_\epsilon(u_{n_\epsilon}) = J_\infty(u_{n_\epsilon}) + \frac{1}{2} \int_{\mathbb{R}^N} \langle K(x) - K_\infty \rangle |u_{n_\epsilon}|^2 \, dx$$

$$\geq J_\infty(u_{n_\epsilon}) + \frac{1}{2} \int_{B_\epsilon(0)} \langle K(x) - K_\infty \rangle |u_{n_\epsilon}|^2 \, dx - \frac{\epsilon}{2} \int_{B_\epsilon(0)} |u_{n_\epsilon}|^2 \, dx,$$

we have $\Phi_{c,\epsilon} \geq \Phi_{c,\infty} - C\zeta$. Note that $\zeta > 0$ is arbitrary, we infer that $\Phi_{c,\epsilon} \geq \Phi_{c,\infty}$, which contradicts with Remark 3.6. Hence $u_\epsilon \neq 0$. ⊓⊔

**Proposition 3.8.** For each $c > 0$, there exist $\sigma = \sigma(c) > 0$, $\tilde{a} > 0$ and $\tilde{\epsilon} = \tilde{\epsilon}(c) > 0$ such that if $K_\infty < \sigma$ and $a < \tilde{a}$, then $\Phi_{c,\tilde{\epsilon}} < 0$ and $\Phi_{c,\epsilon}$ is attained by a positive function for all $\epsilon \in (0, \tilde{\epsilon})$.

**Proof.** By Lemma 2.7 and Remark 3.6, for any $c > 0$, there are $\sigma = \sigma(c) > 0$, $\tilde{a} > 0$ and $\tilde{\epsilon} = \tilde{\epsilon}(c) > 0$ such that $\Phi_{c,\epsilon} < 0$ for $K_\infty < \sigma$, $a < \tilde{a}$ and all $\epsilon \in (0, \tilde{\epsilon})$. By Lemma 3.7, there exists a minimizing sequence $(u_{n_\epsilon}) \subset S(c)$ of $\Phi_{c,\epsilon}$ such that $u_{n_\epsilon} \in H^1(\mathbb{R}^N)$ and $u_{n_\epsilon}(x) \to u_\epsilon(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$ with $u_\epsilon \neq 0$. By Lemma 2.9, we see that $u_\epsilon \in S(c)$, $J_\epsilon(u_\epsilon) = \Phi_{c,\epsilon}$ and $u_{n_\epsilon} \to u_\epsilon$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$. The following arguments are similar to that of Proposition 3.2. ⊓⊔

**D. Proof of Theorem 1.4**

We assume (V4) holds in this subsection. For $u \in H^1(\mathbb{R}^N)$, we choose the equivalent norm

$$|u|^2 \equiv \int_{\mathbb{R}^N} \langle \nabla u \rangle^2 \, dx = \int_{\mathbb{R}^N} \langle \mu W(x) + 1 \rangle |u|^2 \, dx.$$ 

We study the energy functional $J_\mu : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$J_\mu(u) := \frac{a}{2} \int_{\mathbb{R}^N} \langle \nabla u \rangle^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} \langle \nabla u \rangle^2 \, dx \right)^2$$

$$+ \frac{\mu}{2} \int_{\mathbb{R}^N} W(x) |u|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\theta * |u|^p) |u|^2 \, dx$$

and the minimization problem

$$\Phi_{c,\epsilon} := \inf_{u \in S(c)} J_\mu(u).$$

Actually, $J_\mu$ is bounded from below on $S(c)$ for any $c > 0$ by Lemma 2.6. Let $(u_{n_\epsilon}) \subset S(c)$ be a minimizing sequence of $\Phi_{c,\epsilon}$. Now we study the properties of $(u_{n_\epsilon})$.

**Lemma 3.9.** For any $c > 0$, there exist $\sigma = \sigma(c) > 0$, $\tilde{a} > 0$ and $\gamma = \gamma(c) > 0$ such that if $\mu |W|_\infty < \sigma$ and $a < \tilde{a}$, then $\Phi_{c,\epsilon} < 0$ and

$$\liminf_{n \to \infty} \int_{\mathbb{R}^N} (I_\theta * |u_{n_\epsilon}|^p) |u_{n_\epsilon}|^2 \, dx \geq \gamma.$$  \hspace{1cm} (3.5)

