A Generalization of Special Lorentz Transformation in de Sitter Space-time

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The transformations of the full Lorentz group are generalized in de Sitter space-time from the standpoint of the group of motions. When the curvature of the space-time tends to zero, the generalized transformations tend to the well-known forms of the corresponding transformations of the Lorentz group. Some properties of the transformations, especially those of the generalized special Lorentz transformation are studied. In connection with this research the important rôle the group of motions plays in relativity is considered and the difference of the notion of the invariancy in general and special relativities is also studied.

§ 1. Introduction

Special Lorentz transformation plays an important rôle in the special theory of relativity. In this paper we are going to obtain a transformation in de Sitter space-time \([S]\) corresponding to the Lorentz transformation in Minkowski space-time \([M]\). Lorentz transformation as a transformation in \([M]\) has various properties, so we may generalize it from various standpoints such as that of the group of motions, that of the generalized uniform motion\(^*\), etc. In this paper we shall take the first standpoint, i.e. we shall obtain the generalized Lorentz transformation from the standpoint of the group of motions in the sense used in Riemannian geometry\(^1\) using the fact that the Lorentz transformation is a motion in \([M]\). As will be seen later, a generalization of uniform motion in \([S]\) is obtained as a byproduct.

First in § 2 and § 3 some considerations are made about the physical meanings of the group of motions and the coordinate systems used in this paper. In § 4 and § 5 the infinitesimal and the finite forms of the generalized special Lorentz transformation \(L_1\) are determined in the coordinate system of (4.2). Then some properties of this \(L_1\) are studied in § 6 and § 7. Lastly in § 8 the infinitesimal and the finite forms of \(L_1\) are obtained in another coordinate system in which (8.6) holds.

\(^*\) Since Lorentz transformation was first obtained in relativity in connection with the problem of equivalency of the observers moving uniformly relative to each other, we may first generalize this notion of uniform motion suitably and then may define the generalized Lorentz transformation in \([S]\) as a transformation connecting observers in such a generalized uniform motion relative to each other.
§2. Group of motions in a space-time and the equivalency of observers

Mathematically speaking, the fundamental principle of special relativity is that the physical laws are invariant under the transformations of Lorentz group, while one of the leading principles of general relativity is that all the physical laws are invariant under all transformations. However the meaning of the term "invariant" as used in both principles is not necessarily the same. In the following we shall first explain the concept of the invariance in general relativity.*

Now let $A, B, \ldots$ be tensors expressing physical quantities, then a physical law in general relativity is expressed by one or several tensor equations of the form:

$$F(A, B, \ldots; p_i A, \ldots) = 0,$$

(2.1)

where $p_i$ denotes covariant derivative of the tensor. If we regard this (2.1) as an equation for $A, B, \ldots; p_i A, \ldots$, it is form-invariant under any coordinate transformation and in this sense all coordinate systems are equivalent. This is the meaning of the invariance in general relativity. (Here it is to be noticed that some of these coordinate systems have no such physical meaning as observation system.) In general, however, this (2.1) is not necessarily invariant in the sense used in special relativity. To clarify the circumstance we shall give an example.

If we regard the fundamental tensor $g_{ij}$ of the space-time as a physical quantity its components are functions of the coordinate variables and their functional forms depend upon the coordinate system. Hence in the equation $ds^2 = 0$, i.e. $g_{ij}dx^i dx^j = 0$ concerning the path of light, $(i, j=1, \ldots, 4)$, if we regard $ds^2$ as a function of $g_{ij}$ and $dx^i$, this law is form-invariant under any coordinate transformation, while on the other hand if we regard this $ds^2$ as a function of $x^i$ and $dx^i$ the law $ds^2 = 0$ is form-invariant only under conformal transformations in the space-time and not form-invariant under another transformations, and further the quantity $ds^2$ is form-invariant only under the group of motions which forms a sub-group of the above stated conformal group.

Therefore from the standpoint of the form-invariancy of $ds^2$ as a function of $x^i$ and $dx^i$, it is only the observers relatively connected with each other by the motion of the space-time that are equivalent. This is the meaning of invariancy in special relativity. In the theories which treat of such invariancy it is needless to deal with the coordinate systems which have no physical meaning, since motions are, in general, accompanied by some physical meanings.** Moreover it is not only to be noticed that tensor equations are not necessarily form-invariant, but also that invariant laws are not necessarily to be expressed by tensor equations.***

*) See also the references 2) and 3).

