Fast Access to Concepts in Concept Lattices via Bidirectional Associative Memories

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Bidirectional associative memories (BAMs) are shown to be capable of precisely learning concept lattice structures by Radim Bělohlávek. The focus of this letter is to show that the BAM, when set up with a concept lattice by setting up connection weights according to the rule proposed by Bělohlávek, always returns the most specific or most generic concept containing the given set of objects or attributes when a set of objects or attributes is presented as input to the object or attribute layer. A proof of this property is given here, together with an example, and a brief application of the property is provided.

1 Introduction

Bělohlávek (2000) proved that for each concept lattice $B(G, M, I)$, there is a bidirectional associative memory (BAM) such that the set of all concepts of $B(G, M, I)$ is precisely the set of all stable points of this BAM. This is an extremely useful finding that encourages the employment of BAMs to encode concept lattice structures in practical applications. Embedding concept lattices in BAM structures serves an attractive alternative to building concept lattices, as it greatly improves the efficiency while reducing the complexity of concept lattice building.

Concept lattice structures have been recognized as a powerful alternative to the tree structures that dominate the implementation of structural representations of data. Unlike tree structures, a lattice structure allows the same item to be grouped under different categories (i.e., the same item can appear on different nodes), and also a given item to be reached by different routes or paths. These are very powerful properties that we wish to have in structural representations of data, as we often find data items that, by nature, occupy more than one place in categorization structures. And having

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more than one path to reach the same item allows access to an item through an alternative path in the case of a failure to reach it by a given default path.

In the following, first we give brief introductions to the theory of formal concept analysis, concept lattices, and BAMs. Bělohlávek’s theorem on the ability of BAMs to learn concept lattices is given next. The property of the BAM that we report on in this article is described and proved, followed by an example. Finally, a brief application that makes use of this property is provided.

2 Formal Concept Analysis

Formal concept analysis (FCA) was proposed by Rudolf Wille in 1982 (Ganter & Wille, 1999; Wille, 1997) as a mathematical framework for performing data analysis. It structures data into units that are formal abstractions of concepts of human thought, allowing meaningful and comprehensible interpretation. FCA models the world as being composed of objects and attributes. It is assumed that an incidence relation connects objects to attributes. In addition, FCA encodes the specificity-generality relationship between related concepts by means of an order relation. The following definitions are crucial to the theory of FCA.

Definition 1. A formal context $K = (G, M, I)$ is a triplet consisting of two sets $G$ (set of objects) and $M$ (set of attributes) and a relation $I$ between $G$ and $M$. Table 1 shows an example: the context of planets.

Definition 2. A formal concept in a formal context is a pair $(A, B)$ of sets $A \subseteq G$ and $B \subseteq M$ such that $A^\uparrow = B$ and $B^\downarrow = A$ (completeness constraint), where $A^\uparrow = \{m \in M \mid gIm \text{ for all } g \in A\}$ (i.e., the set of attributes common to all the objects in $A$), and $B^\downarrow = \{g \in G \mid gIm \text{ for all } m \in B\}$ (i.e., the set of objects that have all attributes in $B$). By $gIm$ we denote the fact that object $g$ has attribute $m$.

Table 1: Context of the Planets.

<table>
<thead>
<tr>
<th>Planet</th>
<th>Size</th>
<th>Distance from Sun</th>
<th>Moon</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Small</td>
<td>Medium</td>
<td>Large</td>
</tr>
<tr>
<td>Mercury (Me)</td>
<td>(ss)</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Venus (V)</td>
<td>(sm)</td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Earth (E)</td>
<td>(sl)</td>
<td></td>
<td>x</td>
</tr>
<tr>
<td>Mars (Ma)</td>
<td></td>
<td>x</td>
<td>x</td>
</tr>
<tr>
<td>Jupiter (J)</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Saturn (S)</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Uranus (U)</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Neptune (N)</td>
<td></td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Pluto (P)</td>
<td></td>
<td>x</td>
<td></td>
</tr>
</tbody>
</table>
The set of all concepts of a context \((G, M, I)\) is denoted by \(B(G, M, I)\). This consists of all pairs \((A, B)\) such that \(A^\uparrow = B\) and \(B^\downarrow = A\), where \(A \subseteq G\) and \(B \subseteq M\).

