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A note on the explicit asymptotics of rows and of closed to row sequences of classical Padé approximants

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Abstract. In the present paper, results about the explicit asymptotics of sequences of rows and of closed to rows sequences of classical Padé approximants will be provided.

INTRODUCTION

Let $f(z) := \sum_{j=0}^{\infty} f_j z^j$ be a formal power series at the zero, and let $R_0, R_0 \geq 0$ be the radius of holomorphy. Given a nonnegative integer $m, (m \in \mathbb{N})$, denote by $R_m(f) := R_m$ the radius of $m-$meromorphy of f, that is: the radius of the largest disk centered at the zero where the power series f admits a continuation as a meromorphic function with $\leq m$ poles (poles are counted with regard to their multiplicities). Apparently, $R_m \geq R_0$. As it was shown by A. A. Gonchar (see [1]), $R_m > 0$ iff $R_0 > 0$.

Given a pair of integers (n, m), let $\pi_{n,m}$ be the classical Padé approximant of f of order (n, m). Recall that $\pi_{n,m} = p/q$, where p, q are polynomials of degree $\leq n, m$ respectively and such that

$$(fq - p)(z) = 0(z^{n+m+1}).$$

As it is well known (see [2]), the Padé approximant $\pi_{n,m}$ always exists and is uniquely determined by the condition above.

For the sake of accuracy, we recall that the sequence $\{\pi_{n,m}\}, n \to \infty, m-$ fixed, is called the "mth row of the Padé table, associated with the power series f"(comp. [2]).

In the present paper, we assume that the power series f presents a function, holomorphic[1] at the origin (e.g. $R_0 := (\limsup_{n\to\infty} |f_n|^{1/n})^{-1} > 0$.) Also, we suppose that R_0 is finite. We will concentrate on the strong asymptotics of rows of classical Padé approximants, in other words, the number $m \in \mathbb{N}$ will be fixed.

Set

$$\pi_{n,m} := \pi_n = \frac{P_n}{Q_n},$$

where $(P_n, Q_n) = 1$ and Q_n is monic. The zeros of the polynomial Q_n are called *the free poles* of the rational function π_n. Set $\mathcal{P}_n$ for the set containing all free poles π_n, and $\mathcal{L}$ for the set of the concentration points of $\mathcal{P}_n$ in $\overline{\mathbb{C}}$, $n \to \infty$.

Let

$$f(z) - \pi_n(z) = 0(z^{n+m+1-\tau_{n,m}}),\ \tau_{n,m} \geq 0.$$

The number $\tau_{n,m}$ is *the defect* in the approximation by $\pi_{n,m}$, $\tau_{n,m} = \min(n - \deg P_n, m - \deg Q_n)$. Apparently, $\deg P_n \leq n - \tau_{n,m}$, $\deg Q_n \leq m - \tau_{n,m}$. Denote by A_n the leading coefficient of the polynomial P_n, that is:

$$P_n(z) = A_n z^{\deg P_n} + \cdots.$$

Following the terminology of G. A. Baker and P. Gr. Morris (see [3]), we say that the rational function $\pi_{n,m}$ exists iff $\tau_{n,m} = 0$. As it is known , if $\tau_{n,m} > 0$, then $\pi_{n-l,m-k} \equiv \pi_{n,m}$ for $k, l = 0, \cdots, \tau_{n,m}$ ([2], for details, the reader is referred to

[1] *a simple valued branch of an analytic function*

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the structure of the Padé-table). Therefore, without loosing the generality, we will assume that $\tau_{n,m} = 0$ for all n large enough, as well as that $\deg Q_{n,m} = m$ and $\deg P_n = n$ for all $n \in \mathbb{N}$. (Recall that we are considering the case when the number m is fixed). Let

$$Q_{n,m}(z) = \prod_{l=1}^{m}(z - \zeta_{n,l}) \tag{1}$$

and

$$P_{n,m}(z) = A_n z^n + \cdots. \tag{2}$$

Our main result is

Theorem 1: *Given a power series* $f(z) := \sum_{j=0}^{\infty} f_j z^j$ *and a fixed integer m, suppose that* $0 < R_m < \infty$. *Assume that the set* $\mathcal{L}$ *consists of a finite number of points in* $\overline{\mathbb{C}}$.

