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Note on Random Raindrops and Sampling Error

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A number of variables of interest in precipitation studies can be interpreted as sums of random numbers of random variables, where the summed quantity is a power transformation of raindrop diameter. Included in this category are raingauge measurements and radar reflectivity. A general expression is given for the coefficient of variation of such variables when the number of terms per sum and the transformed variable follow any specified probability distribution. Some simple expressions are obtained for the special case of an exponential distribution of raindrop diameters.

Introduction

Many measured variables of interest in precipitation studies represent sample sums of variables derived from a power transformation of raindrop diameter. Included in this category are the depth of rainfall measured from a raingauge, and the radar reflectivity of a given volume of air containing raindrops. Gertzman and Atlas (1977) derived an expression for the coefficient of variation of such variables when the number of drops per sample follow a particular type of mixed Poisson distribution. In this paper, it is shown that some statistical results obtained by Robbins (1948) can be used to obtain a more general expression for the coefficient of variation, permitting any combination of sample size and drop distributions. Some results of Gertzman and Atlas (1977) are obtained as special cases of the general expression.

Assumptions

Following Gertzman and Atlas (1977), the basic assumptions are:

1) One or more samples of N raindrops are obtained. Such samples are assumed to have been accumulated in collectors over some time interval Δt, or alternatively the raindrops may have been obtained from an instantaneous volume sample.
2) The magnitude of N in any one sample is independent of the magnitude of N in any other, i.e. N is a random variable.
3) The drop size distribution remains constant through space, and through the time Δt in the case of accumulated samples. The present treatment differs from that of Gertzman and Atlas (1977) in that no assumption is made with respect to the probability function governing the distribution of N.

Derivations

Defining D as a random variable representing raindrop diameter, the probability density function of drop diameter can be written

$$f_D(d), \qquad 0 < d \leqslant u, \qquad 0 < u \leqslant \infty \tag{1}$$

A derived variable x is related to drop diameter by the power transformation

$$x = a\,D^c; \qquad a, c > 0 \tag{2}$$

and the random variable y is defined as the sample sum of x

$$y = \sum_{i=0}^{N} x_i \tag{3}$$

It is evident that y represents the sum of a random number of random variables. From Robbins (1948, Eq.(5)) the cofficient of variation of y is obtained immediately as

$$\frac{\sigma_y}{\mu_y} = \frac{(\mu_N \sigma_x^2 + \sigma_N^2 \mu_x^2)^{\frac{1}{2}}}{\mu_N \mu_x} \tag{4}$$

where μ and σ denote mean and standard deviation with respect to the subscripted variable. The expression (4) is of course quite general, and x could represent any random variable.

If N is a Poisson random variable with expectation λ, then Eq. (4) simplifies to

$$\frac{[E(x^2)]^{\frac{1}{2}}}{\lambda^{\frac{1}{2}} \mu_x} \tag{5}$$

where E denotes expectation. It will be noted that Eq. (5) is the same as Eq. (14) of Gertzman and Atlas (1977). If λ is large, y is approximately normally distributed (Robbins 1948).

Exponential Drop Size

For the special case where the raindrop diameters follow a negative exponential distribution, the transformed variable x follows a Weibull distribution with shape parameter c^{-1} (Johnson and Kotz 1970, p. 221). The cofficient of variation of y is obtained by substituting μ_x and σ_x in Eq. (4) with the expressions for the Weibull mean and standard deviation. This gives

$$\frac{[\mu_N\, \Gamma(2c+1) + (\sigma_N^2 - \mu_N)\Gamma^2(c+1)]^{\frac{1}{2}}}{\mu_N\, \Gamma(c+1)} \tag{6}$$

where Γ denotes the gamma function. It will be noted that Eq. (6) is independent of the parameter of the exponential distribution.

If the raindrops are randomly distributed in space (over a surface or through a volume), then the number of raindrops in samples collected from equal areas or volumes will be Poisson distributed with some expectation λ. The coefficient of variation Eq. (6) then simplifies to

$$\frac{\Gamma^{\frac{1}{2}}(2c+1)}{\lambda^{\frac{1}{2}}\Gamma(c+1)} \tag{7}$$

If the raindrops are obtained from volumes or areas which are not constant but follow some probability distribution then this implies a probability distribution of λ. In this case μ_N and σ_N in Eq. (6) will represent the mean and standard deviation of a mixed Poisson distribution, as defined by Haight (1967, p. 35).

It should be pointed out that Eq. (7) is a special case of the equivalent expression of Gertzman and Atlas (1977), corresponding to their Eq. (29). The unnecessary complexity of the latter equation results from the erroneous conclusion of Gertzman and Atlas (1977) that area sampling devices (raingauges, hailpads etc) accumulate samples from volumes of air, presumably with defined boundaries. Given the turbulent nature of atmospheric motion and the variable fall velocities of individual raindrops, it is apparent that such hypothetical »volumes« cannot have any physical meaning. Areal collectors, therefore, can only be viewed as devices for obtaining samples from particular portions of surface.

The use of Eq. (7) leads to particularly simple expressions for the coefficient of variation of a number of variables of interest. For example, an expression can be directly obtained from Eq. (7) for the coefficient of variation of rainfall depth as measured by a raingauge. The assumptions are that at the end of the sampling period, the raindrop impact points are distributed randomly over the surface of the area of interest and the drop diameters follow a common exponential distribution. The required transformation is given by

$$x = g^{-1} \frac{1}{6} \pi D^3 \qquad (8)$$

where g is the gauge aperture and x the depth increment contributed by a single raindrop. The coefficient of variation of the observed depth is obtained from Eq. (7) with $c=3$ as

$$4.47 \lambda^{-\frac{1}{2}} \qquad (9)$$

The raingauge aperture size is implicitly included in Eq. (9) since λ is directly proportional to g.

It is apparent from Eq. (7) that the coefficient of variation in this case will always be of the form $h\lambda^{-1/2}$, where h is some constant determined by the magnitu-

Table 1 - Some **c** and **h** values for some quantities resulting from Poisson sums of power transformations on raindrop diameters.

c	quantity	h
2	optical extinction	2.45
3	water volume	4.47
6	radar reflectivity within the Rayleigh scattering region	30.40

de of c. As noted by Gertzman and Atlas (1977), a number of important variables in precipitation studies can be defined with respect to particular values of c. Some of these variables are listed in Table 1, together with their associated h values. These expressions are only applicable to the particular case of exponential drop size distribution and Poisson distributed sample size.

Conclusions

In the case of raingauge measures, the sampling period is often long, and observed rainfall depth must represent the cumulative effect of a sequence of m different drop size distributions. In this situation, a given measured depth represents the sum of m variously distributed random variables, each one of which represents the sum of a random number of random variables. It would be useful, therefore, to preserve the discrete raindrops as they enter the gauge, perhaps through the use of a suitable chemical. The resulting sample could then be interpreted as a vertical core through a stratified sedimentary deposit, and the mean and variance of the total rainfall depth could then be obtained as the sum of the mean and variance of each individual layer.

The interpretation of some hydrologically important variables as representing sums of random numbers of random variables provides a useful base for investigating their stochastic behaviour. If required, more sophisticated models could be constructed by allowing the drop size distribution to vary through space and time. It is emphasised that the results given in this paper apply only to random sampling variation. Systematic components of variability may or may not be of greater importance, depending on the physical environment and the particular variable sampled.

References

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