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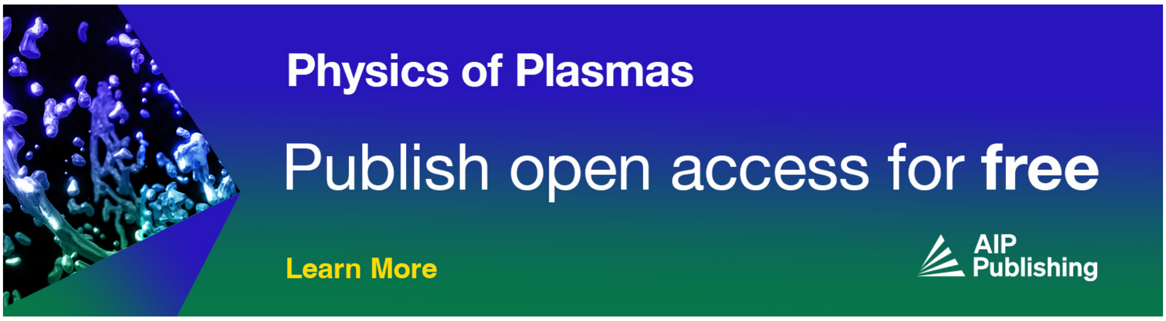

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Fractional-dimensional Child-Langmuir law for a rough cathode

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This work presents a self-consistent model of space charge limited current transport in a gap combined of free-space and fractional-dimensional space (F^{α}), where α is the fractional dimension in the range $0 < \alpha \leq 1$. In this approach, a closed-form fractional-dimensional generalization of Child-Langmuir (CL) law is derived in classical regime which is then used to model the effect of cathode surface roughness in a vacuum diode by replacing the rough cathode with a smooth cathode placed in a layer of effective fractional-dimensional space. Smooth transition of CL law from the fractional-dimensional to integer-dimensional space is also demonstrated. The model has been validated by comparing results with an experiment. *Published by AIP Publishing.*
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I. INTRODUCTION

The classical Child-Langmuir (CL) law[1,2] gives the maximum current density allowed for steady-state electron flow across a planar gap in terms of gap spacing and gap voltage. Due to contemporary needs on the studies of nano-scale devices, the one-dimensional (1D) classical CL law has been extended to various regimes, including quantum regime.[3–5] The 1D CL law has also been extended to include other effects, such as multi-dimensional models,[6–10] single electron regimes, short pulse limit,[11–13] single-electron limit,[14] and new scaling in other geometries.[15] Most of the space charge limited (SCL) current models do not consider the effect of imperfection or roughness of the cathode surface in vacuum diodes. In the devices where the quality of high current electron beam is important, the effects of roughness may no longer be neglected. In theory, the study of these effects requires rigorous computations due to irregular boundary conditions in the solution of governing equations. Thus, in this context, a simplified effective model of the SCL current with low complexity would be of particular interest to characterize the amount and quality of electron beam by the order of irregularity of the cathode surface.

There is an increasing interest in fractional order modeling of complexity in physical systems.[16,17] Recently, the concept of fractional-dimensional space has been used as an effective physical description of restraint conditions in complex physical systems.[18–20] The approaches to describe the fractional dimensions include fractal geometry,[21] fractional calculus,[22,23] and the integration over fractional-dimensional space.[24,25] The axiomatic basis of spaces with fractional dimension had been introduced by Stillinger,[24] where he described the integration on a space with non-integer dimension, and provided a generalization of second order Laplace operators. This approach has been widely applied in quantum field theory,[18,26,27] general relativity,[28] thermodynamics,[29] mechanics,[30–32] hydrodynamics,[33] and electrodynamics.[20,34–43] To expand the range of possible applications of models with fractional-dimensional spaces, a complete generalization of vector calculus operators has been reported recently.[19,20] The fractional-dimensional space generalization of vector calculus operators allows us to describe the complex problem of SCL current involving devices with rough surface cathodes by replacing such complexities with an effective system embedded in α-dimensional fractional space, where the fractional dimension α is the measure of complexity in the real system.

