



RESEARCH ARTICLE | SEPTEMBER 29 2016

Introduction to focus issue: Patterns of network synchronization **FREE**

Daniel M. Abrams; Louis M. Pecora ; Adilson E. Motter 



Chaos 26, 094601 (2016)

<https://doi.org/10.1063/1.4962970>



Articles You May Be Interested In

Introduction to Focus Issue: Dynamics of oscillator populations

Chaos (January 2023)

Introduction to Focus Issue: Symmetry and optimization in the synchronization and collective behavior of complex systems

Chaos (June 2020)

Introduction to focus issue: In memory of Vadim S. Anishchenko: Statistical physics and nonlinear dynamics of complex systems

Chaos (January 2022)



APL Quantum
Latest Articles Now Online
Read Now



Introduction to focus issue: Patterns of network synchronization

Daniel M. Abrams,¹ Louis M. Pecora,² and Adilson E. Motter³

¹*Department of Engineering Sciences and Applied Mathematics and Northwestern Institute on Complex Systems (NICO), Northwestern University, Evanston, Illinois 60208, USA*

²*U.S. Naval Research Laboratory, Washington, DC 20375, USA*

³*Department of Physics and Astronomy and Northwestern Institute on Complex Systems (NICO), Northwestern University, Evanston, Illinois 60208, USA*

(Received 6 September 2016; accepted 6 September 2016; published online 29 September 2016)

The study of synchronization of coupled systems is currently undergoing a major surge fueled by recent discoveries of new forms of collective dynamics and the development of techniques to characterize a myriad of new patterns of network synchronization. This includes chimera states, phenomena determined by symmetry, remote synchronization, and asymmetry-induced synchronization. This Focus Issue presents a selection of contributions at the forefront of these developments, to which this introduction is intended to offer an up-to-date foundation. *Published by AIP Publishing.* [<http://dx.doi.org/10.1063/1.4962970>]

Synchronization is an inherently collective dynamics phenomenon most suitably investigated within the framework of networks of coupled dynamical entities. Observed in a range of systems and conditions, synchronization manifests itself in a multitude of different scenarios, ranging from complete synchronization of all variables of all entities, to partial synchronization of some variables of some entities, to long lived but transient synchrony, and to various forms of generalized synchronization. As in many other network phenomena, the nature of the collective dynamics depends on the properties of the entities, properties of the interactions (structure and dynamics), system size, coupling with external factors, and initial conditions. Characterizing the new, and often surprising, forms of synchronization that might emerge as a result of the interplay between these various factors is a major topic of current research. Below, we give an introduction that attempts to show the various trajectories in dynamical systems research that led to the current interests in patterns of synchronization dynamics in networks. We do not intend to present a comprehensive review, but rather to give the reader a sense of the research paths that converged to this issue.

I. INTRODUCTION

In 1967, Winfree published a study on the modeling of biological rhythms and the behavior of populations of coupled oscillators,¹ which inadvertently inspired volumes of research not only in biology but also in the physical, mathematical, and engineering sciences. In particular, it motivated Kuramoto to introduce in 1975 his celebrated solvable model,² which established a firm connection between the onset of synchronization in coupled oscillators and phase transitions in statistical physics. By the time Winfree wrote his paper, the phenomenon of synchronization had been known for at least 300 years, with the original discovery often attributed to Huygens' 1665 fortuitous observation of synchronization of pendulum clocks³—incidentally, the

same year Hooke reported his discovery of the living cell.⁴ Its pertinence to biological systems had also long been known. George Mines, for example, whose premature death at age 28 apparently resulted from experimentation with self-induced fibrillation, had developed by as early as 1914 experimental techniques to investigate cardiac arrhythmias resulting from loss of synchrony in the heart induced by small electric stimuli.⁵ More recently, in 1959, Belousov published his observations of long-lived patterns of oscillatory dynamics in a nonlinear chemical reaction system.⁶ While previous research had been largely observational and qualitative, the theme of Winfree and Kuramoto's work was theoretical and quantitative.

The legacy of those early studies combined with the ever increasing availability of computer resources and, more recently, concepts from network science has led to significant new insights over the past few decades. It is now well established that systems of coupled entities can exhibit various forms of synchronization phenomena. This is the case for many types of coupled systems and forms of interactions, whether the entities are conscious, animate, or inanimate; periodic, aperiodic, or even chaotic; oscillatory, pulse-like, or excitatory; whether the system is of finite size or in the thermodynamic limit; whether entities are coupled locally, globally, or through a more general network; forming a discrete or continuous medium; described by iterated maps or differential equations, as deterministic or stochastic systems; coupled in isolation or driven by external chemical, thermal, optical, electrical, or mechanical stimuli; whether the interactions are pairwise or through higher-order (hypergraph) structures; whether the coupling is weak or strong, static or time dependent; whether the entities and interactions are all identical or heterogeneous. Various forms of synchronization phenomenon have been shown, with their own particularities, in each of these scenarios.

