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Coupled fixed points for Chatterjea type maps with the mixed monotone property in partially ordered metric spaces

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Abstract. We apply generalization of Ekeland's variational principle in partially ordered complete metric spaces on sets defined by maps with the mix monotone property to obtain sufficient conditions for the existence of coupled fixed points for Chatterjea type of maps. If in addition every ordered pair of elements from the underlying space, considered as an element of the produced space endowed with the classical sum metric, has an upper or lower bound then the coupled fixed point is unique one.

INTRODUCTION

Fixed point theorems, initiated by Banach's Contraction Principle has proved to be a powerful tool in nonlinear analysis. We can not mention all kinds of generalizations of Banach's Contraction Principle. One direction for generalization of it is the notion of coupled fixed points [13], where mixed monotone maps in partially ordered by a cone Banach spaces are investigated. Later this idea was developed for mixed monotone maps in partially ordered metric spaces [2]. It is impossible to summarize all generalizations of the ideas of coupled fixed points, for mixed monotone maps, in partially ordered metric spaces. The investigation on the subject continuous as seen [1, 14, 20], which is far from exhausting the most recent results. Another kind of maps considered in partially ordered complete metric spaces are for monotone maps without the mixed monotone property [7, 8, 16, 22].

Another direction is by altering the contraction map conditions. Some classical type of contractive type conditions are Kannan maps [15], Chaterjea maps [5], Zamfirescu maps [23], ets.

The Ekeland's variation principle is well known [9, 10, 11, 12]. It has many generalizations and applications in different fields of Mathematics ([3], [6], [18]).

Let us mention that Ekeland's variational principle holds for any l.s.c maps $T : X \times X \to \mathbb{R}$, provided that X is a partially ordered complete metric space. Unfortunately, when investigating contraction type of maps $F : X \times X \to X$, satisfying the mixed monotone property in a partially ordered complete metric space $X \times X$. the contraction conditions holds only for part of the points $(x, y), (u, v) \in X \times X$. Thus we can not apply Ekeland's variational principle, as it is done in [12]. In [24] Ekeland's variational principle is generalized on classes of subsets of partially ordered complete metric spaces $(X, \preceq)$ with a function $F : X \times X \to X$, which have the mixed monotone property.

We apply the main result from [24] to investigate an existence and uniqueness of coupled fixed points for mixed monotone maps of Chatterjea type in partially ordered metric spaces.

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PRELIMINARIES

Definition 1 *([2, 13]) Let $(X, \preceq)$ be a partially ordered set and let $F : X \times X \to X$. The function F is said to have the mixed monotone property if*

$$\text{for any } x_1, x_2, y \in X \text{ such that } x_1 \preceq x_2 \\ \text{there holds } F(x_1, y) \preceq F(x_2, y)$$

and

$$\text{for any } y_1, y_2, x \in X \text{ such that } y_1 \preceq y_2 \\ \text{there holds } F(x, y_1) \succeq F(x, y_2).$$

Definition 2 *([2, 13]) Let $F : X \times X \to X$. An ordered pair $(x, y) \in X \times X$ is called coupled fixed point of F if $x = F(x, y)$ and $y = F(y, x)$.*

Let $(X, \rho, \preceq)$ be a partially ordered complete metric space. We endow the product space $X \times X$ with the following partial order $(u, v) \preceq (x, y)$, provided that $x \succeq u$ and $y \preceq v$ holds simultaneously and with the following metric $d((x, y), (u, v)) = \rho(x, u) + \rho(y, v)$ for $(x, y), (u, v) \in X \times X$.

Every where for a partially ordered metric space $(X, \rho, \preceq)$ we will consider the product space $(X \times X, d, \preceq)$ endowed with the mentiond above partial order and metric.

Just to fit some of the formulas in the text field we will use the notation $u = \left(u^{(1)}, u^{(2)}\right) \in X \times X$ and for any $u \in X \times X$ let us denote $\overline{u} = \left(u^{(2)}, u^{(1)}\right)$.

Theorem 1 *([24]) Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let*

$$V \times V = \{x = (x^{(1)}, x^{(2)}) \in X \times X : x^{(1)} \preceq F(x) \text{ and } x^{(2)} \succeq F(\overline{x})\} \neq \emptyset.$$

Let $T : X \times X \to \mathbb{R} \cup \{+\infty\}$ be a proper, l.s.c, bounded from below function. Let $\varepsilon > 0$ be arbitrary chosen and fixed and let $u_0 \in V \times V$ be an ordered pair such that the inequality

$$T(u_0) \le \inf_{V \times V} T(v) + \varepsilon \tag{1}$$

holds. Then there exists an ordered pair $x \in V \times V$, such that

(i) $T(x) \le \inf_{u \in V \times V} T(u)$;