In what follows, after an introduction to vector calculus in fractional-dimensional space, we will derive the closed form fractional-dimensional generalization of 1D classical CL law and its application to study the SCL current enhancement due to cathode surface roughness. In order to validate the presented model, we will compare the calculated SCL enhancement factor due to rough surface cathode with the results reported in an experiment. A smooth transition of fractional dimensional CL law to integer-dimensional scaling will also be demonstrated.

II. VECTOR CALCULUS IN FRACTIONAL-DIMENSIONAL SPACE

In Stillinger's work,[24] only the second-order scalar Laplace operator for fractional-dimensional space is suggested. The fractional-dimensional generalization of the first order Laplace operators was then reported by Zubair[20] as approximations of the square of the fractional-dimensional Laplace operator given in the literature.[18,24] Recently, a complete generalization of the first and second order Laplace operators is proposed by Tarasov,[19] which is summarized in the following. In fractional-dimensional space ($F^{\alpha} \subseteq E^{n}$), it is convenient to work with physically dimensionless space variables $x/R_0 \rightarrow x$, $y/R_0 \rightarrow y$, $z/R_0 \rightarrow z$, $\mathbf{r}/R_0 \rightarrow \mathbf{r}$, where R_0 is a characteristic size of the considered model. This provides dimensionless integration and dimensionless differentiation in α-dimensional space which leads to correct physical

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[b)] Electronic mail: ricky_ang@sutd.edu.sg dimensions of quantities. We can define a differential operator that takes into account the density of states $c(\alpha_k, x_k)$ by

$$\partial_{\alpha_k,x_k} = \frac{\partial}{\partial X_k} = \frac{1}{c(\alpha_k,x_k)}\frac{\partial}{\partial x_k}, \tag{1}$$

where $c(\alpha_k, x_k)$ corresponds to the non-integer dimensionality along the X_k-axis and it is defined by[19]

$$c(\alpha_k,x_k) = \frac{\pi^{\alpha_k/2}}{\Gamma(\alpha_k/2)}|x_k|^{\alpha_k-1}. \tag{2}$$

Note that these derivatives cannot be considered as derivatives of the non-integer order (also called fractional derivatives). The operators in Eq. (1) are usual differential operators of the first order that are defined on differentiable functions in $\mathbb{R}^3$. Using these operators, we can generalize vector differential operators in an α-dimensional space. The gradient of a scalar function $\varphi(\mathbf{r})$ in fractional-dimensional space is the vector field

$$\nabla_\alpha \varphi(\mathbf{r}) = \sum_{k=1}^{3} \mathbf{e}_k \partial_{\alpha_k,x_k}\varphi(\mathbf{r}), \tag{3}$$

where $\mathbf{e}_k$ are unit base vectors of the Cartesian coordinate system. The divergence of the vector field $\mathbf{f}(\mathbf{r}) = \mathbf{e}_k f_k(\mathbf{r})$ is

$$\nabla_\alpha \cdot \mathbf{f}(\mathbf{r}) = \sum_{k=1}^{3} \partial_{\alpha_k,x_k} f(\mathbf{r}). \tag{4}$$

The curl for the vector field $\mathbf{f}(\mathbf{r})$ is

$$\nabla_\alpha \times \mathbf{f}(\mathbf{r}) = \sum_{k,i,l=1}^{3} \mathbf{e}_i \varepsilon_{ikl} \partial_{\alpha_k,x_k} f(\mathbf{r}), \tag{5}$$

where ε_{ikl} is the Levi-Civita symbol. Using Eqs. (3) and (4), the scalar Laplacian in the fractional-dimensional-space has the form[19]

$$\begin{aligned}\nabla_\alpha^2 \varphi(\mathbf{r}) &= \nabla_\alpha \cdot \nabla_\alpha \varphi(\mathbf{r}) \\ &= \sum_{k=1}^{3} \frac{1}{c^2(\alpha_k,x_k)}\left(\frac{\partial^2}{\partial x_k^2} - \frac{\alpha_k - 1}{x_k}\frac{\partial}{\partial x_k}\right).\end{aligned} \tag{6}$$

These generalized differential operators allow us to describe complexity, anisotropy, inhomogeneity, roughness, or disorder in the framework of continuum models with fractional spatial dimensions (e.g., see Refs. 19, 20, 25, and 41 and references therein).