What is then left to be discovered? From a basic science standpoint, a major outstanding question is to determine how different patterns of synchronization (including yet-to-be discovered ones) relate to the properties of the entities, the properties of the interactions (both internal and external),

and the properties of the initial conditions. The significance of this question has become even more evident by a series of discoveries made over the past 15 years. For example, the discovery of chimera states—spatiotemporal patterns of coexisting synchronized and incoherent populations in networks of identically coupled identical oscillators^{7,8}—shows that multi-stability and symmetry breaking can lead to counterintuitive behavior not anticipated in previous studies of global synchronization. The discovery of relay and remote synchronization,^{9,10} where mutually synchronized oscillators are connected through oscillators in asynchronous states, exposes yet another class of subtle collective behavior. The latter as well as numerous other patterns of cluster synchronization can now be systematically understood using a recently developed framework that relates stable cluster synchronization with symmetries in the underlying network of interactions.¹¹ More recently, synchronization in a network of identically coupled Stuart-Landau oscillators was used to demonstrate the converse of symmetry breaking: a scenario in which the system cannot synchronize when the oscillators are identical but synchronizes stably when the oscillators' parameters are tuned to be suitably different from each other.¹² As these examples abundantly illustrate, the relationship between the various factors that influence synchronization and synchronization dynamics is anything but obvious.

Because the concept of synchronization lies at the heart of many network dynamical patterns, we offer below a little more insight into the history of two main types of synchronization (phase and identical), which lead up to the papers in this Focus Issue. We then briefly summarize the content of the issue.

II. PHASE SYNCHRONIZATION (PS)

Phase synchronization (PS) is most often discussed in the context of phase oscillators, which in the absence of coupling satisfy the extremely simple linear ordinary differential equation $d\varphi_i/dt = \omega_i$: the i th oscillator increases its phase φ_i at a constant rate ω_i . Given an ensemble of such oscillators, coupling may induce synchronization such that all phases become equal at some point or points in time. That is, a state we refer to as phase-synchronized: $\varphi_i = \varphi_j$ for all i, j . Most often, PS (sometimes referred to as phase lock) refers to a *persistent* state in which all oscillators converge to the same (usually time-varying) phase. The related concept of frequency lock (confusingly, sometimes also called phase lock) refers to a state where coupling induces oscillators to converge to the same instantaneous frequency $d\varphi_i/dt$, but not necessarily the same phase. These concepts can also be applied, sometimes with slight modification, to more complicated oscillators. Important examples include oscillators with nonconstant ω_i (nonautonomous or nonlinear oscillators), oscillators with nonconstant amplitudes (higher-dimensional oscillators), and oscillators with higher-order derivatives in their differential equations (e.g., oscillators with inertia).

The foundational work in the study of PS was done by Kuramoto beginning in the 1970s, when he developed what has come to be known as the Kuramoto model² while studying nonequilibrium statistical mechanics. His reduction to

phase oscillations and ingenious solution of that simplified model created a new paradigm for understanding collective behavior in nature. Much of the history of the research on PS is tied up with the study of the Kuramoto model; Strogatz's 2000 review paper¹³ does an excellent job of summarizing progress through that point and highlighting open questions. Dörfler and Bullo¹⁴ updated progress in 2014.

In 1986, Kuramoto and his student Sakaguchi generalized the Kuramoto model to allow for arbitrary coupling networks and the presence of a phase-lag term. This formulation is now often referred to as the Kuramoto-Sakaguchi model.¹⁵ Exploration of that model first focused primarily on the case of discrete lattice networks [e.g., Refs. 16–18], often motivated by oscillations in arrays of Josephson junctions. As interest in complex networks grew, more attempts were made to understand the impact of a variety of structures on synchronization properties [e.g., Refs. 19–24]. Several papers in this Focus Issue continue to examine the interplay between network structure and synchronization dynamics.

In 2002, Kuramoto and Battogtokh published a paper showing the existence and apparent stability of a state with broken spatial symmetry in a ring of nonlocally coupled oscillators.⁷ Two years later, motivated by the surprising combination of synchrony and incoherence within a fully symmetric system, Abrams and Strogatz dubbed this a chimera state⁸ and attempted to explain its origin. Much work has since been done to understand the properties of these so-called chimera states; a review article published last year²⁵ summarizes some of it, but research is ongoing, and many papers in the current issue address chimera states.

The Kuramoto and Kuramoto-Sakaguchi models have motivated a variety of generalizations that continue to be of interest, including, for example, delay-coupling, unidirectional coupling, non-pairwise coupling, and heterogeneity in phase lags. These systems, along with comparisons to experimental data, comprise several more of the papers in our Focus Issue.