Then there is an infinite sequence Λ *and a function* χ, $\chi \not\equiv 0$ *meromorphic in* $D^c_{R_m} := \{z, |z| \geq R_m\}^c$ *and having not more than m poles such that*

$$\frac{\pi_n(z)}{A_n z^n} \to \chi(z),\ n \in \Lambda$$

locally uniformly in $D^c_{R_m} \setminus \mathcal{L}$.

This result expresses the explicit asymptotics (or, as it si called, the *strong asymptotics*) of the sequence $\pi_{n,m}$, $n \in \Lambda$ outside the disk of the $m-$ meroorphic continuation of f.

Proof of Theorem 1

The proofs will be preceded by an auxiliary lemma.

Lemma 1 (Kakehashi's Regularization Lemma, [4]. *Let* $\{c_n\}$ *be an infinite sequence of complex numbers such that*

$$\limsup_{n\to\infty} |c_n|^{1/n} = c \in (0, \infty).$$

Then there exists a monotone sequence $\{\lambda_n\}$ *such that*
1) $\lim_{n\to\infty} \lambda_{n+1}/\lambda_n \to 1$ *as* $n \to \infty$;
2) setting $c_n^* := \lambda_n c^n$, *we have*

$$\begin{cases} c_n^* \geq |c_n| \ for \ \ every\ n \\ |c_n| = c_n^* \ for\ a\ subsequence\, \Lambda, \end{cases}$$

Set $\Gamma_R := \{z, |z| = R\}$ and $D_R : 0\{z, |z| < R\}$.

It is known that (see [1]), the sequence $\{\pi_{n,m}\}$ converges to the function f, as $n \to \infty$ and m fixed, almost uniformly in σ-content on compact subsets of the disk D_{R_m}. Furthermore (see [5]), the infinite series $\sum A_n z^{n+m+1-\tau_{n,m}}/Q_n(z)Q_{n+1}(z)$ converges and diverges together with the sequence $\{\pi_{n,m}\}_{n=1}^{\infty}$. For the coefficients A_n (comp. (2)) we have (see [5])

$$\limsup_{n\to\infty} |A_n|^{1/n} = 1/R_m. \tag{3}$$

The next equation follows from the definition of the Padé functions (comp. (1))

$$\pi_k(z) - \pi_{k-1}(z) = \frac{A_k z^{k+m}}{Q_k(z)Q_{k-1}(z)}. \tag{4}$$

From here, we arrive at

$$\pi_n(z) = \sum_{k=n_0+1}^{n} (\pi_k(z) - \pi_{k-1}(z)) + \pi_{n_0}(z) = \frac{A_n z^{n+m}}{Q_n(z)Q_{n-1}(z)} + \sum_{k=n_0+1}^{n-1} \frac{A_k z^{k+m}...}{Q_k(z)Q_{k-1}(z)}.$$

In what follows, we will be utilizing the method presented in [6], namely, we apply Kakehashi's lemma to the infinite sequence $\{|A_n|^{1/n}\}$ (comp. (4)).

Consider now the sequence

$$\{\pi_n(z)/A_n z^n\}, n \in \Lambda.$$

Recall that $|A_n| = A_n^*$ for $n \in \Lambda$. We have for $Q_k(z) \neq 0, k = n_0, ..., n$

$$\frac{P_n(z)}{A_n z^n} = Q_n(z)(\frac{A_n z^{n+m}}{A_n Q_{n-1}(z)Q_n(z)z^n} + \sum_{k=n_0-1}^{n-1} \frac{A_k z^{k+m} + ...}{A_n Q_k(z)Q_{k-1}(z)z^n}).$$

