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Common Fixed Point Results on Modular $\mathcal{F}$-Metric Spaces

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Abstract. Jleli and Samet[1] introduced a new concept, named a $\mathcal{F}$-metric space, as a generalization of the notion of metric space. We define new generalization of modular metric space as modular $\mathcal{F}$-metric space. We compare the topology produced by modular metric and by modular $\mathcal{F}$-metric, then cover some useful properties of this topology for fixed point theorems for future studies. In the end, we prove Banach contraction principle for modular $\mathcal{F}$-metric space.
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INTRODUCTION

Generalization of metric structure by using many different ways is always been interesting for many authors. Recently we defined generalized modular metric on any abstract set and gave some examples about its connection with other metrics[2]. Now we use f functions and $\mathcal{F}$-metric definition to generalize modular metric spaces and give its own topology.

Let $\mathcal{F}$ be the set of functions $f : (0, \infty) \to \mathbb{R}$ satisfying the following conditions:

($\mathcal{F}_1$) f is non-decreasing, i.e., $0 < s < t$ implies $f(s) \leq f(t)$.
($\mathcal{F}_2$) For every sequence $\{t_n\} \subset (0, \infty)$, we have $\lim_{n\to\infty} t_n = 0 \iff \lim_{n\to\infty} f(t_n) = -\infty$.

Jleli and Samet[1] gave the definiton of $\mathcal{F}$-metric space as below by using f function:

Definition 1 *Let X be a nonempty set and $D : X \times X \to [0, \infty)$ be a function. If there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, such that*

(F1) $D(x, y) = 0 \iff x = y$, *for all* $(x, y) \in X \times X$;
(F2) $D(x, y) = D(y, x)$ *for all* $(x, y) \in X \times X$;
(F3) *For all* $(x, y) \in X \times X$, $p \in \mathbb{N}$ *with* $p \geq 2$ *and for all* $(v_i)_{i=1}^{p} \subset X$ *with* $(v_1, v_p) = (x, y)$, *we have*

$$D(x, y) > 0 \;\; implies \;\; f(D(x, y)) \leq f\left(\sum_{j=1}^{p-1} D(v_j, v_{j+1})\right) + \alpha,$$

then F is called an $\mathcal{F}$-metric on X. The pair (X, D) is called an $\mathcal{F}$-metric space.

We start with the definition of modular $\mathcal{F}$-metric as more generalized version of metric, modular metric and $\mathcal{F}$-metric.

Definition 2 *Let X be a nonempty set and $F_\lambda : (0, \infty) \times X \times X \to [0, \infty]$ be a function. If there exists $(f, \alpha) \in \mathcal{F} \times [0, \infty)$, such that*

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(F1) $F_\lambda(x,y) = 0 \iff x = y$, *for all* $(x,y) \in X \times X$;
(F2) $F_\lambda(x,y) = F_\lambda(y,x)$ *for all* $(x,y) \in X \times X$;
(F3) *For all* $(x,y) \in X \times X$, $p \in \mathbb{N}$ *with* $p \geq 2$ *and for all* $(v_i)_{i=1}^p \subset X$ *with* $(v_1, v_p) = (x,y)$, *we have;*

$$F_\lambda(x,y) > 0 \;\; implies \;\; f(F_\lambda(x,y)) \leq f\left(\sum_{j=1}^{p-1} F_{\frac{\lambda}{j}}(v_j, v_{j+1})\right) + \alpha,$$

then F_λ *is called an modular* $\mathcal{F}$*-metric on X. The pair* (X, F_λ) *is called an modular* $\mathcal{F}$*-metric space.*

Now, we start to built a topology on modular $\mathcal{F}$-metric space by definitions and theorems step by step. Very nice work will lead idea of building topology in this section[3]. The first step is giving boundary conditions for modular $\mathcal{F}$-metric.