A final twist in the study of PS has come with the recent discovery by Nishikawa and one of us (A.E.M.) of networks of amplitude-phase oscillators that can only synchronize identically when the oscillators themselves are not identical.¹² This is interesting because it has been generally assumed that individual entities are more likely to exhibit the same or similar behavior if they are equal to each other (e.g., think of lasers pulsing at the same frequency, birds singing the same notes, and agents reaching consensus). But as the example in Fig. 1 shows, this assumption is generally false in oscillator networks. In its strongest form, this phenomenon can be shown for *zero-lag* synchronization and oscillators identically coupled in the network, meaning that the state of the system can be symmetric only when the system itself is not symmetric. For this reason, the phenomenon was termed *asymmetry-induced symmetry* and can be seen as the converse of the well-studied phenomenon of symmetry breaking.

III. IDENTICAL SYNCHRONIZATION (IS)

Generally, when we speak of identical synchronization (IS), we mean two or more coupled identical dynamical systems whose dynamics are exactly the same; that is, they

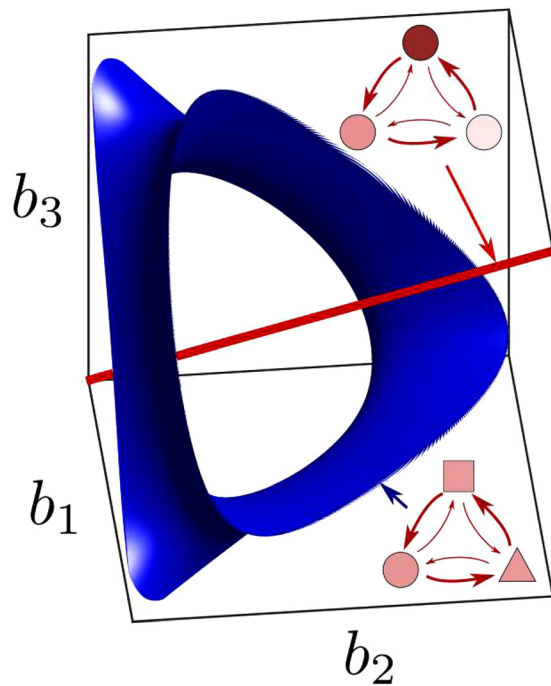


FIG. 1. Symmetric state requiring system asymmetry in three-oscillator networks. Inside the blue surface is the region of stable synchronization in the three-dimensional parameter space, in which all nodes exhibit identical dynamics. Since all networks with identical node parameters b_1 , b_2 , and b_3 (illustrated by identical node shapes) lie on the red line, which does not intersect the blue region, the nodes can achieve identical dynamics only if the nodes themselves are non-identical. In this example, the nodes are amplitude-phase oscillators—a class that includes Stuart-Landau oscillators as special cases. (Adapted with permission from T. Nishikawa and A. E. Motter, *Phys. Rev. Lett.* **117**, 114101 (2016). Copyright 2016 American Physical Society.)

follow the same state space trajectory (for IS among non-identical systems, see Ref. 12). This can mean that they are phase or time lagged, but the same trajectory is followed for all coupled systems or, perhaps, certain subsets of a larger network.

A surprising form of IS comes from the identical synchronization of chaotic systems. This is initially counterintuitive since positive Lyapunov exponents tend to push two nearly synchronized oscillators apart in state space exponentially fast in time. The first papers we know of to show that chaotic IS is possible are those of Fujisaka and Yamada.^{26–28} They actually considered simple networks of identical systems coupled diffusively through all their dynamical variables. These were followed closely, but sporadically, by several other papers out of the Soviet Union. Pikovsky coupled two chaotic oscillators and showed in 1984 that two attractors could synchronize their motion.²⁹ A little later, a now-famous paper appeared by Afraimovich, Verichev, and Rabinovich,³⁰ which showed that two Duffing-like systems could exactly synchronize when linked by diffusive coupling. Note that early Soviet papers often referred to chaotic motion as stochastic motion. This can lead to some confusion and unfortunate exclusion of the works by others familiar with only the "chaotic" adjective. Finally, work by Volkovskii and Rul'kov³¹ closed out the decade of the 1980s with another take on IS.

In 1990, one of us (L.M.P.) along with Carroll³² found a new way to synchronize two oscillators in a one-way driving

scheme. The goal was to eventually think of the system as a communication setup with the drive system being the transmitter and the driven system (also called the response) as the receiver. It was this potential application and the possibility that some type of privacy or even encryption could be added using the chaos that led soon after to a growing research effort in investigating IS and a growing body of literature. The topic of synchronizing a few coupled (usually chaotic) systems still shows up in different guises in the literature at this time. However, up to here, there are no real networks or dynamics patterns in the system as a whole except for the synchronized motion of the few coupled oscillators. This changed in two stages, one with the introduction of networks to the synchronization scene and the other with the discovery of cluster synchronization.