**Definition 3.** Specificity-generality order relationship. If \((A_1, B_1)\) and \((A_2, B_2)\) are concepts of a context, then \((A_1, B_1)\) is called a subconcept of \((A_2, B_2)\), if \(A_1 \subseteq A_2\) (or equivalently \(B_1 \supseteq B_2\)). This sub-super concept relation is written as \((A_1, B_1) \leq (A_2, B_2)\). According to this definition, a subconcept always contain fewer objects and greater attributes than any of its superconcepts.

### 3 Concept Lattice

A lattice is an ordered set \(V\) with an order relation in which for any given two elements \(x\) and \(y\), the supremum and the infimum elements always exist in \(V\). Furthermore, such a lattice is called a complete lattice if supremum and infimum elements exist for any subset \(X\) of \(V\) (Ganter & Wille, 1999; Wille, 1997). The fundamental theorem of FCA (Ganter & Wille, 1999; Wille, 1997) states that the set of all formal concepts of a formal context forms a complete lattice. A complete lattice of formal concepts is called a concept lattice. Figure 1 shows the concept lattice of the context given in Table 1 as a line diagram.

### 4 Bidirectional Associative Memories

Based on the early associative memory models (Hopfield, 1984), Kosko (1987, 1988) proposed a bidirectional associative neural network called the
bidirectional associative memory (BAM). It consists of two layers of neurons. The states (activities) of the neurons in the two layers are denoted by \( x_i (i = 1, \ldots, k) \) and \( y_j (j = 1, \ldots, l) \), respectively, where \( k \) and \( l \) are the number of neurons in the two layers. The states \( x_i \) and \( y_j \) can be represented as either a binary (0 or 1) or a bipolar (+1 or −1) encoding. Each \( i \)th neuron of the first layer is connected to each \( j \)th neuron of the second layer by a connection weight. A real threshold \( \theta^x_i \) (\( \theta^y_j \)) is assigned to the \( i \)th neuron of the first layer (\( j \)th neuron of the second layer), respectively.

In contrast to an autoassociator, the BAM is a hetero-associator that associates pairs of patterns together. A number of training rules have been developed for training BAMs. We use the rule proposed by Bělohlávek (2000). This rule is proven to set up a concept lattice precisely in a BAM.

5 BAMs for the Representation and Storage of Concept Lattices:
Bělohlávek’s Theorem

In general, there are BAM stable points that cannot be interpreted as formal concepts. However, using the weight computation given below, Bělohlávek (2000) has shown not only that the BAMs can learn concept lattices, but also that the set of stable points in the BAM is precisely the set of nodes of the underlying concept lattice. In the following, we state Bělohlávek’s theorem on this property; the reader is referred to Bělohlávek (2000) for details and for the proof of the theorem.

**Bělohlávek’s Theorem.** Let \( B(G, M, I) \) be a concept lattice given by the context \( (G, M, I) \) with \( G \) and \( M \) finite. Then there is a BAM given by the weights \( W \) and thresholds \( \theta \) such that \( \text{Stab}(W, \theta) = \{ \langle S_G(A), S_M(B) \rangle | \langle A, B \rangle \in B(G, M, I) \} \); where \( S_G(A) = \langle a_1, \ldots, a_n \rangle \in \{0, 1\}^n \) denotes the characteristic vector of \( A \) (\( A \subseteq Z \)), that is, \( a_i = 1 \) if \( z_i \in A \) and \( a_i = 0 \) otherwise.