**) Møller laid stress on the motion in a curved space-time and called it a generalized Lorentz transformation**. It is to be noticed, however, that his terminology differs from the present one.

*** For instance, if we take $(\gamma^I \partial / \partial \gamma^I + m)\phi = 0$ where $\gamma(\gamma + 1) = g^{44}$ as the generalized Dirac equation, this equation is form-invariant under any motion, though it is not of tensor form. If we wish to treat Dirac equation of tensor form, we must take $(\gamma^I p_i + \gamma m)\phi = 0$ using covariant derivative $p_i$ in place of $\partial / \partial x^i$. 
From the above consideration we can conclude that when we wish to discuss about the invariance of a physical law we must first make clear in what sense the invariance is treated, otherwise the deduction will be meaningless.

It was first emphasized by Robertson in his relativistic cosmology that the concept of group of motions plays an important rôle in the theory of equivalency of observers in a curved space-time. We may rather say that the equivalency of observers relatively connected with each other by motions was the starting point of his theory.\(^5\)

As is well known, the group of motions of \([M]\) is the full Lorentz group with 10 parameters, and in this sense we may regard the special relativity as a theory concerning the physical laws form-invariant under the group of motions of \([M]\). Hence in order to construct in a curved space-time a theory which corresponds to the special theory of relativity in \([M]\), we have to deal with the physical laws form-invariant (in the sense used in the special relativity) under the group of motions of the space-time. For this purpose, we must first make clear the group of motions. In Riemannian geometry, however, it is well known that the number of the parameters of the group is 10 only when the space-time is \([M]\) or \([S]\), and is less than 10 in another space-time. Accordingly from the standpoint of the group of motions, \([S]\) is the curved space-time in which the generalization of the Lorentz transformation is dealt with most naturally.

In this paper, we shall first determine the forms of the transformations of the group of motions in this \([S]\) so that they may tend to the well-known forms of the corresponding transformations of the full Lorentz group when the curvature of the space-time tends to zero, and then we shall study some properties of the transformation corresponding to the special Lorentz transformation.

\(\S\) 3. Standard coordinate system in a curved space-time

In Minkowski space-time \([M]\) there exists a standard coordinate system in which the values of the coordinate variables correspond directly to the results of our measurement or experiment. Namely, it is the coordinate system in which the metric \(ds^2\) takes the form:

\[
   ds^2 = -dx^2 - dy^2 - dz^2 + dt^2, \tag{3.1}
\]

where, for brevity’s sake, the units of length and time are taken so as the light velocity in vacuum is unity. On the other hand in de Sitter space-time or, generally speaking, in a curved space-time, such a standard coordinate system does not exist. Furthermore it is very difficult to determine theoretically such a coordinate system. This problem will not be solved till we succeed in clarifying the law which connects the values of the coordinate variables and the results of our observations in a curved space-time.

Therefore we can not decide what coordinate system in \([S]\) is the most suitable one for our present investigation. In the following we shall explain this in detail: In the coordinate systems of \([S]\) treated later in which \(ds^3\) takes the forms \((4.2)\) and \((8.6)\) respectively, if the constant \(k^2\) which gives the curvature of the space-time tends to zero, both \(ds^3\)'s tend to the same \((3.1)\). And it is impossible to determine theoretically which coordinate system is the fundamental one corresponding to \((3.1)\) of \([M]\). To answer
this it is necessary to compare both systems by experimenting in some way or other. But it seems to the writer that at the present stage even by experiment it is difficult to solve this problem completely, and moreover it should be noticed that there exist many coordinate systems with similar qualifications. Hence in this paper we shall not deal with this problem further and shall content ourselves only with the investigation of the transformation in \([S]\) corresponding to Lorentz transformation in the two coordinate systems stated above.

\section{§ 4. Infinitesimal form}

As is well known, \([S]\) is a Riemannian space of constant curvature whose signature of the fundamental form is given by the type \((- - - +).\) Its curvature tensor \(K_{ijlm}\) satisfies

\[ K_{ijlm} = k^2 (g_{im} g_{jl} - g_{il} g_{jm}), \]  

(4.1)

where \(k\) is a constant, and its line element is reducible to the form

\[ ds^2 = -e^{2k} (d\theta^2 + dy^2 + dz^2) + dt^2 \]  

(4.2)

by a suitable choice of the coordinate system. In (4.2) the units of \(x, y\) and \(z\) are taken so as the light velocity may become unity when \(k\) tends to 0. If \(k\) tends to 0 \([S]\) tends to \([M]\) and (4.2) to (3.1).

