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ABSTRACT

The sound speed of the relativistic free electron gas is derived and investigated. We examine the non-relativistic and ultra-relativistic limits. It is shown from first principles that the sound speed is lower than the velocity of light divided by $\sqrt{3}$. The question is open as whether taking into account positrons and photons in thermodynamic equilibrium with electrons, including other particles, or describing different fundamental forces can modify this result.

I. INTRODUCTION

We investigate the sound speed of the relativistic free electron gas, which is confined by some very strong gravitational field at high and very high temperature.[1,2] This can be also of interest in the early Universe where very dense and very hot regime can be encountered.[3,4] The basic thermodynamic properties of this system have been studied thoroughly recently.[5] Various formulas have been established for the density, pressure, energy, or entropy of the relativistic free electron gas. The non-relativistic and ultra-relativistic limits have been also derived. The reduced chemical potential $\eta = \mu/k_B T$ of such a gas can be either positive, negative, or equal to zero. In this expression, μ is the chemical potential, T is the temperature of the gas, and k_B is the Boltzmann constant. The expressions of interest when normalized adequately depend on the reduced chemical potential η and on the reduced temperature $\theta = k_B T/m_e c^2$, where m_e is the electron mass and c the velocity of light. Thanks to the work of Gong *et al.*,[6] we have Fortran subroutines to calculate the generalized relativistic Fermi–Dirac integrals $F_k(\eta, \theta)$ and their partial derivatives with respect to η and θ up to third order. Doing so, we can calculate numerically the basic formulas that cannot be done analytically.

The work concerning the relativistic free electron gas presented in Ref. 5 covers a large amount of facts but unfortunately does not say a word concerning the sound speed c_s of the relativistic free electron gas in the non-degenerate or degenerate domains at low or high temperature. This is the goal of the present paper to establish the general formula of the sound speed of the relativistic free electron gas as a function of the two parameters η and θ. The derivation is quite tedious but can be done. We consider also the non-relativistic and ultra-relativistic limits where formulas for c_s can be found. Our first-principles approach is quite general and is based on the key definition of the sound speed of a gas provided by Landau and Lifchitz.[7] This is important to start from a firm base since many variants can be found in the literature that are not obtained from clear first-principles. Once the general expression of the sound speed of the relativistic free electron gas obtained, we can study the very important fact about its value with respect to the velocity of light. Very comforting, we found that c_s is lower than $c/\sqrt{3}$ at fixed η for a given θ, so that lower than the speed of light.

The present paper is organized as follows. In Sec. II, we present the theoretical derivations of the speed of sound of the relativistic free electron gas. Numerical applications are presented in Sec. III. Section IV is the conclusion.

II. THEORY

By definition, the sound speed c_s is given by the following expression:[1,7,8]

$$c_s^2 \equiv \left.\frac{\partial P}{\partial \rho}\right|_{S/N}, \tag{1}$$

where P is the pressure of the gas, ρ the mass density, S the entropy, and N the number of electrons. We cannot calculate the sound speed in the relativistic regime using Eq. (1). Let us change variables from $(\rho, S/N)$ to (ρ, T). So,

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_T + \left.\frac{\partial T}{\partial \rho}\right|_{S/N} \left.\frac{\partial P}{\partial T}\right|_\rho. \tag{2}$$

Since S/N is constant, we have

$$\delta\rho \left.\frac{\partial (S/N)}{\partial \rho}\right|_T + \delta T \left.\frac{\partial (S/N)}{\partial T}\right|_\rho = 0. \tag{3}$$

Consequently,

$$\left.\frac{\partial T}{\partial \rho}\right|_{S/N} = -\frac{\left.\frac{\partial (S/N)}{\partial \rho}\right|_T}{\left.\frac{\partial (S/N)}{\partial T}\right|_\rho}. \tag{4}$$