(ii) $d(x, u_0) \le 1$;

(iii) *For every $w \in V \times V$ different from $x \in V \times V$ holds the inequality*

$$T(w) > T(x) - \varepsilon d(w, v).$$

Proposition 1 *([24]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $(x, y) \in X \times X$ satisfies the inequalities $x \preceq F(x, y)$, $y \succeq F(y, x)$ and let us put $u = F(x, y)$ and $v = F(y, x)$. Then there hold $u \preceq F(u, v)$, $v \succeq F(v, u)$, $u \succeq x$ and $v \preceq y$.*

Proposition 2 *([24]) Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ be a map with the mixed monotone property. Let $(x, y) \in X \times X$ be a coupled fixed point, i.e. $x = F(x, y)$, $y = F(y, x)$ and let (ξ_0, η_0) be comparable with (x, y). Then (ξ_n, η_n) is comparable with $(x, y) = (F(x, y), F(y, x))$ and (η_n, ξ_n) is comparable with $(y, x) = (F(y, x), F(x, y))$.*

COUPLED FIXED POINTS FOR CHATTERJEA TYPE MAPS

Theorem 2 *Let $(X, \rho, \preceq)$ be a partially ordered complete metric space, $(X \times X, d, \preceq)$ and $F : X \times X \to X$ be a continuous map with the mixed monotone property. Let there exists $\alpha \in [0, 1/2)$, so that the inequality*

$$\rho(F(x, y), F(u, v)) \le \alpha\rho(x, F(u, v)) + \alpha\rho(u, F(x, y)) \tag{2}$$

holds for all $x \succeq u$ and $y \preceq v$. If there exists at least one ordered pair (x, y), such that $x \preceq F(x, y)$ and $y \succeq F(y, x)$, then there exists a coupled fixed point (x, y) of F.

Proof. It is well known that $\frac{\alpha}{1-\alpha} \in [0,1)$ for any $\alpha \in [0,1/2)$. Let us consider the function $T : X \times X \to \mathbb{R}$, defined by

$$T(z) = d(z,(F(z),F(\overline{z})) = \rho(x,F(x,y)) + \rho(y,F(y,x)),$$

where $z = (x,y) \in X \times X$. The map T satisfies the conditions of Theorem 1, as far as T is continuous, proper function, bounded from below and the set of all $z \in X \times X$, such that $x \preceq F(z)$ and $y \succeq F(\overline{z})$ is not empty. Let us choose $\varepsilon \in (0, 1-\alpha)$. By Theorem 1 there exists (x,y), satisfying $x \preceq F(x,y)$ and $y \succeq F(y,x)$, such that there holds the inequality

$$T(x,y) \le T(u,v) + \varepsilon d((x,y),(u,v)) \tag{3}$$

for every $u \preceq F(u,v)$ and $v \succeq F(v,u)$.

Let us put $u = F(x,y)$, $v = F(y,x)$ and $w = (u,v)$. By Proposition 1 it follows that $u \preceq F(u,v)$, $v \succeq F(v,u)$, $u \succeq x$ and $v \preceq y$. From (2) using the symmetry of $\rho(\cdot,\cdot)$ we obtain

$$\begin{array}{rcl} \rho(F(x,y),F(F(x,y),F(y,x))) & = & \rho(F(F(x,y),F(y,x)),F(x,y)) \\ & \le & \alpha\rho(F(x,y),F(x,y)) + \alpha\rho(x,F(F(x,y),F(y,x))) = 0 + \alpha\rho(x,F(F(x,y),F(y,x))) \\ & \le & \alpha\rho(x,F(x,y)) + \alpha\rho(F(x,y),F(F(x,y),F(y,x))), \end{array}$$

because $F(x,y) \succeq x$ and $F(y,x) \preceq y$ and thus

$$\rho(F(x,y),F(F(x,y),F(y,x))) \le \frac{\alpha}{1-\alpha}\rho(x,F(x,y)). \tag{4}$$

Similarly from (2) we get

$$\begin{array}{rcl} \rho(F(y,x),F(F(y,x),F(x,y))) & \le & \alpha\rho(y,F(F(y,x),F(x,y))) + \alpha\rho(F(y,x),F(y,x)) \\ & = & \alpha\rho(y,F(F(y,x),F(x,y))) + 0 \le \alpha\rho(y,F(y,x)) + \alpha\rho(F(y,x),F(F(y,x),F(x,y))) \end{array}$$

and consequently

$$\rho(F(y,x),F(F(y,x),F(x,y))) \le \frac{\alpha}{1-\alpha}\rho(y,F(y,x)). \tag{5}$$