Transformations which keep (3.1) form-invariant constitute the full Lorentz group \(\text{O}\) or the group of motions in \([M]\) using the terminology of geometry.* \(\text{O}\) is a continuous group of 10 essential parameters and in the coordinate system of (3.1) its 10 independent transformations are given by

\[ \text{O} : \quad T_1, T_2, T_3; \quad R_1, R_2, R_3; \quad U; \quad \bar{L}_1, \bar{L}_2, \bar{L}_3 \]  

(4.3)

using the operator forms of the corresponding infinitesimal transformations, where \(T_1 = \partial_x, T_2 = \partial_y\) and \(T_3 = \partial_z\) give the translations of the space-frame; \(R_1 = -y\partial_x + x\partial_y, R_2 = -x\partial_x + y\partial_y\) and \(R_3 = -y\partial_x + x\partial_y\) give the rotations of the space-frame; \(U = \partial_t\) gives the translation of the time-frame; and

\[ \bar{L}_1 = \partial_x + x\partial_t, \quad \bar{L}_2 = \partial_y + y\partial_t, \quad \bar{L}_3 = \partial_z + z\partial_t \]  

(4.4)

give the special Lorentz transformations. Finite forms of these transformations are also well known. For later use we shall give the finite forms of \(\bar{L}_a, (a = 1, 2, 3)\):

\[ \bar{L}_1 : \quad x' = (x + vt) / \sqrt{1 - \beta^2}, \quad y' = y, \quad z' = z, \quad t' = (t + vx) / \sqrt{1 - \beta^2}. \]  

(4.5)

The corresponding forms of \(\bar{L}_2\) and \(\bar{L}_3\) are given from \(\bar{L}_1\) by the cyclic changes of \(x, y\) and \(z\).

In \([S]\), the transformations which keep its \(ds^2\) form-invariant (i.e. its motions) also form a group of 10 parameters as in the case of \([M]\) and the concrete forms of the 10

* We study the problem from the standpoint of the continuous group of transformations so that we do not deal with the transformation which has no infinitesimal transformation such as a reflection \(x' = -x\).
independent elements in the coordinate system of \((4.2)\) are given by the following operator forms:*  
\[ \Theta : \ T_1, T_2, T_3; \ R_1, R_2, R_3; \ U_0; S_1, S_2, S_3 \]  
where \(T_a\) and \(R_a\), \((a = 1, 2, 3)\), are the same as those in \(\Theta\), and  
\[ U_0 = k(x \partial_x + y \partial_y + z \partial_z) - \partial_t, \]  
\[ S_1 = (e^{-2ikt} + k^2(x^2 - y^2 - z^2)) \partial_x + 2k^2(x \partial_y + z \partial_z) - 2kx \partial_t, \text{ etc.} \]  
Any transformation of \(\Theta\) is given by a linear combination with constant coefficients of 10 operators given in \((4.6)\). Hence in order to obtain the generalizations of the transformations in \((4.3)\) we have only to determine the linear combinations of \((4.6)\) which tend to the operators of \((4.3)\) respectively. By this method we can easily obtain the following 10 transformations in place of those given by \((4.6)\):  
\[ \Theta : \ T_1, T_2, T_3; \ R_1, R_2, R_3; \ U; L_1, L_2, L_3, \]  
where  
\[ U = -U_0 = -k(x \partial_x + y \partial_y + z \partial_z) + \partial_t, \]  
\[ L_1 = \frac{1}{2k}(T_1 - S_1) = \left\{ \frac{1}{2k}(1 - e^{-2ikt}) + \frac{k}{2}(y^2 + z^2 - x^2) \right\} \partial_x \]  
\[ -kx(y \partial_y + z \partial_z) + x \partial_t, \text{ etc.} \]  
Evidently when \(k \rightarrow 0\) we have \(U \rightarrow \bar{U}\) and \(L_a \rightarrow \bar{L}_a\) \((a = 1, 2, 3)\).