Thus,

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_T - \frac{\left.\frac{\partial (S/N)}{\partial \rho}\right|_T}{\left.\frac{\partial (S/N)}{\partial T}\right|_\rho}\left.\frac{\partial P}{\partial T}\right|_\rho. \tag{5}$$

In hydrodynamic simulations, the set of variables (ρ, T) is more adapted than the set (V, T, N),[7,9] where V is the volume of the system. We can introduce the reduced entropy $s = S/Nk_B$ and Eq. (5) reads

$$c_s^2 = \left.\frac{\partial P}{\partial \rho}\right|_T - \frac{\left.\frac{\partial s}{\partial \rho}\right|_T}{\left.\frac{\partial s}{\partial T}\right|_\rho}\left.\frac{\partial P}{\partial T}\right|_\rho. \tag{6}$$

Equation (6) is the fundamental equation of this work. Let us calculate the various quantities $\frac{\partial P}{\partial \rho}|_T$, $\frac{\partial s}{\partial \rho}|_T$, $\frac{\partial s}{\partial T}|_\rho$, and $\frac{\partial P}{\partial T}|_\rho$. The pressure, the electron density, and the kinetic energy density are given, respectively, by[5]

$$P = \frac{2\sqrt{2}m_e^4c^5\theta^{5/2}}{3\pi^2\hbar^3}\left[F_{3/2}(\eta,\theta) + \frac{\theta}{2}F_{5/2}(\eta,\theta)\right], \tag{7}$$

$$N_e = \frac{\sqrt{2}m_e^3c^3\theta^{3/2}}{\pi^2\hbar^3}\left[F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)\right], \tag{8}$$

$$K = \frac{\sqrt{2}m_e^4c^5\theta^{5/2}}{\pi^2\hbar^3}\left[F_{3/2}(\eta,\theta) + \theta F_{5/2}(\eta,\theta)\right], \tag{9}$$

where

$$F_k(\eta,\theta) = \int_0^{+\infty} dx \frac{x^k\sqrt{1+\frac{\theta x}{2}}}{1+e^{x-\eta}}. \tag{10}$$

$\hbar$ is the reduced Planck constant. In Ref. 5, the kinetic energy is noted as U. This is the energy of the relativistic free electron gas without the contribution of the rest mass. Let us mention a very important point. To calculate ρ in the relativistic regime, one should add[1,7] to the rest-mass density m_eN_e the quantity K/c^2. If not, one finds the absurd result that the sound speed can exceed the velocity of light. So,

$$\rho = \frac{\sqrt{2}m_e^4c^3\theta^{3/2}}{\pi^2\hbar^3}\left[F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)\right] + \frac{\sqrt{2}m_e^4c^3\theta^{5/2}}{\pi^2\hbar^3}\left[F_{3/2}(\eta,\theta) + \theta F_{5/2}(\eta,\theta)\right]. \tag{11}$$

In the non-relativistic regime, where $\theta \ll 1$, one recovers the well-known result,

$$\rho \approx m_eN_e. \tag{12}$$

The reduced entropy s is a function of η and θ only,[5] i.e.,

$$s = -\eta + \frac{\frac{5}{3}F_{3/2}(\eta,\theta) + \frac{4\theta}{3}F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta)}. \tag{13}$$

So,

$$\left.\frac{\partial s}{\partial \rho}\right|_T = \frac{\partial s}{\partial \eta}\left.\frac{\partial \eta}{\partial \rho}\right|_T \tag{14}$$

and

$$\left.\frac{\partial s}{\partial T}\right|_\rho = \frac{\partial s}{\partial \eta}\left.\frac{\partial \eta}{\partial T}\right|_\rho + \frac{\partial s}{\partial \theta}\left.\frac{\partial \theta}{\partial T}\right|_\rho, \tag{15}$$

or

$$\left.\frac{\partial s}{\partial T}\right|_\rho = \frac{\partial s}{\partial \eta}\left.\frac{\partial \eta}{\partial T}\right|_\rho + \frac{\partial s}{\partial \theta}\frac{\theta}{T}, \tag{16}$$

since the sound speed of the relativistic free electron gas is

$$\left.\frac{\partial \theta}{\partial T}\right|_\rho = \frac{\theta}{T}. \tag{17}$$