Early work on global synchronization in networks of diffusively coupled systems of many oscillators is presented in the paper by Heagy *et al.*³³ Although this was still global synchronization, some of the techniques that emerged would later morph into general analysis tools. More sophisticated approaches appear in the work by Hu *et al.*,³⁴ where it was shown how to deal with more general networks and parameter variations in their effects on global synchronization. Another important paper at this time was by Gade,³⁵ who looked at much more complicated networks of randomly coupled maps. Some general notions of global synchronization, such as looking at the eigenvalues of the coupling matrix as depending on the number of connections in the network, showed that the two are related in some classes of networks.³⁶ Finally, in 1998, Pecora and Carroll³⁷ developed an analysis technique for closely linking the eigenvalue spectrum of the coupling matrix to the linear stability of the identical oscillator systems that works well for any type of dynamical motion (time-independent, periodic, quasiperiodic, and chaotic). This allowed for the separation of the network structure from the dynamics of the node oscillators, which simplified the problem since the latter only needed to be calculated once as what is now called the master stability function. Then, a change in network only meant a change in eigenvalue spectrum and the stability of the system could be determined directly with no more dynamical calculations. These papers and the many that came after as well as the blossoming field of network science^{38,39} inspired much synchronization in networks research from 1998 up to the present.

The concept of clusters of synchronized nodes appearing in a network followed a few years later. This is the early work that led to many of the papers in this Focus Issue. Typical early works are analyzed and/or demonstrated in Refs. 10, 40–50. These and other papers in this time period were very interesting, but mostly for special networks which were constructed to exhibit cluster synchronization intentionally. Earlier work in the famous book by Golubitsky and colleagues⁵¹ essentially showed that networks of identical oscillators could have cluster synchronization (although it was not called that) if the network had symmetries. Symmetries imply that certain subsets of nodes in the network can be permuted among themselves, which implies immediately that the subset of nodes can have IS dynamics,

although its dynamics will, in general, differ from nodes in other subsets. The subsets are then cluster patterns. The only problem is that one has to know the symmetries beforehand. This is also true for a recent publication on remote synchronization,¹⁰ which can be seen as symmetry-induced cluster synchronization, although the latter did develop some parts of a scheme to calculate the stability of the clusters. More recently, a very general scheme for discovering cluster synchronization was developed.¹¹ The approach uses computer efficient discrete algebra software⁵² to find the symmetries of a network, which are called automorphisms of the adjacency matrix. This allows one to study networks for which symmetries are not obvious or are so numerous that they cannot be counted by a human. Moreover, the machinery of group representations is also available in the software and this allows for the block diagonalization of the variational equations, greatly simplifying the calculation of the stability of the clusters.

Finally, another approach to finding cluster synchronization in networks was developed in Ref. 53 and earlier papers (see also the book by Golubitsky and Stewart⁵⁴). In this approach, it is assumed that nodes have the same intrinsic dynamics and whether two nodes will synchronize depends only on the inputs to them. If they have the same number of inputs from the same synchronized clusters (not necessarily the same nodes), they will synchronize. This approach is applied in Judd's work⁴⁹ and in work of O'Clery *et al.*⁵⁰ The input approach is more general than the symmetry approach in finding synchronization clusters since there exist in some systems synchronization clusters that are not symmetry clusters. And there do exist efficient algorithms that will find the minimal number of input/synchronization clusters,⁵⁵ although the problem of finding sub-synchronous clusters from there is a very hard, unsolved problem. However, in the calculation of stability of the clusters, the input method does not fully block diagonalize the variational equations like the symmetry method since powerful computational machinery of irreducible representations of groups has, so far, no counterpart in the graph theory associated with the input approach. At this time, it is best to say that the symmetry and input methods complement each other.

Many of the above-mentioned authors have contributed to this Focus Issue. The reader now has a good chance to see where all these studies have led to today and get a good sense of what the new, unsolved problems are.

IV. THIS FOCUS ISSUE

Here, we briefly comment on the papers in this issue. The contributions cover a wide range of topics—a tribute to the diversity of research in the field—as reflected, for example, in the following papers:

- Ottino-Löffler and Strogatz⁵⁶ report on localized patterns in a lattice of Kuramoto oscillators, which are characterized by the instantaneous frequencies of the oscillators and manifest themselves as rotating spirals.
- Wang and Chen⁵⁷ introduce a metric-topological model to characterize global synchronization of moving agents on a plane.

- Lu *et al.*⁵⁸ consider a synchronization-based model to explain the asymmetry of human response to jet lag caused by eastward versus westward travel.
- Bick *et al.*⁵⁹ study the emergence of chaos in coupled oscillator networks that may involve simultaneous interaction of more than two oscillators.
- DeVille and Ermentrout⁶⁰ study the Kuramoto model on special sparse networks and find that a variety of stable non-synchronized states can exist.
- Fujiwara *et al.*⁶¹ study conditions for synchronization in time-varying networks of coupled oscillators.