III. FRACTIONAL-DIMENSIONAL GENERALIZATION OF THE CHILD-LANGMUIR LAW

Given a simple infinite parallel plate diode in fractional-dimensional space (F^α) with $0 < \alpha \leq 1$. In F^α, the magnitude of the electric field E in the diode is given in terms of potential V as[20]

$$E = -\nabla_\alpha V(x) = -\frac{1}{c(\alpha,x)}\frac{dV(x)}{dx}, \tag{7}$$

with

$$c(\alpha,x) = \frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}|x|^{\alpha-1}. \tag{8}$$

From the energy conservation of electrons, we get

$$v(x) = \sqrt{\frac{2eV(x)}{m}}, \tag{9}$$

where m and v are the respective mass and velocity of electrons and e is the elementary charge. We can also write down the Poisson's equation in fractional-dimensional space[19,20]

$$\nabla_\alpha^2 V(x) = \frac{-\rho(x)}{\epsilon_0}, \tag{10}$$

where ∇_α^2 is the Laplacian in fractional-dimensional space defined in Eq. (6) as

$$\nabla_\alpha^2 = \frac{1}{c^2(\alpha,x)}\left(\frac{d^2}{dx^2} - \frac{\alpha-1}{x}\frac{d}{dx}\right), \tag{11}$$

where ϵ_0 is the permittivity of free space and ρ is the charge density. Now, if we write J in terms of ρ and v and note that $J(x)$ is constant in steady state, $J(x) = \rho(x)v(x) = -J$. We combine Eqs. (9) and (10) to get the differential equation

$$\frac{d^2V}{dx^2} - \frac{\alpha-1}{x}\frac{dV}{dx} = \gamma x^{2(\alpha-1)}V^{-1/2}, \tag{12}$$

where

$$\gamma = \left[\frac{\pi^{\alpha/2}}{\Gamma(\alpha/2)}\right]^2 \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}. \tag{13}$$

Equation (12) is a modified Emden-Fowler equation,[44] which can be reduced to Emden-Fowler equation under substitution $z = x^\alpha$

$$\frac{d^2V}{dz^2} = \left(\frac{\sqrt{\gamma}}{\alpha}\right)^2 V^{-1/2}. \tag{14}$$

The system is solved with the boundary conditions, $V(0) = 0$, $V(L) = V_0$, where V_0 is the voltage applied to the diode and L is the electrode separation which, to get the solution as

$$z = \frac{2}{3}\frac{\alpha}{\sqrt{\gamma}}V^{3/4}, \tag{15}$$

and after back substitution, leads to the following limiting current at $x = L$

$$J(\alpha) = \frac{4\epsilon_0}{9}\alpha^2\left[\frac{\Gamma(\alpha/2)}{\pi^{\alpha/2}}\right]^2\sqrt{\frac{2e}{m}}\frac{V_0^{3/2}}{L^{2\alpha}}. \tag{16}$$

Equation (16) is the fractional-dimensional Child-Langmuir (CL) law with varying dimension $0 < \alpha \leq 1$.

For $\alpha = 1$, Eq. (16) reduces to the Child-Langmuir (CL) law in integer dimensional space[1,2]

$$J(1) = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}}\frac{V_0^{3/2}}{L^2}. \tag{17}$$

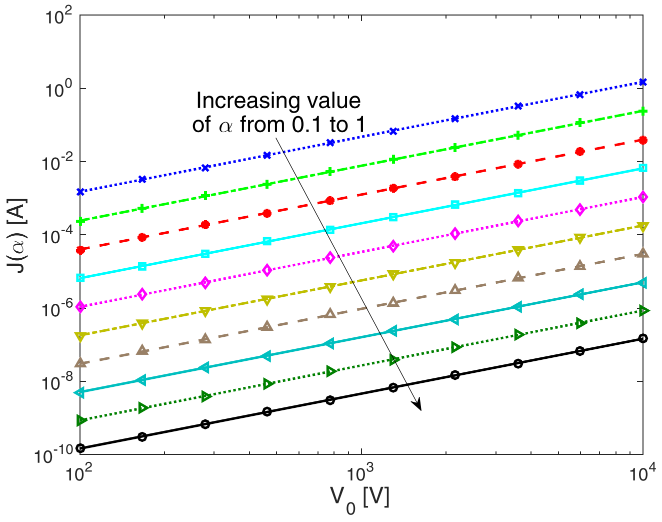

FIG. 1. SCL current versus applied voltage for vacuum diode embedded in fractional-dimension space of varying dimensions.