This theorem says that there is essentially a one-to-one correspondence between stable points and formal concepts. Given below is Bělohlávek’s weight-setting rule:

\[
\begin{align*}
    w_{ij} = \begin{cases} 
        1 & \text{if } \langle g_i, m_j \rangle \in I \\
        -q & \text{if } \langle g_i, m_j \rangle \notin I
    \end{cases}
\end{align*}
\]

for \( i = 1, \ldots, k \), \( j = 1, \ldots, l \); where \( q = \max \{k, l\} + 1 \) and thresholds are set to \(-1/2\).

6 Accessing Concepts in the Concept Hierarchy of a Concepts Lattice

Encoding concept lattices in BAM structures in a way that ensures there are no spurious states is an important and attractive alternative to the complex lattice-building algorithms. However, building a concept lattice solves only part of the problem. Accessing the concepts in the concept hierarchy of the
underlying concept lattice is equally important. In the following (theorem), we report how concepts can be accessed from a concept lattice representation encoded in a BAM. The proof of the theorem follows.

**Theorem.** Let $X$ denote a BAM trained with the concept lattice $B(G,M,I)$ given by the context $⟨G,M,I⟩$ with $G$ and $M$ finite. Then, given a set of objects $A$ when presented to the object layer $X(G)$ of $X$ results in $X$ stabilizing on the most specific concept containing the given set of objects. Conversely, given a set of attributes $B$ when presented to the attribute layer $X(M)$ of $X$ results in $X$ being stabilized on the most generic concept containing the given set of attributes.

In its simplest case, when only one object or one attribute is presented to the corresponding layer of the BAM, it returns the object or attribute concept, respectively. Note that the object concept of a given object is defined as the most specific concept in the lattice containing the given object—that is, the concept represented by the lowest node where the given object appears. The attribute concept of a given attribute is defined as the most generic concept in the lattice containing the given attribute—that is, the concept represented by the highest node where the given attribute appears. Note that the proof that follows proves that given an object, the BAM stabilizes on the object concept. The converse of this—given an attribute, the BAM stabilizes on the attribute concept—can be proved by a similar argument.

Therefore, the concepts in different parts of the concept hierarchy can be accessed by presenting appropriate sets of objects or attributes to the corresponding layer of the BAM. For instance, adding more objects to the input allows traversing up the concept lattice, and adding more attributes allows traversing down the concept lattice.

**Proof.** Let $B(G,M,I)$ be the set of all concepts of the context $⟨G,M,I⟩$, where $G$ is the set of all objects in the context, $M$ is the set of all attributes in the context, and $I$ is the incident relation. Let us assume that a BAM (say, $X$) is setup according to Bělohlávek’s weight-setting rule embedding the concept lattice of this context in $X$. Then the stable points in $X$ represent (with a one-to-one correspondence) all formal concepts $(A,B)$ of the context $⟨G,M,I⟩$, and thus they hold the completeness constraint $A↑ = B$ and $B↓ = A$, where $A↑ = \{m ∈ M | ∀g ∈ A \text{ s.t. } ⟨g,m⟩ ∈ I\}$ and $B↓ = \{g ∈ G | ∀m ∈ B \text{ s.t. } ⟨g,m⟩ ∈ I\}$. Now, let us take an arbitrary object $g_o$. Assume, without loss of generality, that the BAM gives $\{g_o, g_i\} → \{m_x, m_y\}$ when the object $g_o$ is presented to the object layer of the BAM. This means $\{g_o, g_i\} → \{m_x, m_y\}$ is a formal concept in the context. Therefore, by the completeness constraint ($A↑ = B$ and $B↓ = A$) we get:

$$\{g_o, g_i\}↑ = \{m_x, m_y\}$$ (6.1)
Now we will prove that no other formal concept that contains the object \( g_o \) and is more specific to the concept \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) exists in the concept lattice.