Many papers in the issue can be partially grouped into (nonexclusive) themes. In particular, a number of contributions cover the topic of chimera states.

- Hart *et al.*⁶² experimentally observe chimera states in a minimal network of four globally coupled identical optoelectronic oscillators with time-delayed feedback loops.
- Kemeth *et al.*⁶³ propose a new scheme for defining and classifying a wide range of chimera states.
- Martens *et al.*⁶⁴ generalize a two-cluster system that has previously been used to study chimera states by allowing for distinct phase lags within and across clusters.
- Belykh *et al.*⁶⁵ study a model similar to one often studied in the context of chimera state, but generalize it by allowing for distinct numbers of oscillators in each node and by giving the oscillators inertia.
- Nkomo *et al.*⁶⁶ report experimental chimera states in a system of coupled Belousov–Zhabotinsky chemical oscillators and compare their experimental results with numerical simulations of a mathematical model.
- Ulonska *et al.*⁶⁷ study van der Pol oscillators coupled via networks of increasingly hierarchical nature and find that large network clustering coefficients can promote the existence of chimera states of different types.

Several other contributions explicitly consider implications of symmetries on synchronization patterns:

- Golubitsky and Stewart⁶⁸ survey, within the formalism of coupled cell systems, symmetry and other mechanisms for patterns of synchrony and phase locking.
- Nishikawa and Motter⁶⁹ study the properties of networks optimized for the global synchronizability of coupled oscillators and find that such networks exhibit symmetrical structures and can also support cluster synchronization.
- Schaub *et al.*⁷⁰ apply the graph-theoretical concept of external equitable partitions to find conditions for cluster synchronization to occur in general systems of Pecora–Carroll coupled oscillators.
- Sorrentino and Pecora⁷¹ examine a network of coupled oscillators with small parameter mismatches both in the oscillator properties and network couplings and develop low-dimensional methods for predicting synchronization error.

There are also several contributions (in addition to some above) that address problems of optimization and/or control in synchronization dynamics:

- Skardal *et al.*⁷² study optimization of network structures for synchronization of coupled phase oscillators in

directed networks with a given distribution of natural frequencies.

- Nagao *et al.*⁷³ experimentally study the use of delay-coupled networks of electrochemical reactions to show that changes in coupling can induce changes in behavior from oscillation death to anti-phase synchronization.
- Skardal and Arenas⁷⁴ analyze two methods of control for Stuart-Landau coupled oscillators, with the goal of directing the system to a desired state with a minimal intervention.
- Deng *et al.*⁷⁵ study the optimization of coupling strengths to suppress amplitude death in a chain of coupled Stuart-Landau oscillators.
- Tandon *et al.*⁷⁶ propose a control strategy in which chaotic nonidentical oscillators can be synchronized by temporarily uncoupling them when they enter certain regions of the phase space.

Finally, a number of other contributions focus on the impact of time delays, phase lags, and noise:

- Laing studies⁷⁷ the impact of delays on the existence of stable travelling waves in non-locally coupled networks of theta neurons and phase oscillators.
- Kobayashi and Kori⁷⁸ show that, in networks of noisy phase oscillators, synchronizability is reduced as degree heterogeneity is increased and that both noise and strong coupling tend to induce phase slip desynchronization.
- Focusing on the impact of phase lags on the synchronization transitions in the Kuramoto model, Omel'chenko and Wolfrum⁷⁹ show that there are scenarios for which synchrony may decrease for increasing coupling strength.
- D'Huys *et al.*⁸⁰ examine networks of Boolean switches with time-delayed coupling both experimentally and analytically, finding surprisingly long transients that scale exponentially with the time delay.
- Emenheiser *et al.*⁸¹ investigate the deterministic and stochastic dynamics of nearest-neighbor-coupled micromechanical resonators and characterize the switching that can occur between distinct attracting states.

ACKNOWLEDGMENTS

We thank all the authors of this Focus Issue for their invaluable contributions. We apologize to colleagues who feel their papers should also be mentioned in this brief introduction—a comprehensive review would take many more pages.

¹A. T. Winfree, "Biological rhythms and the behavior of populations of coupled oscillators," *J. Theor. Biol.* **16**, 15 (1967).

²Y. Kuramoto, "Self-entrainment of a population of coupled non-linear oscillators," in *International Symposium on Mathematical Problems in Theoretical Physics*, Lecture Notes in Physics Vol. 39, edited by H. Araki (Springer-Verlag, 1975), p. 420.

³C. Huygens, Letters to de Sluse (Letters No. 1333, No. 1335, and No. 1345) (1665).

⁴R. Hooke, *Micrographia: Some Physiological Descriptions of Minute Bodies Made by Magnifying Glasses with Observations and Inquiries Thereupon* (Jo. Martin and Ja. Allestry, London, 1665).

⁵R. A. DeSilva, "George Ralph Mines, ventricular fibrillation and the discovery of the vulnerable period," *J. Am. Coll. Cardiol.* **29**, 1397 (1997).