**Proof.** Similar to the Proof of Lemma 2.7, we can find $\sigma = \sigma(c) > 0$, $\tilde{a} > 0$ and $\gamma = \gamma(c) > 0$ such that $\Phi_{c,\epsilon} < -\frac{\gamma}{2p}$ when $\mu |W|_\infty < \sigma$ and $a < \tilde{a}$. Since

$$\Phi_{c,\epsilon} + o_\epsilon(1) = \frac{a}{2} \int_{\mathbb{R}^N} \langle \nabla u_{n_\epsilon} \rangle^2 \, dx + \frac{b}{4} \left( \int_{\mathbb{R}^N} \langle \nabla u_{n_\epsilon} \rangle^2 \, dx \right)^2$$

$$+ \frac{\mu}{2} \int_{\mathbb{R}^N} W(x) |u_{n_\epsilon}|^2 \, dx - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\theta * |u_{n_\epsilon}|^p) |u_{n_\epsilon}|^2 \, dx,$$

then for $\mu |W|_\infty < \sigma$ and $a < \tilde{a}$, we have

$$-\frac{\gamma}{2p} + o_\epsilon(1) \geq - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\theta * |u_{n_\epsilon}|^p) |u_{n_\epsilon}|^2 \, dx,$$

which implies (3.5). ⊓⊔

**Lemma 3.10.** If $\Phi_{c,\mu} < 0$, $u_{n_\epsilon} \to u_\mu$ in $H^1(\mathbb{R}^N)$, $u_{n_\mu}(x) \to u_\mu(x)$ a.e. in $\mathbb{R}^N$ as $n \to \infty$, and $u_\mu \neq 0$, then $u_{n_\epsilon} \to u_\mu$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$, $u_\epsilon \in S(c)$ and $J_\mu(u_\mu) = \Phi_{c,\mu}$.
The Proof of Lemma 3.10 is similar to that of Lemma 2.9 and so is omitted.

Lemma 3.11. \( \{u_{\mu(n)}\} \) is uniformly bounded with respect to \( \mu \) in \( H^1(\mathbb{R}^N) \) when \( \mu|W|_\infty < \sigma \) and \( a < \tilde{a} \), where \( \sigma \) and \( \tilde{a} \) are given in Lemma 3.9.

Proof. Due to Lemma 3.9, \( \Phi_{c_0} < 0 \). It follows from \( p \in \left( \frac{N+\theta}{N}, \frac{N+\theta+\delta}{N} \right) \) and (2.1) that \( |\nabla u_{\mu(n)}|_2 \) is uniformly bounded with respect to \( \mu \). Moreover, the sequence \( \left\{ \int_{\mathbb{R}^N} \mu W(x)|u_{\mu(n)}|^2 \, dx \right\} \) is uniformly bounded with respect to \( \mu \). Thus \( \{u_{\mu(n)}\} \) is uniformly bounded with respect to \( \mu \) in \( H^1(\mathbb{R}^N) \).

\( \square \)

Lemma 3.12. There exist \( \tilde{r} = \tilde{r}(c) > 0 \) and \( \tilde{\mu} = \tilde{\mu}(c) > 0 \) such that if \( \mu \in [\tilde{\mu}, +\infty) \) and \( \mu|W|_\infty < \sigma \), then
\[
\limsup_{n \to \infty} \int_{B(0)} (I_\theta * |u_{\mu(n)}|^\theta)|u_{\mu(n)}|^2 \, dx \leq \frac{\gamma}{2},
\]
where \( \sigma \) and \( \gamma \) are given in Lemma 3.9.

Proof. We refer to Ref. 4 for part of the proof. For \( r > 0 \), we consider the sets
\[
A(r) := \{x \in \mathbb{R}^N : |x| > r, W(x) \geq W_0 \}, \quad B(r) := \{x \in \mathbb{R}^N : |x| > r, W(x) < W_0 \},
\]
where \( W_0 \) is given in (V4). By Lemma 3.11, there is \( M > 0 \) independent of \( \mu \) such that \( |u_{\mu(n)}|_1 \leq M \). Thus we obtain
\[
\int_{A(r)} |u_{\mu(n)}|^2 \, dx \leq \frac{1}{\mu W_0 + 1} \int_{\mathbb{R}^N} (\mu W(x) + 1)|u_{\mu(n)}|^2 \, dx \leq \frac{M^2}{\mu W_0 + 1}.
\]

By the Hölder inequality, we choose \( q \in [1, 2] \) and get
\[
\int_{B(r)} |u_{\mu(n)}|^2 \, dx \leq \left( \int_{B(r)} |u_{\mu(n)}|^{2q} \, dx \right)^{\frac{2}{q}} \left( \int_{B(r)} |\nabla u_{\mu(n)}|^2 \, dx \right)^{\frac{1}{q}} \leq CM^q |B(r)|^{\frac{2}{q}}.
\]