According to the order relation, a formal concept that is more specific to a second formal concept should contain fewer objects and more attributes than that of the second formal concept. We will prove that no such subconcept containing the object \( g_o \) can exist in the lattice, given the assumption that the BAM returns \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) for the input \( \{g_o\} \).

There are three forms that a subconcept of a given concept can take. The first two actually contradict the completeness constraint, but they are included in the proof for the sake of completeness, as they satisfy the order relation.

**Case 1:** A concept with fewer objects and same attributes
**Case 2:** A concept with the same objects and greater attributes
**Case 3:** A concept with fewer objects and greater attributes.

**Case 1.** Take \( \{g_o\} \rightarrow \{m_x, m_y\} \) to be a formal concept in the concept lattice. This is more specific to the concept \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) as it satisfies the subsumption relation \( A_1 \subseteq A_2 \) (and \( B_1 \supseteq B_2 \)). If this is a formal concept of the context, from the completeness constraint, we get:

\[
\{m_x, m_y\} \uparrow = \{g_o\}. \tag{6.3}
\]

Equations 6.2 and 6.3 lead to a contradiction, meaning that \( \{g_o\} \rightarrow \{m_x, m_y\} \) cannot be a formal concept of the context given the assumption that \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) is a formal concept.

**Case 2.** Now take \( \{g_o, g_i\} \rightarrow \{m_x, m_y, m_z\} \) to be a formal concept of the context. This is more specific to the concept \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) as it satisfies the subsumption relation \( B_1 \supseteq B_2 \) (and \( A_1 \subseteq A_2 \)). If this is a formal concept, from the completeness constraint we get:

\[
\{g_o, g_i\} \uparrow = \{m_x, m_y, m_z\}. \tag{6.4}
\]

Equations 6.1 and 6.4 lead to a contradiction, meaning that \( \{g_o, g_i\} \rightarrow \{m_x, m_y, m_z\} \) cannot be a formal concept of the content given the assumption that \( \{g_o, g_i\} \rightarrow \{m_x, m_y\} \) is a formal concept.
**Case 3.** Finally, take without loss of generality that \( \{ g_0 \} \rightarrow \{ m_x, m_y, m_z \} \) is a formal concept in the concept lattice. This does not violate the completeness constraint, and therefore does not contradict conditions 6.1 and 6.2. Hence, it is a potential candidate for a formal concept that can possibly exist in the concept lattice. Also, this is a more specific concept to \( \{ g_0, g_i \} \rightarrow \{ m_x, m_y \} \) as it satisfies the subsumption relation \( B_1 \supseteq B_2 \) (and, equivalently, \( A_1 \subseteq A_2 \)).

Now we will prove that this formal concept cannot exist in the concept lattice, given the assumption that the concept \( \{ g_0, g_i \} \rightarrow \{ m_x, m_y \} \) is present and is returned when the object \( g_0 \) (alone) is presented to the object layer of \( X \).

Since \( \{ g_0, g_i \} \rightarrow \{ m_x, m_y \} \) was assumed to be a formal concept, both \( g_0 \) and \( g_i \) must possess at least the attributes \( m_x \) and \( m_y \). The concept \( \{ g_0 \} \rightarrow \{ m_x, m_y, m_z \} \) says that the object \( g_0 \) possesses \( m_z \) as well.

Let us consider the dynamics of the BAM to see what it returns when the object \( g_0 \) is presented to the object layer. Note that we use Bělohlávek’s weight computation rule to set up connection weights to encode the concept lattice in \( X \). Recall that a weight between an object and an attribute node is set to one (1) if the object possesses the attribute; otherwise, the weight is set to the negative value of the maximum number of nodes in either layer +1 (see Figure 2). A node fires if the weighted sum of incomes \( \left( \sum x_i w_i \right) \) exceeds \(-1/2\).