⁶B. P. Belousov, "Periodicheski deistvuyushchaya reaktsia i ee mekhanizm" [Periodically acting reaction and its mechanism], *Sbornik Referatov po Radiatsionnoi Meditsine*, 1958 [Collection of Abstracts on Radiation Medicine, 1958] p. 145 (1959).

⁷Y. Kuramoto and D. Battogtokh, "Coexistence of coherence and incoherence in nonlocally, coupled phase oscillators," *Nonlinear Phenom. Complex Syst.* **5**, 380 (2002).

⁸D. M. Abrams and S. H. Strogatz, "Chimera states for coupled oscillators," *Phys. Rev. Lett.* **93**, 174102 (2004).

⁹I. Fischer, R. Vicente, J. M. Buldú, M. Peil, C. R. Mirasso, M. C. Torrent, and J. García-Ojalvo, "Zero-lag long-range synchronization via dynamical relaying," *Phys. Rev. Lett.* **97**, 123902 (2006).

¹⁰V. Nicosia, M. Valencia, M. Chavez, A. Díaz-Guilera, and V. Latora, "Remote synchronization reveals network symmetries and functional modules," *Phys. Rev. Lett.* **110**, 174102 (2013).

¹¹L. M. Pecora, F. Sorrentino, A. M. Hagerstrom, T. E. Murphy, and R. Roy, "Cluster synchronization and isolated desynchronization in complex networks with symmetries," *Nat. Commun.* **5**, 4079 (2014).

¹²T. Nishikawa and A. E. Motter, "Symmetric states requiring system asymmetry," *Phys. Rev. Lett.* **117**, 114101 (2016).

¹³S. H. Strogatz, "From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators," *Physica D* **143**, 1 (2000).

¹⁴F. Dörfler and F. Bullo, "Synchronization in complex networks of phase oscillators: A survey," *Automatica* **50**, 1539 (2014).

¹⁵H. Sakaguchi and Y. Kuramoto, "A soluble active rotator model showing phase transitions via mutual entertainment," *Prog. Theor. Phys.* **76**, 576 (1986).

¹⁶H. Sakaguchi, S. Shinomoto, and Y. Kuramoto, "Local and global self-entrainments in oscillator lattices," *Prog. Theor. Phys.* **77**, 1005 (1987).

¹⁷S. H. Strogatz and R. E. Mirollo, "Phase-locking and critical phenomena in lattices of coupled nonlinear oscillators with random intrinsic frequencies," *Physica D* **31**, 143 (1988).

¹⁸K. Wiesenfeld, P. Colet, and S. H. Strogatz, "Synchronization transitions in a disordered Josephson series array," *Phys. Rev. Lett.* **76**, 404 (1996).

¹⁹E. Montbrió, J. Kurths, and B. Blasius, "Synchronization of two interacting populations of oscillators," *Phys. Rev. E* **70**, 056125 (2004).

²⁰A. Jadbabaie, N. Motee, and M. Barahona, "On the stability of the Kuramoto model of coupled nonlinear oscillators," *Proc. Am. Control Conf.* **5**, 4296 (2004).

²¹Y. Moren and A. F. Pacheco, "Synchronization of Kuramoto oscillators in scale-free networks," *EPL* **68**, 603 (2004).

²²E. Oh, K. Rho, H. Hong, and B. Kahng, "Modular synchronization in complex networks," *Phys. Rev. E* **72**, 047101 (2005).

²³A. Arenas, A. Díaz-Guilera, and C. J. Pérez-Vicente, "Synchronization processes in complex networks," *Physica D* **224**, 27 (2006).

²⁴S. Boccaletti *et al.*, "Detecting complex network modularity by dynamical clustering," *Phys. Rev. E* **75**, 045102 (2007).

²⁵M. J. Panaggio and D. M. Abrams, "Chimera states: Coexistence of coherence and incoherence in networks of coupled oscillators," *Nonlinearity* **28**, R67 (2015).

²⁶T. Yamada and H. Fujisaka, "Stability theory of synchronized motion in coupled-oscillator systems. II," *Prog. Theor. Phys.* **70**, 1240 (1983).

²⁷T. Yamada and H. Fujisaka, "Stability theory of synchronized motion in coupled-oscillator systems. III," *Prog. Theor. Phys.* **72**, 885 (1984).

²⁸H. Fujisaka and T. Yamada, "Stability theory of synchronized motion in coupled-oscillator systems. IV," *Prog. Theor. Phys.* **74**, 918 (1985).

²⁹A. Pikovskii, "On the interaction of strange attractors," *Z. Phys. B* **55**, 149 (1984).

³⁰V. S. Afraimovich, N. N. Verichev, and M. I. Rabinovich, "Stochastic synchronization of oscillation in dissipative systems," *Radiophys. Quantum Electron.* **29**, 795 (1986).