From (4) and (5) we obtain

$$\begin{array}{rcl} T(w) & = & d(w,(F(w),F(\overline{w})) = \rho(F(x,y),F(F(x,y),F(y,x))) + \rho(F(y,x),F(F(y,x),F(x,y)))) \\ & \le & \frac{\alpha}{1-\alpha}\left(\rho(x,F(x,y)) + \rho(y,F(y,x))\right) = \frac{\alpha}{1-\alpha}T(x,y) \end{array} \tag{6}$$

Consequently from (4) using (6) we get

$$T(x,y) \le T(u,v) + \varepsilon d((x,y),(u,v)) \le \frac{\alpha}{1-\alpha}T(x,y) + \varepsilon T(x,y) = \left(\frac{\alpha}{1-\alpha} + \varepsilon\right)T(x,y).$$

From the choice of $\varepsilon \in \left(0, 1 - \frac{\alpha}{1-\alpha}\right)$ we obtain $T(x,y) < T(x,y)$ From the last inequality it follows that $T(x,y) = d((x,y),(F(x,y),F(y,x))) = 0$, i.e. $\rho(x,F(x,y)) + \rho(y,F(y,x)) = 0$. Therefore (x,y) is a coupled fixed points of F.

Let there are two coupled fixed points $(x,y),(u,v) \in X \times X$, then $x = F(x,y)$, $y = F(y,x)$, $u = F(u,v)$ and $v = F(v,u)$. By the assumption that any element has an lower or an upper bound it follows from [19] that there exists (ξ_0,η_0) comparable with (x,y) and (u,v). From Proposition 2 it follows that (ξ_n,η_n) is comparable with both $(x,y) = (F(x,y),F(y,x))$ and $(u,v) = (F(u,v),F(v,u))$ and (η_n,ξ_n) is comparable with both (y,x) and (v,u).

We will apply inequality (2). If $(\xi_n,\eta_n) \succeq (x,y)$, then it satisfies the assumptions of the theorem.

If $(\xi_n,\eta_n) \preceq (x,y)$, using the symmetry of the metrics ρ we get

$$\rho(F(\xi_n,\eta_n),F(x,y)) = \rho(F(x,y),F(\xi_n,\eta_n)) \le \alpha\rho(x,F(\xi_n,\eta_n)) + \alpha\rho(\xi_n,F(x,y)).$$

Consequently we can apply (2) when (ξ_n,η_n) is comparable with

$$(F(x,y),F(y,x)).$$

There exists $n_0 \in \mathbb{N}$, such that $\left(\frac{\alpha}{1-\alpha}\right)^{n_0} < \dfrac{\rho(\xi_0,x) + \rho(\eta_0,y) + \rho(\xi_0,u) + \rho(\eta_0,v)}{\rho(x,u) + \rho(y,v)}$.

Let us denote $I_n = \rho(\xi_n, x)$ and $J_n = \rho(\eta_n, y)$. Use inequality (2) we get that

$$I_n = \rho(\xi_n, x) = \rho(F(\xi_{n-1}, \eta_{n-1}), F(x, y)) \leq \alpha\rho(\xi_{n-1}, F(x, y)) + \alpha\rho(\xi_{n-1}, F(x, y)) = \alpha\rho(\xi_{n-1}, x) + \alpha\rho(x, \xi_n)$$

and

$$J_n = \rho(\eta_n, y) = \rho(F(\eta_{n-1}, \xi_{n-1}), F(y, x)) \leq \alpha\rho(\eta_{n-1}, F(y, x)) + \alpha\rho(y, F(\eta_{n-1}, \xi_{n-1})) = \alpha\rho(\eta_{n-1}, y) + \alpha\rho(y, \eta_n).$$

Summing the last two inequalities we obtain

$$I_n + J_n \leq \alpha\left(\rho(\xi_{n-1}, x) + \rho(\eta_{n-1}, y) + \rho(x, \xi_n) + \rho(y, \eta_n)\right) = \alpha(I_{n-1} + J_{n-1}) + \alpha(I_n + J_n).$$

Consequently $I_n + J_n \leq \frac{\alpha}{1-\alpha}(I_{n-1} + J_{n-1})$ and thus

$$I_n + J_n \leq \left(\frac{\alpha}{1-\alpha}\right)^n (\rho(\xi_0, x) + \rho(\eta_0, y)).$$

Then we obtain

$$\begin{array}{rcl} \rho(x, u) + \rho(y, v) & \leq & \rho(x, \xi_{n_0}) + \rho(\xi_{n_0}, u) + \rho(y, \eta_{n_0}) + \rho(\eta_{n_0}, v) \\ & \leq & \left(\frac{\alpha}{1-\alpha}\right)^{n_0} (\rho(\xi_0, x) + \rho(\eta_0, y) + \rho(\xi_0, u) + \rho(\eta_0, v)) < \rho(x, u) + \rho(y, v), \end{array}$$

which is a contradiction and that $(x, y) = (u, v)$. □

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