Definition 3 *Let X be a nonempty set and* $F_\lambda : (0,+\infty) \times X \times X \to [0,\infty)$ *be a mapping that satisfies* $(F1)$ *and* $(F2)$ *with recpect to* $(f,\alpha) \in \mathcal{F} \times [0,\infty)$*, than we say that* (X, F_λ) *is modular* $\mathcal{F}$*-metric bounded space, if there is a metric* d_ω *on X such that;*

$$(x,y) \in X \times X, F_\lambda(x,y) > 0 \;\; implies \;\; f(d_\omega(x,y)) \leq f(F_\lambda(x,y)) \leq f(d_\omega(x,y)) + \alpha.$$

If we have modular $\mathcal{F}$-metric with (f,α), then we have bounded modular $\mathcal{F}$-metric with (f,α) as below:

Theorem 1 *Let X be a nonempty set, and let* $F_\lambda : (0,+\infty) \times X \times X \to [0,\infty)$ *e a given mapping satisfying* $(F1)$ *and* $(F2)$*. Let* $(f,\alpha) \in \mathcal{F} \times [0,\infty)$ *and suppose that f is continuous from the right. Then the following statements are equivalent:*

(i) (X, F_λ) *is an modular* $\mathcal{F}$*-metric on X with* (f,α) *defined above.*
(ii) (X, F_λ) *is an modular* $\mathcal{F}$*-metric bounded on X with recpect to* (f,α)*.*

Very important to define what is open(closed) set(ball) on a topology. Which conditions are necessary for this definition and if we have any set how to determine a closure of a set, how to find a convergent sequence in the space, and so on.

Definition 4 *Let* (X, F_λ) *is a modular* $\mathcal{F}$*-metric space and let* $U \subset X$ *is a nonempty subset on it.* $U \subset X$ *is called modular* F_λ*-open, if for every* $x \in U$*, there is some* $r > 0$ *such that* $B_{F_\lambda}(x,r) \subset U$ *where* $B_{F_\lambda}(x,r) = \{y \in X : F_\lambda(x,y) < r\}$*. Then, if* $X - U = F$ *is modular* F_λ*-open subset,* $F \subset X$ *is called modular* F_λ*-closed. For the family of all modular* F_λ*-open subsets of X we determine* τ_{F_λ}*.*

Proposition 1 *Let* (X, F_λ) *be a modular* $\mathcal{F}$*-metric space. Then* τ_{F_λ} *is a topology on X.*

Proposition 2 *Let* (X, F_λ) *be a modular* $\mathcal{F}$*-metric space. Then, for any nonempty* $A \subset X$*, the following statements are equivalent:*

(i) *A is modular* F_λ*-closed.*
(ii) *For any sequence* $\{x_n\}_{n\in\mathbb{N}} \subset A$ *we have* $\lim_{n\to\infty} F_\lambda(x_n, x) = 0, x \in X$ *implies* $x \in A$*.*

Definition 5 *Let* (X, F_λ) *be a modular* $\mathcal{F}$*-metric space. Let a nonempty subset* $A \subset X$*. We denote by* $\overline{A}$ *the closure of A with respect to the topology* τ_{F_λ}*, i.e.* $\overline{A}$ *is the intersection of all modular* $\mathcal{F}$*-metric closed subsets of X containing A. Clearly, A is the smallest modular* $\mathcal{F}$*-metric closed subset which contains itself.*

Proposition 3 *Let* (X, F_λ) *be a modular* $\mathcal{F}$*-metric space. Then, for any nonempty subset* $A \subset X$*, we have* $x \in \overline{A}, r > 0$ *implies* $B_{F_\lambda}(x,r) \cap A \neq \emptyset$*.*

Definition 6 *Let* (X, F_λ) *be a modular* $\mathcal{F}$*-metric space. Let* $\{x_n\} \subset X$ *be a sequence in X. We say that* $\{x_n\} \subset X$ *is modular* $\mathcal{F}$*-convergent to point* $x \in X$*, if* $\{x_n\} \subset X$ *is convergent to x with respect to the topology* τ_{F_λ}*, i.e. for every modular* $\mathcal{F}$*-open subset* $U_x \subset X$ *containing x,there exists some* $n \geq N$*,* $N \in \mathbb{N}$ *such that* $x_n \in U_x$*, for all n. Then,* $\lim_{n\to\infty} F_\lambda(x_n, x) = 0, x \in X$ *implies x is the limit of* $\{x_n\} \subset X$*.*

Proposition 4 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $\{x_n\} \subset X$ be a sequence in X, and $x \in X$. The following statements are equivalent:*

(i) $\{x_n\} \subset X$ *modular $\mathcal{F}$-convergent to $x \in X$.*
(ii) $\lim_{n\to\infty} F_\lambda(x_n, x) = 0.$

Now, let us define modular $\mathcal{F}$-Cauchy sequence, then completeness definition and conditions for modular $\mathcal{F}$-metric space.