³¹A. R. Volkovskii and N. F. Rul'kov, "Experimental study of bifurcations at the threshold for stochastic locking," *Sov. Tech. Phys. Lett.* **15**, 249 (1989).

³²L. M. Pecora and T. L. Carroll, "Synchronization in chaotic systems," *Phys. Rev. Lett.* **64**, 821 (1990).

³³J. F. Heagy, T. L. Carroll, and L. M. Pecora, "Synchronous chaos in coupled oscillator systems," *Phys. Rev. E* **50**, 1874 (1994).

³⁴G. Hu, J. Yang, and W. Liu, "Instability and controllability of linearly coupled oscillators: Eigenvalue analysis," *Phys. Rev. E* **58**, 4440 (1998).

³⁵P. M. Gade, "Synchronization in coupled map lattices with random nonlocal connectivity," *Phys. Rev. E* **54**, 64 (1996).

- ³⁶C. W. Wu and L. O. Chua, "On a conjecture regarding the synchronization in an array of linearly coupled dynamical systems," *IEEE Trans. Circuits Syst., I* **43**, 161 (1996).
- ³⁷L. M. Pecora and T. L. Carroll, "Master stability functions for synchronized coupled systems," *Phys. Rev. Lett.* **80**, 2109 (1998).
- ³⁸D. J. Watts and S. Strogatz, "Collective dynamics of 'small-world' networks," *Nature* **393**, 440 (1998).
- ³⁹A.-L. Barabási and R. Albert, "Emergence of scaling in random networks," *Science* **286**, 509 (1999).
- ⁴⁰C. Zhou and J. Kurths, "Hierarchical synchronization in complex networks with heterogeneous degrees," *Chaos* **16**, 015104 (2006).
- ⁴¹C. Fu *et al.*, "Topological control of synchronous patterns in systems of networked chaotic oscillators," *Phys. Rev. E* **87**, 032909 (2013).
- ⁴²I. Kanter *et al.*, "Synchronization of unidirectional time delay chaotic networks and the greatest common divisor," *EPL* **93**, 60003 (2011).
- ⁴³D. P. Rosin, D. Rontani, and D. J. Gauthier, "Control of synchronization patterns in neural-like Boolean networks," *Phys. Rev. Lett.* **110**, 104102 (2013).
- ⁴⁴F. Sorrentino and E. Ott, "Network synchronization of groups," *Phys. Rev. E* **76**, 056114 (2007).
- ⁴⁵C. Williams *et al.*, "Experimental observations of group synchrony in a system of chaotic optoelectronic oscillators," *Phys. Rev. Lett.* **110**, 064104 (2013).
- ⁴⁶V. Belykh *et al.*, "Cluster synchronization in oscillatory networks," *Chaos* **18**, 037106 (2008).
- ⁴⁷O. D'Huys *et al.*, "Synchronization properties of network motifs: Influence of coupling delay and symmetry," *Chaos* **18**, 037116 (2008).
- ⁴⁸G. Russo and J.-J. E. Slotine, "Symmetries, stability, and control in nonlinear systems and networks," *Phys. Rev. E* **84**, 041929 (2011).
- ⁴⁹K. Judd, "Networked dynamical systems with linear coupling: Synchronisation patterns, coherence and other behaviours," *Chaos* **23**, 043112 (2013).
- ⁵⁰N. O'Clery, Y. Yuan, G.-B. Stan, and M. Barahona, "Observability and coarse graining of consensus dynamics through the external equitable partition," *Phys. Rev. E* **88**, 042805 (2013).
- ⁵¹M. Golubitsky, I. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory* (Springer-Verlag, 1985).
- ⁵²W. Stein, *SAGE: Software for Algebra and Geometry Experimentation*, see <http://www.sagemath.org/sage/>; <http://sage.scipy.org/> (2013).
- ⁵³M. Golubitsky, I. Stewart, and A. Török, "Patterns of synchrony in coupled cell networks with multiple arrows," *SIAM J. Appl. Dyn. Syst.* **4**(1), 78 (2005).
- ⁵⁴M. Golubitsky and I. Stewart, *The Symmetry Perspective: From Equilibrium to Chaos in Phase Space and Physical Space* (Berkhäuser-Verlag, 2002).
- ⁵⁵I. Belykh and M. Hasler, "Mesoscale and clusters of synchrony in networks of bursting neurons," *Chaos* **21**, 016106 (2011).
- ⁵⁶B. Ottino-Löffler and S. H. Strogatz, "Frequency spirals," *Chaos* **26**, 094804 (2016).
- ⁵⁷L. Wang and G. Chen, "Synchronization of multi-agent systems with metric-topological interactions," *Chaos* **26**, 094809 (2016).