We have to calculate $\frac{\partial \eta}{\partial T}|_\rho$, $\frac{\partial \eta}{\partial \rho}|_T$, $\frac{\partial P}{\partial T}|_\rho$, $\frac{\partial P}{\partial \rho}|_T$, $\frac{\partial s}{\partial \eta}$, and $\frac{\partial s}{\partial \theta}$. Let us start with $\frac{\partial \eta}{\partial \rho}|_T$. From Eq. (11), we find that

$$\left.\frac{\partial \eta}{\partial \rho}\right|_T = \frac{1}{\rho}\frac{F_{1/2}(\eta,\theta) + \theta F_{3/2}(\eta,\theta) + \theta\left[F_{3/2}(\eta,\theta) + \theta F_{5/2}(\eta,\theta)\right]}{\partial_\eta F_{1/2}(\eta,\theta) + \theta\partial_\eta F_{3/2}(\eta,\theta) + \theta\left[\partial_\eta F_{3/2}(\eta,\theta) + \theta\partial_\eta F_{5/2}(\eta,\theta)\right]}. \tag{18}$$

Let us now calculate $\frac{\partial \eta}{\partial T}|_\rho$. As above, we start with Eq. (11). We find that

$$\left.\frac{\partial \eta}{\partial T}\right|_\rho = -\frac{1}{T}\tilde{G}_0(\eta,\theta), \tag{19}$$

where[5]

$$\tilde{G}_0(\eta,\theta)=\frac{\frac{3}{2}\left[F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)\right]+\theta\partial_\theta F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)+\theta^2\partial_\theta F_{3/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}$$
$$+\frac{\frac{5}{2}\theta\left[F_{3/2}(\eta,\theta)+\theta F_{5/2}(\eta,\theta)\right]+\theta\left[\theta\partial_\theta F_{3/2}(\eta,\theta)+\theta F_{5/2}(\eta,\theta)+\theta^2\partial_\theta F_{5/2}(\eta,\theta)\right]}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]} \tag{20}$$

Let us now calculate $\frac{\partial P}{\partial\rho}|_T$. From Eq. (7), we find that

$$\left.\frac{\partial P}{\partial\rho}\right|_T=\frac{2c^2\theta}{3}\frac{\partial_\eta F_{3/2}(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}, \tag{21}$$

using Eqs. (11) and (18). Finally, let us calculate $\frac{\partial P}{\partial T}|_\rho$. Starting again from Eq. (7), we find that using Eqs. (11) and (19)

$$\left.\frac{\partial P}{\partial T}\right|_\rho=\frac{5c^2\rho\theta}{3T}\frac{F_{3/2}(\eta,\theta)+\frac{\theta}{2}F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)+\theta\left[F_{3/2}(\eta,\theta)+\theta F_{5/2}(\eta,\theta)\right]}+\frac{2c^2\rho\theta^2}{3T}\frac{\partial_\theta F_{3/2}(\eta,\theta)+\frac{1}{2}F_{5/2}(\eta,\theta)+\frac{\theta}{2}\partial_\theta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)+\theta\left[F_{3/2}(\eta,\theta)+\theta F_{5/2}(\eta,\theta)\right]}$$
$$-\frac{2c^2\rho\theta}{3T}\frac{\partial_\eta F_{3/2}(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)+\theta\left[F_{3/2}(\eta,\theta)+\theta F_{5/2}(\eta,\theta)\right]}\tilde{G}_0(\eta,\theta). \tag{22}$$