Hence we must conclude that in \([S]\) the transformation \(I_n\) defined by \((4.10)\) are the generalized special Lorentz transformations. In the following sections we shall obtain the finite form of \(L_n\) which corresponds to \((4.5)\) of \(\bar{L}_n\), and then shall study some of its properties.

\(U\) given by \((4.9)\) is the generalization of the translation of the time-frame \(\bar{U}\) in \([S]\) and its finite form will be given in the appendix.

\[ \text{§ 5. Finite form of } L_1 \]

In this section we shall determine the finite form of \(L_1\). In order to make clear the correspondence between this process and that of obtaining the finite form \((4.5)\) of \(\bar{L}_1\) in \([M]\), we shall first give a brief description of the latter:

From \((4.4)\) we have the simultaneous ordinary differential equations  
\[ \frac{dx'}{d\tau} = t', \quad \frac{dy'}{d\tau} = 0, \quad \frac{dz'}{d\tau} = 0, \quad \frac{dt'}{d\tau} = x'. \]  
Solving \((5.1)\) under the initial condition  
\[ x' = x, \quad y' = y, \quad z' = z, \quad t' = t \quad \text{for} \quad \tau = 0, \]  
we have  
\[ x' = ax + bt, \quad y' = y, \quad z' = z, \quad t' = at + bx, \]  
\[ \text{for} \quad \tau = 0. \]  

*) Here we give the results obtained by the present writer.\(^{(07)}\)

**) If we use Sibata's notations, we have \(L_1 = (U_1 - U_0)/2k\). His expressions of \(U_1\) and \(U_0\) were given in the coordinate system of \((8.6)\).\(^{(08)}\)
where \(a = \cosh \tau\) and \(b = \sinh \tau\). Then putting \(b/a = \tanh \tau = \nu\) we get (4.5).

We have only to follow the same process with respect to (4.10) in place of (4.4). From (4.10) we have the following equations corresponding to (5.1):

\[
\begin{align*}
\frac{dx'}{d\tau} &= \left(\frac{1}{2k}\right) \left(1 - e^{-2kt}\right) + \left(\frac{k}{2}\right) \left(y'^2 + z'^2 - x'^2\right), \\
\frac{dy'}{d\tau} &= -kx'y', \quad \frac{dz'}{d\tau} = -kx'z', \quad \frac{dt'}{d\tau} = x'.
\end{align*}
\]

From (5.42), (5.43) and (5.44), we have

\[
y'/y = z'/z = \exp\left\{-k(t' - t)\right\},
\]

and from (5.4),

\[
dx'^2/d\tau^2 + k/2 \cdot dx'/d\tau + Pe^{-2kt} - 1/2k = 0, \quad (P = 1/2k - k/2 \cdot (y^2 + z^2)e^{2kt}).
\]

Integrating this we have

\[
\frac{(dt')^2}{d\tau^2} = -\frac{2}{k} Pe^{-kt} - \frac{1}{k^2} e^{2kt} = e^{kt} \left(r^2 - 1/k^2\right) - 1/k^2 \cdot e^{-kt} (\equiv Q),
\]

where \(r = \sqrt{x^2 + y^2 + z^2}\), by virtue of (5.2). From (5.7) we can get \(dt'/d\tau\) and can integrate this easily.* By using (5.2) and (5.4), we finally obtain

\[
\begin{align*}
2L x' e^{kt} &= (Lx^2 - 1/2 \cdot k^2 Q)^2 - 2kP', \\
x' e^{kt} &= \eta \sqrt{(2k^2 + k^2 Qe^{kt} + 2kP)},
\end{align*}
\]

where

\[
L = 1/2 \cdot (1 + k^2 \eta^2)e^{kt} - 1/2 \cdot e^{-kt} + \eta/kxe^{kt},
\]

and \(\eta = \pm 1\). Since the result for \(\eta = -1\) is obtained from that for \(\eta = +1\) by reversing the orientations of \(x\) and \(x'\) axes, we shall assume that \(\eta = +1\).

(5.5) and (5.8) together give the finite form of \(L_1\) which we are seeking for. In fact we can easily verify that when \(k \to 0\) (5.5) tends to the second and the third equations of (4.5) and that (5.81) and (5.82) tend to the fourth and the first of (4.5) respectively.