- ⁵⁸Z. Lu, K. Klein-Cardena, S. Lee, T. M. Antonsen, M. Girvan, and E. Ott, "Resynchronization of circadian oscillators and the east-west asymmetry of jet-lag," *Chaos* **26**, 094811 (2016).
- ⁵⁹C. Bick, P. Ashwin, and A. Rodrigues, "Chaos in generically coupled phase oscillator networks with nonpairwise interactions," *Chaos* **26**, 094814 (2016).
- ⁶⁰L. DeVillie and B. Ermentrout, "Phase-locked patterns of the Kuramoto model on 3-regular graphs," *Chaos* **26**, 094820 (2016).
- ⁶¹N. Fujiwara, J. Kurths, and A. Díaz-Guilera, "Synchronization of mobile chaotic oscillator networks," *Chaos* **26**, 094824 (2016).
- ⁶²J. D. Hart, K. Bansal, T. E. Murphy, and R. Roy, "Experimental observation of chimera and cluster states in a minimal globally coupled network," *Chaos* **26**, 094801 (2016).
- ⁶³F. P. Kemeth, S. W. Haugland, L. Schmidt, I. G. Kevrekidis, and K. Krischer, "A classification scheme for chimera states," *Chaos* **26**, 094815 (2016).
- ⁶⁴E. A. Martens, C. Bick, and M. J. Panaggio, "Chimera states in two populations with heterogeneous phase-lag," *Chaos* **26**, 094819 (2016).
- ⁶⁵I. Belykh, B. N. Brister, and V. N. Belykh, "Bistability of patterns of synchrony in Kuramoto oscillators with inertia," *Chaos* **26**, 094822 (2016).
- ⁶⁶S. Nkomo, M. R. Tinsley, and K. Showalter, "Chimera and chimera-like states in populations of nonlocally coupled homogeneous and heterogeneous chemical oscillators," *Chaos* **26**, 094826 (2016).
- ⁶⁷S. Ulonska, I. Omelchenko, A. Zakharova, and E. Schöll, "Chimera states in networks of Van der Pol oscillators with hierarchical connectivities," *Chaos* **26**, 094825 (2016).
- ⁶⁸M. Golubitsky and I. Stewart, "Rigid patterns of synchrony for equilibria and periodic cycles in network dynamics," *Chaos* **26**, 094803 (2016).
- ⁶⁹T. Nishikawa and A. E. Motter, "Network-complement transitions, symmetries, and cluster synchronization," *Chaos* **26**, 094818 (2016).
- ⁷⁰M. T. Schaub, N. O'Clery, Y. N. Billeh, J.-C. Delvenne, R. Lambiotte, and M. Barahona, "Graph partitions and cluster synchronization in networks of oscillators," *Chaos* **26**, 094821 (2016).
- ⁷¹F. Sorrentino and L. Pecora, "Approximate cluster synchronization in networks with symmetries and parameter mismatches," *Chaos* **26**, 094823 (2016).
- ⁷²P. S. Skardal, D. Taylor, and J. Sun, "Optimal synchronization of directed complex networks," *Chaos* **26**, 094807 (2016).
- ⁷³R. Nagao, W. Zou, J. Kurths, and I. Z. Kiss, "Restoring oscillatory behavior from amplitude death with anti-phase synchronization patterns in networks of electrochemical oscillations," *Chaos* **26**, 094808 (2016).
- ⁷⁴P. S. Skardal and A. Arenas, "On controlling networks of limit-cycle oscillators," *Chaos* **26**, 094812 (2016).
- ⁷⁵T. Deng, W. Liu, Y. Zhu, J. Xiao, and J. Kurths, "Reviving oscillation with optimal spatial period of frequency distribution in coupled oscillators," *Chaos* **26**, 094813 (2016).
- ⁷⁶A. Tandon, M. Schröder, M. Mannattil, M. Timme, and S. Chakraborty, "Synchronizing noisy nonidentical oscillators by transient uncoupling," *Chaos* **26**, 094817 (2016).
- ⁷⁷C. R. Laing, "Travelling waves in arrays of delay-coupled phase oscillators," *Chaos* **26**, 094802 (2016).
- ⁷⁸Y. Kobayashi and H. Kori, "Synchronization failure caused by interplay between noise and network heterogeneity," *Chaos* **26**, 094805 (2016).
- ⁷⁹O. E. Omel'chenko and M. Wolfrum, "Is there an impact of small phase lags in the Kuramoto model?," *Chaos* **26**, 094806 (2016).
- ⁸⁰O. D'Huys, J. Lohmann, N. D. Haynes, and D. J. Gauthier, "Super-transient scaling in time-delay autonomous Boolean network motifs," *Chaos* **26**, 094810 (2016).
- ⁸¹J. Emenheiser, A. Chapman, M. Pósfai, J. P. Crutchfield, M. Mesbahi, and R. M. D'Souza, "Patterns of patterns of synchronization: Noise induced attractor switching in rings of coupled nonlinear oscillators," *Chaos* **26**, 094816 (2016).