Let us now finish this part by calculating $\frac{\partial s}{\partial\eta}$ and $\frac{\partial s}{\partial\theta}$. Let us use Eq. (13). So,

$$\frac{\partial s}{\partial\eta}=-1+\frac{\frac{5}{3}\partial_\eta F_{3/2}(\eta,\theta)+\frac{4\theta}{3}\partial_\eta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}-\frac{\frac{5}{3}F_{3/2}(\eta,\theta)+\frac{4\theta}{3}F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}\times\frac{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)} \tag{23}$$

and

$$\frac{\partial s}{\partial\theta}=\frac{\frac{5}{3}\partial_\theta F_{3/2}(\eta,\theta)+\frac{4}{3}F_{5/2}(\eta,\theta)+\frac{4\theta}{3}\partial_\theta F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}, \tag{24}$$

$$-\frac{\frac{5}{3}F_{3/2}(\eta,\theta)+\frac{4\theta}{3}F_{5/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}\times\frac{\partial_\theta F_{1/2}(\eta,\theta)+F_{3/2}(\eta,\theta)+\theta\partial_\theta F_{3/2}(\eta,\theta)}{F_{1/2}(\eta,\theta)+\theta F_{3/2}(\eta,\theta)}. \tag{25}$$

We can check that we recover the non-relativistic and the ultra-relativistic limits. The values of $F_k(\eta,\theta)$, $\partial_\eta F_k(\eta,\theta)$, and $\partial_\theta F_k(\eta,\theta)$ in the non-relativistic and ultra-relativistic regimes are given in Table III of Ref. 5. Note that $\partial_\eta F_k(\eta,\theta)=\frac{\partial F_k(\eta,\theta)}{\partial\eta}$ and $\partial_\theta F_k(\eta,\theta)=\frac{\partial F_k(\eta,\theta)}{\partial\theta}$. We can now calculate the sound speed using Eqs. (6), (14), (16), and (18)–(24). We find that

$$\frac{c_s^2}{c^2}=\frac{2\theta}{3}\frac{\partial_\eta F_{3/2}(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}+\frac{\frac{\partial s}{\partial\eta}}{\frac{\partial s}{\partial\eta}\tilde{G}_0(\eta,\theta)-\theta\frac{\partial s}{\partial\theta}}$$
$$\times\left[\frac{5\theta}{3}\frac{F_{3/2}(\eta,\theta)+\frac{\theta}{2}F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}\right.$$
$$+\frac{2\theta^2}{3}\frac{\partial_\theta F_{3/2}(\eta,\theta)+\frac{1}{2}F_{5/2}(\eta,\theta)+\frac{\theta}{2}\partial_\theta F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}$$
$$\left.-\frac{2\theta}{3}\frac{\partial_\eta F_{3/2}(\eta,\theta)+\frac{\theta}{2}\partial_\eta F_{5/2}(\eta,\theta)}{\partial_\eta F_{1/2}(\eta,\theta)+\theta\partial_\eta F_{3/2}(\eta,\theta)+\theta\left[\partial_\eta F_{3/2}(\eta,\theta)+\theta\partial_\eta F_{5/2}(\eta,\theta)\right]}\tilde{G}_0(\eta,\theta)\right]. \tag{26}$$

The expression of c_s^2/c^2 is quite involved. In the non-relativistic limit when $\theta\ll 1$, we have

$$\frac{c_s^2}{c^2} = \frac{10}{9}\theta\frac{I_{3/2}(\eta)}{I_{1/2}(\eta)}, \tag{27}$$

and in the ultra-relativistic limit when $\theta \gg 1$, we have

$$\frac{c_s^2}{c^2} = \frac{1}{3}, \tag{28}$$

where $I_k(\eta)$ is the Fermi–Dirac integral of k, i.e.,

$$I_k(\eta) = \int_0^{+\infty} \frac{x^k}{1+e^{x-\eta}}dx. \tag{29}$$

Note that in the non-relativistic regime, since[5]

$$\frac{3}{2}\frac{m_e}{\rho k_B T}P = \frac{I_{3/2}(\eta)}{I_{1/2}(\eta)}, \tag{30}$$

we have also

$$c_s^2 = \frac{5}{3}\frac{P}{\rho}. \tag{31}$$

This is consistent with Ref. 10.