From the mathematical standpoint, the above obtained transformation \{(5.5), (5.8)\} is somewhat complicated than (4.5) in that \(y\) and \(z\) are contained in the expressions of \(x'\) and \(t'\) besides \(x\) and \(t\) and that \(y\) and \(z\) are not invariant under the transformation.

**§ 6. Finite form of \(L_1\) expressed in a symmetric form**

In this section we shall show that we can express the finite form \{(5.5), (5.8)\} of \(L_1\) obtained in the last section in a more symmetrical form by using three invariants and one relative invariant.

From (5.5) it is easily seen that both

\[
\Delta_1 = y e^{kt} \quad \text{and} \quad \Delta_2 = z e^{kt}
\]

are invariants of \(L_1\), i.e.

\* \(dx'/dt\) is determined to within a sign. The result for minus sign, however, becomes identical with that for plus sign by changing the sign of the parameter \(\tau\).
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Next from (5.8) we have

\begin{equation}
\varepsilon^{2k\mu} (\delta^2_+ \delta_-^2 - 1) - \varepsilon^{k\mu} \{ \varepsilon^{k\mu} (\delta^2_+ \delta_-^2 - 1) - c^{-k\mu} \} = 1 + \varepsilon^{2k\mu} (y^2 + z^2) = 0.
\end{equation}

From (6.2) and (6.3), we obtain

\begin{equation}
\Delta_{s'} = \Delta_{s} \text{ where } \Delta_{s} = e^{kt} (1 - k^2y^2) + e^{-kt},
\end{equation}

and \(\Delta_{s'}\) is the corresponding quantity in \((x',\ldots,s')\)-system. Hence this \(\Delta_{s}\) is the third invariant of \(L_{\gamma}\). Every invariant of \(L_{\gamma}\) is expressible as a function of these three invariants.

Then using (6.1) and (6.4) we can show that (5.8) can be rewritten in the form

\begin{equation}
\Delta_{s'} = e^{ct} \Delta_{s} \text{ where } \Delta_{s} = 2e^{kt} (1 + kx) - \Delta_{\gamma}.
\end{equation}

Hence \(\Delta_{s}\) is a relative invariant of \(L_{\gamma}\).

Evidently \(\Delta_{s}, \ldots, \Delta_{s}\) are relatively independent, so we can finally express the finite form of \(L_{\gamma}\) by the equation:

\begin{equation}
L_{\gamma}: \Delta_{s'} = \Delta_{s}, \quad \Delta_{s'} = \Delta_{s}, \quad \Delta_{s'} = \Delta_{s}, \quad \Delta_{s'} = e^{ct} \Delta_{s}.
\end{equation}

This is the symmetrical form of \(L_{\gamma}\). If we denote (6.6) by \(L_{\gamma}(\tau)\), the identical transformation and the inverse one are given by \(L_{\gamma}(0)\) and \(L_{\gamma}(-\tau)\) respectively and further it holds that:

\begin{equation}
L_{\gamma}(\tau) L_{\gamma}(\tau') = L_{\gamma}(\tau') L_{\gamma}(\tau) = L_{\gamma}(\tau + \tau').
\end{equation}

This is also evident from the theory of the continuous group of transformations.

If we put \(k \to 0\) in order to make clear what quantities in \([M]\) correspond to \(\Delta_{s}, \ldots, \Delta_{s}\) in \([S]\), then we have

\begin{equation}
\Delta_{s} \to y, \quad \Delta_{s} \to z, \quad (\Delta_{s} - 2) / k^2 \to \{t^2 - r^2\}, \quad \Delta_{s} / k \to (t + x).
\end{equation}

Evidently the quantities \(y, z, (t^2 - r^2); (t + x)\) are three invariants and one relative invariant of \(L_{\gamma}\) respectively.

By the investigation of this section we also know that the finite form (4.5) of the special Lorentz transformation \(\bar{L}_{\gamma}\) can be rewritten as (6.6) with \(\Delta_{s} = y, \Delta_{s} = z, \Delta_{s} = (t^2 - r^2), \Delta_{s} = (t + x)\) and \(v = \tanh \tau\). As will be seen from the investigation of \(L_{\gamma}\) in § 8, this form of \(\bar{L}_{\gamma}\) is convenient to determine the form of \(\bar{L}_{\gamma}\) in the coordinate system in which the metric of \([M]\) is not of the form (3.1).