In the first forward pass, \( g_0 \) is presented to the object layer (i.e., \( g_0 = 1 \) and \( g_i = 0 \)); only the nodes correspond to \( m_x, m_y \), and \( m_z \) fire. In the first backward pass, \( m_x, m_y, \) and \( m_z \) are the inputs to the attribute layer; only the node corresponding to \( g_0 \) is fired. The state of the BAM does not change in subsequent passes, and therefore \( \{1,0\} \rightarrow \{1,1,1\} \) is a stable state. In other words, the BAM stabilizes on \( \{ g_0 \} \rightarrow \{ m_x, m_y, m_z \} \) when \( g_0 \) is presented. This contradicts our first assumption that the BAM returns \( \{ g_0, g_i \} \rightarrow \{ m_x, m_y \} \) for the input object \( g_0 \). Therefore, if the BAM returns the concept \( \{ g_0, g_i \} \rightarrow \{ m_x, m_y \} \) for the input object \( \{ g_0 \} \), as we first assumed, then the concept \( \{ g_0 \} \rightarrow \{ m_x, m_y, m_z \} \) cannot exist in the concept lattice.
Cases 1, 2, and 3 prove that given an input object \( g_o \), a BAM stabilizes on the most specific concept in the lattice containing the given input object \( g_o \). Similarly, we can prove that this property holds even for a given subset of objects rather than a single object. In addition, with a similar argument, we can prove that given a set of attributes as input to the attribute layer, the BAM stabilizes on the most generic concept containing the given set of attributes (proof not given here).

**Example.** Consider the context of planets given in Table 1 and the corresponding concept lattice given in Figure 1. Assume that this concept lattice is encoded in a BAM by setting up the connection weights as described above. Figure 3 shows the forward pass (top) and the backward pass (bottom) of the BAM, given the object \( Ma \) as input to the object layer. Note that only the links that spread out from active nodes are shown. The nodes shown in gray are the active (firing) nodes, dashed lines represent links with negative \((-q)\) weights, and solid lines represent links with weight +1. In the forward pass (when the object \( Ma \) is presented as the input to the object layer), the nodes corresponding to the properties \( ss, dn, \) and \( my \) are activated (see Figure 3, top). This set of properties serves as the input to the attribute layer of the BAM in the backward pass, resulting in the activation of the nodes corresponding to \( Ma \) and \( E \) in the object layer (see Figure 3, bottom). The state of the BAM does not change in the subsequent passes; that is, the BAM stabilizes on the concept \( \{Ma, E\} \rightarrow \{ss, dn, my\} \). This indeed is the most specific concept (containing \( Ma \)) present in the concept lattice.
7 Application

We have employed the theories and ideas described above to develop a novel text retrieval model. Only a brief description of this model is given here to demonstrate the use of the property in a practical application. The details of the model can be found in Rajapakse and Denham (2002a, 2002b).

In this application, each information item (documents and queries) is represented in an individual concept lattice trained to a separate BAM. The elements of these lattices (i.e., objects and attributes) are terms (and phrases) extracted from the corresponding text items. The allocation of roles to the elements as objects or attributes and the identification of the existence of a relation (incident relation) between object terms and attribute terms are done according to a set of rules developed based on the syntactic structures and semantic relations between terms in the English language.

The retrieval of documents (to a query) is based on matching nodes (formal concepts) between the query and document lattices (see Figure 4). Instead of trying out all-to-all node matching, we made use of the property of the BAM discussed in this article to perform a selective node matching. The selection of node pairs to match between two lattices (a query and a document) is based mainly on object concepts (attribute concepts are also used if no objects are common to the query and the document). For each individual (unique) element that appears as an object in both the query lattice and the document lattice, object concepts are extracted from the query and document BAMs. This gives the most specific concepts (about the same object element) present in the two lattices. Our interest is in matching such specific formal concepts between queries and documents. Matching object-attribute element pairs between such query-document node pairs contributes to the similarity between the query and the document.

Figure 4: Concept matching between lattice representations.
References


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