III. NUMERICAL APPLICATIONS

Let us plot in Fig. 1 c_s^2/c^2 as a function of η for various reduced temperatures θ. When $\eta \ll -1$, we are in the non-degenerate limit, and when $\eta \gg 1$, we are in the degenerate limit. c_s^2/c^2 is an increasing function of η at fixed θ. We have the transition between the non-degenerate limit and the degenerate limit. This transition is well marked at low θ. We can see that $c_s^2/c^2 \leq 1/3$. The value of $1/3$ is reached in the degenerate limit for all values of θ. When the reduced temperature is large, $c_s^2/c^2 \approx 1/3$ for all values of η where $\eta \in [-100, 1000]$.

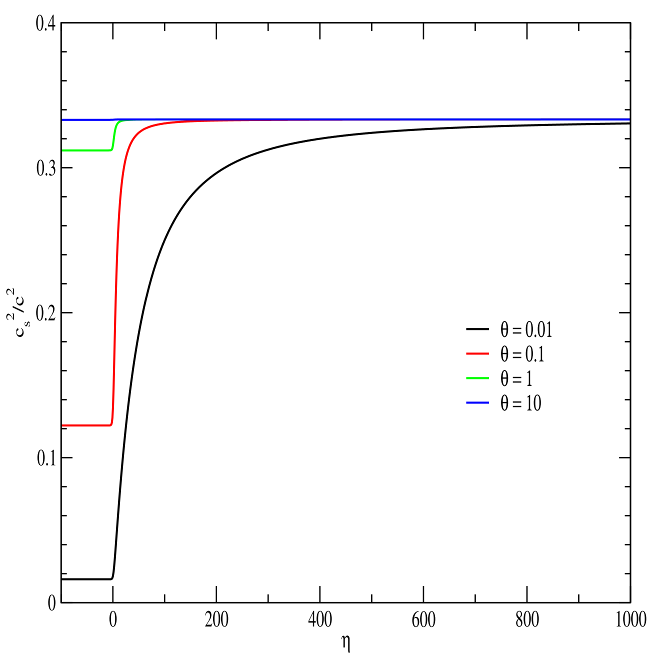

FIG. 1. Square of the sound speed divided by the square of the velocity of light as a function of η of the relativistic free electron gas at various reduced temperature θ.

Since the derivation seems correct from first principles, one may wonder how this result can be modified by including positrons and photons in thermodynamic equilibrium with the electrons or by taking into account the interaction between the different particles, or other particles than electrons, positrons, and photons. This is an astounding work.[11] Indeed, if the author sees what should be done to modify Eq. (11) to include the contribution of positrons and photons, we do not know how we should modify Eq. (1) to include them. If we treat only electrons, we have a pure element gas. However, if we consider the system made by electrons, positrons, and photons, we have a mixture of particles. Again, the author does not know how we can generalize Eq. (1) to mixtures. Due to the Dirac and Breit-Wheeler processes, we cannot just add photons to the electrons, but we must consider positrons too. As noticed by Weinberg[1] and others,[12,13] it is an open question whether the sound velocity remains less than the velocity of light when non-electromagnetic forces are taken into account.[14] This may be of particular interest for the first moments of the Universe after the Big Bang.

IV. CONCLUSION

We have obtained the speed of sound in the relativistic free electron gas. The general expression is consistent with the non-relativistic and ultra-relativistic regimes. We have found that the speed of sound is lower than the velocity of light divided by $\sqrt{3}$. The speed of sound of the relativistic free electron gas is thus smaller than the speed of light.

AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

Author Contributions

Gerald Faussurier: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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