§ 7. Some properties of \(L_{\gamma}\)

Finite forms of \(\bar{L}_{\gamma}\) obtained in § 5 and § 6 are somewhat complicated compared with that of \(\bar{L}_{\gamma}\). In this section we shall obtain some simple properties of \(L_{\gamma}\). For brevity’s sake we denote the observers whose coordinate systems are \((x, y, z, t)\) and \((x', y', z', t')\) by \(K\) and \(K'\) respectively. Then main differences between \(L_{\gamma}\) and \(\bar{L}_{\gamma}\) are as follows: In
all points at rest relative to $K$ move uniformly parallel to the $x'$ axis with common constant velocity relative to $K'$. On the other hand, in $L_1$, as is easily seen from (5.5) and (5.8) any point at rest relative to $K$ moves non-uniformly relative to $K'$ on the curves not parallel to the $x'$ axis in general. Since $L_1^{-1}(\tau) = L_1(\tau)$, the same circumstance holds even if $K$ and $K'$ are interchanged.

The points on the $x$ axis of $K$ are also on the $x'$ axis of $K'$ and both systems have these axes in common. On this common axis, from (5.6), (5.7) and (5.9) we have

$$2\kappa P = 1, \ k^2 Q = \varepsilon^k (k^2 x^2 - 1) - e^{-kt}, \ 2\kappa = \varepsilon^k (1 + \kappa x)^2 - e^{-kt},$$

and the transformation equations for the events on this axis become

$$2\kappa e^{\kappa'} e^{\kappa} = (2\kappa e^{\kappa} - k^2 Q)^2/4 - 1, \ \kappa x' e^{\kappa} = \sqrt{(e^{\kappa'} + k^2 Q e^{\kappa'}) + 1}.$$  

Now we shall study the motion of the spatial origin $O$ of $K$ relative to $K'$. $O$ moves on the $x'$ axis of $K'$. To study the properties of this motion we shall obtain the relation between the coordinates $x'$ and $t'$ of $O$ relative to $K'$. Substituting $x = 0$ into (7.1) we have

$$-1/2 \ k^2 Q = \cosh \kappa t, \ L = \sinh \kappa t.$$  

From (7.2) and (7.3) we have

$$\left\{ \begin{array}{l} e^{\kappa'} = \sinh \kappa t \ \cosh \tau + \cosh \kappa t, \\ 2\cosh \kappa t = e^{\kappa'} + e^{-\kappa'} - k^2 x'' e^{\kappa'}. \end{array} \right.$$  

Eliminating $e^{\kappa'}$ from (7.4), after some calculations, we finally obtain the equation:

$$k x' = 1/2 \ V(1 + k^2 x'^2 - e^{-3\kappa t'}), \quad (V = \tanh \tau),$$

which gives the motion of $O$ relative to $K'$.

When $\kappa \to 0$, (7.5) becomes $x' = V t'$ which coincides with the relations obtained from (4.5). Hence $V$ is the parameter corresponding to $v$ of $L_1$. If we calculate from (7.5) the velocity of $O$ relative to $K'$, we get

$$v = dx' / dt' = V e^{-k T} (1 - k V x'),$$

where $x'$ and $t'$ are coordinates of $O$ with respect to $K'$. We can rewrite this expression of $v$ by using the relation (7.5), and further by using (7.4) we can also express it as a function of $t$, the quantity in $K$, as follows:

$$v = V (1 - k T^2) (1 + k T V) (1 - V^2)^{-1} (1 + k T V) (1 + V^2)^{-1},$$

where $k T = \cosh \kappa t$.

Next we shall obtain a generalization of the well-known composition law of velocities concerning $L_1$. Let $K$, $K'$ and $K''$ be three observers relatively connected by

*) Strictly speaking, we must take $\pm k x' = ...$ in place of (7.5). But we have taken plus sign in order that this equation may coincide with the one obtained from (4.5) when $\kappa \to 0$. The result for minus sign is identical with the one above obtained if we change the sign of the parameter $\tau$.