Consider a vacuum diode with fixed electrode separation L and applied voltage V_0 embedded in F^{α} with $0 < \alpha \leq 1$, where $\alpha = 1$ corresponds to the standard CL law. The SCL current $J(\alpha)$ for this vacuum diode is plotted in Fig. 1 as calculated from Eq. (16). The increasing value of α corresponds to decreasing surface roughness of the cathode. This plot shows the qualitative behavior of SCL current enhancement with increasing surface roughness. It is also clear that the voltage scaling of CL law remains unchanged in F^{α}.

IV. MODELING SCL CURRENT ENHANCEMENT DUE TO A ROUGH CATHODE

Most cathodes for vacuum diodes used in practical applications have nonuniform or rough surfaces. The emitter and the space charge effects near the cathode are usually combined into a so-called virtual emitter by making use of the analytical one-dimensional models of Child or Langmuir.[1,2] Practical diode geometries invariably violate the one dimensionality of the space charge models, for instance, due to the presence of a cathode roughness. An accurate study of the effects of surface roughness requires, at a minimum, a two-dimensional solution of a Child-Langmuir type over a rough surface.[45] Such a solution can reflect the self-consistency between charge distribution and electric field distribution, and an analytic solution does not seem to have been constructed. We propose an effective model to study the SCL current due to cathode surface roughness in a planar vacuum diode with gap L by replacing the rough cathode with a planar cathode placed in a layer of effective α-dimensional space with width x_1 where the fractional dimension α corresponds to the degree of cathode surface roughness. To construct such an effective model, we consider a gap consisting of a fractional-dimensional space region ($x = 0$ to $x = x_1$) and a free-space region ($x = x_1$ to $x = L$). The electrons are injected from the grounded cathode at $x = 0$ to the anode at $x = L$ with an applied voltage V_0 (see Fig. 2). The SCL current in the fractional-dimensional space region, according to the fractional-dimensional CL law derived in Eq. (16), gives

$$V_{x_1} = C x_1^{4/3\alpha}, \tag{18}$$

where

$$C = \left(\frac{3}{2\alpha}\right)^{1/3} \left[\frac{\pi^{\alpha/2}}{\gamma(\alpha/2)}\right]^{4/3} \left(\frac{J}{\epsilon_0}\right)^{2/3} \left(\frac{m}{2e}\right)^{1/3}. \tag{19}$$

The electric potential $V(x_1)$ at the interface ($x = x_1$) gives electric field (using Eq. (7)) as

$$E_{x_1} = \frac{C}{c(\alpha, x_1)} \frac{4\alpha}{3} x_1^{\frac{4\alpha}{3}-1}. \tag{20}$$

In the free-space region ($x = x_1$ to $x = L$), we follow the standard derivation of the CL law to obtain the electric field in the form

$$\frac{dV(x)}{dx} = \left[\frac{2J}{\epsilon_0}\sqrt{\frac{2m}{e}\left(V(x) - V(x_1)\right) + E^2(x_1)}\right], \tag{21}$$

which by integration on both sides, and using boundary conditions $E(x_1) = E_{x_1}$, $V(x_1) = V_{x_1}$, and $V(x_L) = V_0$, can be simplified as

$$\frac{4}{3a^2}\left(a\sqrt{V_0 - V_{x_1}} + b\right)^{3/2} - \frac{4b}{a^2}\left(a\sqrt{V_0 - V_{x_1}} + b\right)^{1/2} + \frac{8}{3}\frac{b^{3/2}}{a^2} = L - x_1, \tag{22}$$