Definition 7 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $\{x_n\} \subset X$ be a sequence in X.*

(i) *We say that $\{x_n\} \subset X$ is modular $\mathcal{F}$-Cauchy , if* $\lim_{n,m\to\infty} F_\lambda(x_n, x_m) = 0.$
(ii) *We say that (X, F_λ) modular $\mathcal{F}$-complete,if every modular $\mathcal{F}$-Cauchy sequence is modular $\mathcal{F}$-convergent to a certain element in X.*

Proposition 5 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. If $\{x_n\} \subset X$ is modular $\mathcal{F}$-convergent, then it is modular $\mathcal{F}$-Cauchy.*

The definiton of modular $\mathcal{F}$-compact set is giving more details about the topology on modular $\mathcal{F}$-metric space.

Definition 8 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $C \subset X$ be a nonempty subset. We say that C is modular $\mathcal{F}$-compact if C is compact with respect to the topology τ_{F_λ} on X.*

Proposition 6 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let C be a nonempty subset of X. Then, the following statements are equivalent:*

(i) *C is modular $\mathcal{F}$-compact.*
(ii) *For any sequence $\{x_n\} \subset C$ there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x \in A$ such that;*

$$\lim_{k\to\infty} F_\lambda(x_{n_k}, x) = 0$$

.

Definition 9 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $C \subset X$ be a nonempty subset. The subset C is called modular sequentially $\mathcal{F}$-compact, if for any sequence $\{x_n\} \subset C$, there exist a subsequence $\{x_{n_k}\} \subset \{x_n\}$ and $x \in C$ such that*

$$\lim_{k\to\infty} F_\lambda(x_{n_k}, x) = 0.$$

Definition 10 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $C \subset X$ be a nonempty subset. The subset C is called modular $\mathcal{F}$-totally bounded, if for any $r > 0$ there exists a sequence (x_j), $j = 1, 2, .., n \subset C$ such that $C \subset \bigcup B_{F_\lambda}(x_j, r)$.*

Proposition 7 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space. Let $C \subset X$ be a nonempty subset. Then*

(i) *C is modular $\mathcal{F}$-compact $\iff$ C is modular sequentially $\mathcal{F}$-compact.*
(ii) *C is modular $\mathcal{F}$-compact $\Rightarrow$ C is modular $\mathcal{F}$-totally bounded.*

Results

In this section, we give Banach Contraction Principle(BCP) and make clear fixed point conditions as below.

Theorem 2 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space, and let $T : X \to X$ be a given mapping. Suppose that the following conditions are satisfied:*

(i) *(X, F_λ) is modular $\mathcal{F}$-complete.*
(ii) *There exists $k \in (0, 1)$ such that;*

$$F_\lambda(T(x), T(y)) \leq k\, F_\lambda(x, y), \ \ (x, y) \in X \times X,$$

Then T has a unique fixed point $x^\star \in X$. Moreover, for any $x_0 \in X$ the sequence $\{x_n\} \subset X$ defined by $x_{n+1} = T(x_n)$, then it is modular $\mathcal{F}$-convergent to $x^\star$.

Proof 1 *First, observe that T has at most one fixed point. Indeed, if $(u, v) \in X \times X$ are two fixed points of T with $u \neq v$, i.e. $F_\lambda(u, v) > 0, T(u) = u, T(v) = v$,then from (ii), we have*

$$F_\lambda(u, v) = F_\lambda(T(u), T(v)) \leq kF_\lambda(u, v) < F_\lambda(u, v),$$