Hence

$$|\frac{P_n(z)}{A_n z^n}| \leq |\frac{z^m}{Q_{n-1}(z)}| + |Q_n(z)|(\sum_{k=n_0+1}^{n-1} |\frac{A_k^*}{A_n}||z|^{k-n} \frac{z^m}{Q_k(z)Q_{k-1}(z)}| \leq$$

$$|\frac{z^m}{Q_{n-1}(z)}| + \sum_{k=n_0+1}^{n-1} \frac{\lambda_k}{\lambda_n} \frac{R_m^{n-k}}{|z|^{n-k}} |\frac{z^m Q_n(z)}{Q_k(z)Q_{k-1}(z)}|.$$

Let ε be a fixed positive number. Thanks to the monotonicity of the sequence λ_k/λ_{k+1}, we may write

$$\frac{\lambda_k}{\lambda_{nk+1}} \leq 1 + \varepsilon$$

for all $k \geq \tilde{k}_0$. We assume that $n_0 \geq \tilde{k}_0$. Hence

$$|\frac{P_n(z)}{A_n z^n}| \leq \frac{|z|^m}{|Q_{n-1}(z)|} + \sum (1+\varepsilon)^{n-k} (\frac{R_m}{|z|})^{n-k} |\frac{z^m Q_n(z)}{Q_k(z)Q_{k-1}(z)}|.$$

Select now a number $r \geq R_m(1+\varepsilon)$ such that dist$\mathcal{L}, \Gamma_r) := d > 0$. Then, for $z \in \Gamma_r$,

$$|\frac{P_n(z)}{A_n z^n}| \leq C_1(1 + C_2(\frac{R_m(1+\varepsilon)}{r})^n).$$

Thus the sequence $\{P_n/A_n z^n\}_{n\in\Lambda}$ is uniformly bounded on Γ_r. Having in mind the maximum principle for holomorphic functions and letting now ε tending to zero, we obtain that $\{P_n/A_n z^n\}_{n\in\Lambda}$ is uniformly bounded on compact subsets of $D^c_{R_m}$. Hence, one can extract from each subsequence a subsequence, converging uniformly on compact subsets to a function, holomorphic in $D^c_{R_m}$. Note that any limit function is not the trivial zero.

We select now a subsequence of Λ which we denote again by Λ such that $Q_n \to Q$. The polynomial Q is of degree $\leq m$. Combining this with the previous conclusion, we complete the proof of the theorem.

Following the same way of considerations, one can prove

Theorem 2: *Under the same conditions of f, as in Theorem 1, assume that the set $\overline{\mathbb{C}} \setminus \mathcal{L}$ is connected. Then the statement of Theorem 1 is valid.*

Given now a power series f as before, denote by R_f the radius of meromorphy of f, that is: the radius of the greatest disk, centered at the zero such that f admits a continuation as a meromorphic function.

Let now the sequence of positive integers $\{m_n\}_{n=1}^{\infty}$ satisfy the conditions

$$m_n \leq m_{n+1} \text{ and } m_n = o(n/\ln n) \text{ as } n \to \infty. \quad (5)$$

Set

$$\pi_{n,m_n}(f) := \pi_{n,m_n} := \pi_n = P_n/Q_n$$

and denote by A_n the leading coefficient of P_n. On the base of the results of [7], one can prove

Theorem 3: *Let f be a power series holomorphic at the zero, and suppose that $R_f < \infty$. Assume that f has a multivalued singularity on the circle Γ_{R_f}. Suppose that the sequence $\{m_n\}_{n=1}^{\infty}$ satisfies condition (5). Then, if the set $\overline{\mathbb{C}} \setminus \mathcal{L}$ is connected, then $P_n(z)/A_n z^n$ converges to a holomorphic function χ, $\chi \not\equiv 0$ uniformly on compact subsets of $\overline{D}^c_{R_f}$ at least for a sequence $\Lambda \subset \mathbb{N}$.*

In the formulation of Theorem 3, we supposed, avoiding technical notations, that $P_n(z) = A_n z^n + ...$ with $A_n \neq 0$. Apparently, we are preserving the generality.

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