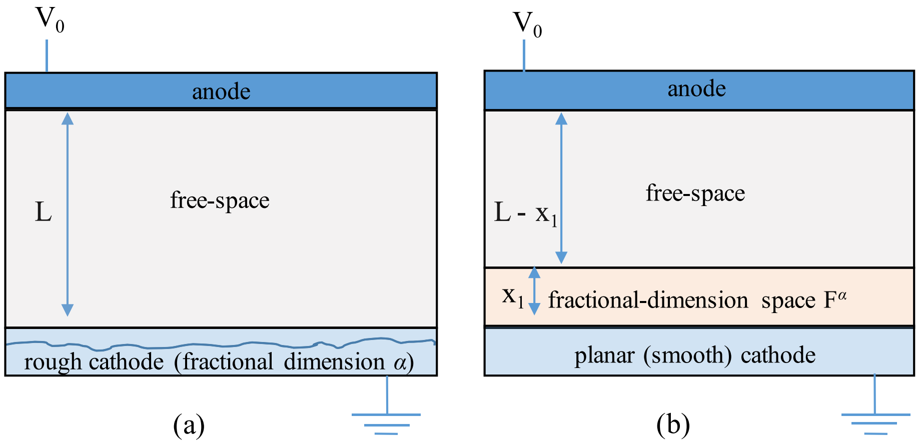

FIG. 2. (a) Schematic diagram of realistic vacuum diode with gap length L and having rough-surface cathode characterized with fractional-dimension α. (b) Schematic diagram of the effective model by replacing the rough-surface cathode with a Planar cathode placed in a layer of fractional-dimensional space (F^{α}) with width x_1.

where

$$a = \left(\frac{2J}{\epsilon_0}\right)\left(\frac{2m}{e}\right)^{1/2}, \tag{23}$$

$$b = E_{x_1}^2. \tag{24}$$

In doing so, Eq. (22) can be solved numerically to obtain the SCL current J as a function of V_0 for a given L, α, and x_1.

We solve Eq. (22) to calculate the SCL current enhancement factor as a function of gap L at fixed voltage V_0 and $\alpha = 0.9$ for varying x_1. The results are shown in Fig. 3. The decreasing value of parameter x_1 corresponds to decreasing width of fractional-dimensional space layer which leads to reduced enhancement factor as expected. For practical applications, we can use the x_1 as the fitting parameter in the model, while α is approximated from the roughness profile of the cathode. The effect of dimension α on SCL current enhancement factor for varying x_1 is shown in Fig. 4.

In an experiment,[46] the generation and the characterization of high current electron beams from rough photocathodes were investigated for electron emission. The cathodes were rough Cu disks. The cathode surface roughness was characterized with a roughness parameter R_a based on the roughness data of the cathode taken from scanning electron microscopy (SEM) micrographs of the cathodes used. We studied the SEM micrographs of three cathode profiles with roughness parameter, $R_a = 0.05$, 0.12, and 0.17, to measure the Hausdorff (fractal) dimension using the box-counting method,[21] and found the fractal dimensions as 0.957, 0.916, and 0.883, respectively. However, no data were provided on current enhancement in SCL regime for this experiment. In another work,[47] an experiment was performed to understand the propagation of SCL electron beams generated by a niobium photocathode illuminated by different wavelength excimer lasers in the space charge regime. The cathode used was a polycrystalline disc with surface roughness parameter $R_a = 0.09$ as defined similar to previous experiment.[46] The average current enhancement factors for this experiment, replacing smooth cathode with a rough cathode keeping diode gap 4 mm and 8 mm, were reported to be 1.495 and 1.24, respectively. We found the fractal dimension of the rough cathode as 0.934 by interpolating the fractal dimension data of three cathodes described above and calculated the current enhancement factor using our model (Eq. (22)) with $\alpha = 0.934$, at fixed $V_0 = 1$ kV and $x_1 = 1$ mm, as shown in Fig. 5. These calculations give enhancement factors of 1.50 and 1.245 for gap 4 mm and 8 mm, respectively, which are in good agreement with those approximated from the experimental results.