According to Theorem 5.8 in Ref. 1, we have
\[
\int_{B(0)} (I_\theta * |u_{\mu(n)}|^\theta)|u_{\mu(n)}|^2 \, dx \leq C\|u_{\mu(n)}\|_1^{Nq-N-\theta} \left( \int_{A(r)} |u_{\mu(n)}|^2 \, dx + \int_{B(r)} |u_{\mu(n)}|^2 \, dx \right)^{\frac{Nq-N(q-1)}{2}}
\]
\[
\leq CM^{\frac{Nq-N-\theta}{2}} \frac{M^2}{\mu W_0 + 1} + M^q |B(r)|^{\frac{2}{q}}.
\]

The first term on the right-hand side of the above inequality can be arbitrarily small for \( \mu \) large enough, and the second term can also be arbitrarily small if \( r \) is large enough since \( |B(r)| \to 0 \) as \( r \to 0 \).

\( \square \)

Lemma 3.13. For \( \mu \in [\tilde{\mu}, +\infty) \), \( \mu|W|_\infty < \sigma \) and \( a < \tilde{a} \), the minimizing sequence \( \{u_{\mu(n)}\} \) is a subsequence of \( \Phi_{c_0} \) that can be chosen such that \( u_{\mu(n)} \to u_\mu \) in \( H^1(\mathbb{R}^N) \) and \( u_{\mu(n)}(x) \to u_\mu(x) \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \) with \( u_\mu \neq 0 \), where \( \tilde{c} \) is given in Lemma 3.12, \( \sigma \) and \( \tilde{a} \) are given in Lemma 3.9.

Proof. By Lemma 3.11, there is \( u_\mu \in H^1(\mathbb{R}^N) \) such that \( u_{\mu(n)} \to u_\mu \) in \( H^1(\mathbb{R}^N) \) and \( u_{\mu(n)}(x) \to u_\mu(x) \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \) along a subsequence. Assume by contradiction that \( u_\mu = 0 \), then \( u_{\mu(n)} \to 0 \) in \( L^\infty_{\text{loc}}(B_r(0)) \) as \( n \to \infty \) for \( r > 0 \) given in Lemma 3.12. Hence
\[
\int_{B(0)} (I_\theta * |u_{\mu(n)}|^\theta)|u_{\mu(n)}|^2 \, dx \to 0 \text{ as } n \to \infty.
\]

According to Lemma 3.9 and Lemma 3.12, we obtain
\[
\gamma \leq \liminf_{n \to \infty} \int_{\mathbb{R}^N} (I_\theta * |u_{\mu(n)}|^\theta)|u_{\mu(n)}|^2 \, dx \leq \limsup_{n \to \infty} \int_{B(0)} (I_\theta * |u_{\mu(n)}|^\theta)|u_{\mu(n)}|^2 \, dx \leq \frac{\gamma}{2},
\]
which is impossible. Hence \( u_\mu \neq 0 \).

\( \square \)

Proposition 3.14. For each \( c > 0 \), there exist \( \sigma = \sigma(c) > 0 \), \( \tilde{a} = \tilde{a}(c) > 0 \) and \( \tilde{\mu} = \tilde{\mu}(c) > 0 \) such that \( \Phi_{c_0} < 0 \) and \( \Phi_{c_\mu} \) is attained by a positive function for all \( \mu \in [\tilde{\mu}, +\infty) \), \( \mu|W|_\infty < \sigma \) and \( a < \tilde{a} \).

Proof. By Lemma 3.9, for any \( c > 0 \), there are \( \sigma = \sigma(c) > 0 \) and \( a < \tilde{a} \) such that \( \Phi_{c_\mu} < 0 \) when \( \mu|W|_\infty < \sigma \) and \( a < \tilde{a} \). By Lemmas 3.11–3.13, there are a bounded minimizing sequence \( \{u_{\mu(n)}\} \subset C(0) \) of \( \Phi_{c_\mu} \) and \( \tilde{\mu} = \tilde{\mu}(c) > 0 \) such that \( u_{\mu(n)} \to u_\mu \) in \( H^1(\mathbb{R}^N) \), \( u_{\mu(n)}(x) \to u_\mu(x) \) a.e. in \( \mathbb{R}^N \) as \( n \to \infty \) with \( u_\mu \neq 0 \). By Lemma 3.10, we get \( u_\mu \in C(0) \), \( \mathcal{F}_\mu(u_\mu) = \Phi_{c_\mu} \) and \( u_{\mu(n)} \to u_\mu \) in \( H^1(\mathbb{R}^N) \) as \( n \to \infty \). The following arguments are similar to the Proof of Proposition 3.2.

\( \square \)
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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Min Liu: Formal analysis (equal); Investigation (equal); Writing – review & editing (equal). Rui Sun: Investigation (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

REFERENCES