which is a contradiction. Next, let $(f, \alpha) \in F \times [0, +\infty)$ be such that $\mathcal{F}_3$ is satisfied. Let $\epsilon > 0$ be fixed. By $(F2)$, there exists $\delta > 0$ such that $0 < t < \delta$ implies $f(t) < f(\epsilon) - \alpha \ \ldots (\S)$. Let $x_0 \in X$ be an arbitrary element. Let $\{x_n\} \subset X$ be the sequence defined by . Without restriction of the generality, we may suppose that $F_\lambda(x_0, x_1) > 0$. Otherwise, x_0 will be a fixed point of T. It can be easily seen that from (ii), we have $F_\lambda(x_n, x_{n+1}) \leq k_n F_\lambda(x_0, x_1)$, which yields $\sum_{i=n}^{m-1} F_\lambda(x_i, x_{i+1}) \leq \frac{k^n}{1-k} F_\lambda(x_0, x_1), m > n$. Since $\lim_{n\to\infty} \frac{k^n}{1-k} F_\lambda(x_0, x_1) = 0$, we have $0 < \frac{k^n}{1-k} F_\lambda(x_0, x_1) < \delta, n \geq N, n \in \mathbb{N}$. Hence, by$(\S)$ and (F1), we have

$$f\left(\sum_{i=n}^{m-1} F_\lambda(x_i, x_{i+1})\right) \leq f\left(\frac{k^n}{1-k} F_\lambda(x_0, x_1)\right) < f(\epsilon) - \alpha.$$

Using (D3) we obtain $F_\lambda(x_n, x_m) > 0, m > n \geq N$, then

$$f(D(x_n, x_m)) \leq F_\lambda(x_i, x_{i+1}) \leq f(F_\lambda(x_0, x_1)) < f(\epsilon) - \alpha, m > n \geq N.$$

which implies by (F1) that $D(x_i, x_{i+1}) + \alpha < f(\epsilon)$ then $D(x_n, x_m) < \epsilon, m > n \geq N$.
This proves that $\{x_n\}$ is $\mathcal{F}$-Cauchy. Since (X, F_λ) is $\mathcal{F}$-complete, there exists $x^\star \in X$ such that $\{x_n\}$ is $\mathcal{F}$-convergent to $x^\star$, i.e. $\lim_{n\to\infty} F_\lambda(x_n, x^\star) = 0 \ldots (\ddagger)$. We shall prove that $x^\star$ is a fixed point of T. We argue by contradiction by supposing that $F_\lambda(T(x^\star), x^\star) > 0$. By (F3), we have

$$f(F_\lambda(T(x^\star), x^\star)) \leq f(F_\lambda(T(x^\star), T(x_n)) + F_\lambda(T(x_n), x^\star)) + \alpha,$$

Using (ii) and (F1), we obtain

$$f(F_\lambda(T(x^\star), x^\star)) \leq f(kF_\lambda(x^\star, x_n) + F_\lambda(x_{n+1}, x^\star)) + \alpha,$$

On the other hand, using (F2) and $(\ddagger)$, we have

$$\lim_{n\to\infty} f(kF_\lambda(x^\star, x_n) + F_\lambda(x_{n+1}, x^\star)) + \alpha = -\infty, n \in \mathbb{N}$$

which is a contradiction. Therefore, we have $F_\lambda(T(x^\star), x^\star) = 0$, i.e. $T(x^\star) = x^\star$. As a consequence, $x^\star \in X$ is the unique fixed point of T.

Corollary 1 *Let (X, F_λ) be a modular $\mathcal{F}$-metric space, and $(f, \alpha) \in \mathcal{F} \times [0, \infty)$ be such that $(F3)$ is satisfied. Let $T : B_{F_\lambda}(x_0, r) \to X$ be a given mapping, where $x_0 \in X$ and $r > 0$. Suppose that the following conditions are satisfied:*

(i) *For any sequence $\{x_n\} \subset X$;*

$$\lim_{n\to\infty} F_\lambda(x_n, x) = 0, x \in X \Rightarrow F_\lambda(x, y) \leq \lim_{n\to\infty} \sup F_\lambda(x_n, y), y \in X.$$

(ii) *(X, F_λ) is modular $\mathcal{F}$-complete.*

(iii) *There exists $k \in (0, 1)$ such that;*

$$F_\lambda(T(x), T(y)) \leq k\ F_\lambda(x, y),\ (x, y) \in B_{F_\lambda}(x_0, r) \times B_{F_\lambda}(x_0, r).$$

(iv) *There exists $0 < \varepsilon < r$ such that;*

$$f(k\varepsilon + F_\lambda(x_0, T(x_0))) \leq f(\varepsilon) - \alpha$$

Then T has a fixed point.

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