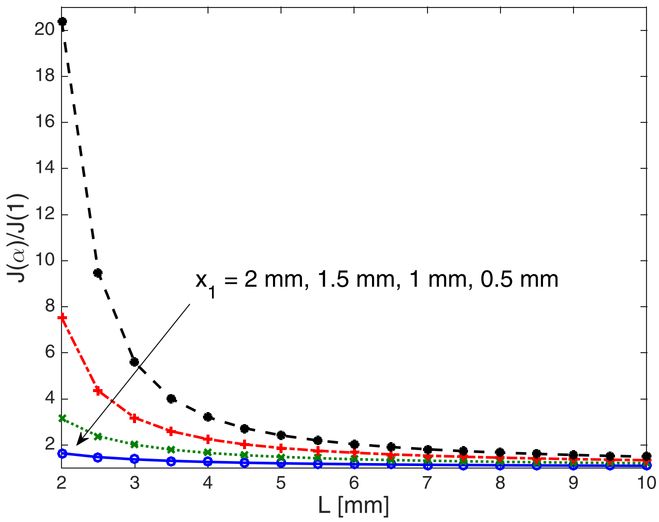

FIG. 3. SCL current enhancement factor versus gap L for $\alpha = 0.9$ and varying x_1.

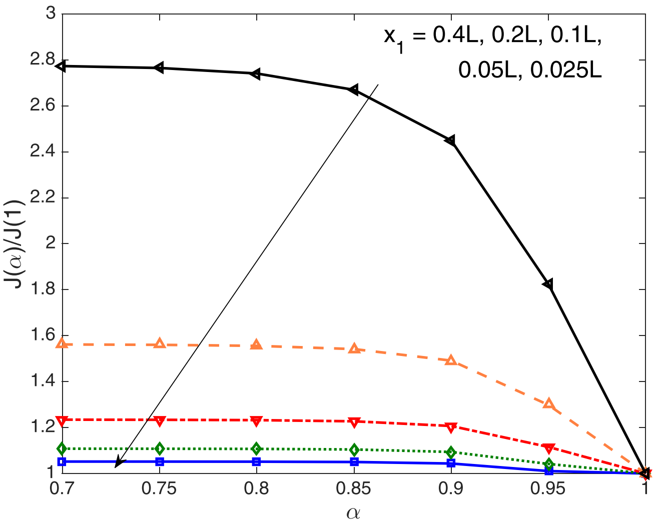

FIG. 4. SCL current enhancement factor versus fractional-dimension α for varying x_1.

V. SUMMARY

In summary, a novel and self-consistent model of CL law has been provided in fractional-dimensional space. This model describes the effect of cathode surface roughness on

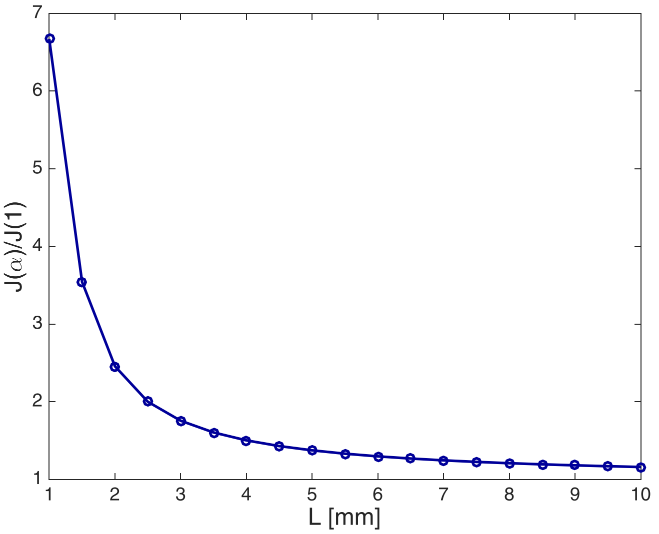

FIG. 5. SCL current enhancement factor versus gap L for vacuum diode of experiment[47] characterized in this work with $\alpha = 0.934$; x_1 is taken as 1 mm. In the experiment, the enhancement factors were approximated to be 1.495 and 1.24 for diode gap 4 mm and 8 mm, respectively, which is in good agreement with the calculated values of 1.50 and 1.245 using this model.

the macroscopic current that can be transmitted across a gap using an effective layer of fractional-dimensional space corresponding to the degree of cathode surface roughness. The fractional-dimensional model of CL law presented in this work is able to simulate the region near a rough cathode's surface without using fine meshing required in the electron gun code.[48]

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