**) See Appendix (1).
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If we assume that they satisfy the relations:

\[ K' = L_1(\tau)K, \quad K'' = L_1(\tau')K', \]

then from (6.7), we have

\[ K'' = L_1(\tau + \tau')K. \tag{7.9} \]

Therefore if we denote the velocities of the spatial origin \( O \) of \( K \) relative to \( K' \) and \( K'' \) by \( v \) and \( \bar{v} \) respectively and that of the spatial origin \( O' \) of \( K' \) relative to \( K'' \) by \( v' \), then by (7.6) we have

\[ v = Ve^{-2kt'}(1 - V'kx')^{-1}, \quad (x' \text{ and } t' \text{ are coordinates of } O \text{ in } K'), \]
\[ v' = V'e^{-2kt''}(1 - V''kx'')^{-1}, \quad (x'' \text{ and } t'' \text{ are } O' \text{ in } K''), \]
\[ \bar{v} = \bar{V}e^{-2kt}(1 - \bar{V}k\bar{x})^{-1}, \quad (\bar{x} \text{ and } \bar{t} \text{ are } O \text{ in } K''), \]

where \( \bar{V} = \text{tanh} \frac{\tau}{2} = \text{tanh} \frac{(\tau + \tau')}{2} = (V + V')/(1 + VV'). \tag{7.11} \]

(7.10) and (7.11) give the composition law sought for. When \( k \to 0 \), \( v \), \( v' \) and \( \bar{v} \) tend to \( V \), \( V' \) and \( \bar{V} \) respectively and (7.11) gives the ordinary composition law of \( L_1 \).

§ 8. \( L_1 \) in another coordinate system

Hitherto we have made clear the infinitesimal and finite forms of \( L_1 \) in the coordinate system of (4.2). From these we can easily obtain the forms of \( L_1 \) in another coordinate system. The following method may be the most simple one:

Let*

\[ x' = f'(x), \quad (|\partial x'/\partial x| = 0), \tag{8.1} \]

be the coordinate transformation from the coordinate system of (4.2) to the new one. If we denote the linear operator corresponding to \( L_1 \) by \( \xi^i \partial/\partial x^i \), then \( '\xi' \) in the new coordinate system is obtained from the original one by the transformation equation of a contravariant vector:

\[ '\xi'i = \partial x'/\partial x^i \cdot \xi^i. \tag{8.2} \]

Hence the infinitesimal form of \( L_1 \) in the new coordinate system is obtained by this method. Next, to obtain the finite form of \( L_1 \) in the new system we have only to use the \( \partial' \)'s introduced in §6, that is, it is given by (6.6) in which the \( \partial' \)'s are those in the new coordinate system.

Now we shall give an example. It is well known that (4.2) is transformed into

\[ ds^2 = -(1 - \kappa^2\nu^2)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1 - \kappa^2\nu^2) dt^2 \tag{8.3} \]

by the transformations:

\[ x = r \cdot \sin\theta \cos\phi, \quad y = r \cdot \sin\theta \sin\phi, \quad z = r \cdot \cos\theta, \tag{8.4} \]

*) Throughout this paper we denote the resulting coordinate variables of motion by \( x^i \) and those of the ordinary coordinate transformation which is independent of the motions by \( f^i \).
and
\[ t' = r e^{\alpha t}, \quad e^{\alpha t} \sqrt{1 - k^2 r^2} = e^{\alpha t}. \] (8.5)

In (8.3) primes are omitted for brevity's sake. Further, transforming (8.3) into \((x, y, z, t)\)-system by using (8.4) again, we finally obtain
\[ ds^2 = -(dx^2 + dy^2 + dz^2) - k^2 \alpha^2 (x dx + y dy + z dz)^2 + \alpha^2 dt^2, \] (8.6)

where \( \alpha = \sqrt{1 - k^2 r^2} \). We shall determine the forms of \( L_1 \) in the coordinate system of (8.6).

By the reason above stated the transformation to (8.6) from (4.2) is given by
\[ xe^{\alpha t} = x', \quad ye^{\alpha t} = y', \quad ze^{\alpha t} = z', \quad e^{\alpha t} = e^{\alpha t} \sqrt{1 - k^2 r^2}. \] (8.7)

Hence from (8.2) and
\[ \xi^1 = (1 - e^{-2\alpha t})/2k + k(y^2 + z^2 - x^2)/2, \]
\[ \xi^2 = -kxy, \quad \xi^3 = -kxz, \quad \xi^4 = x, \] (8.8)

which is obtained from (4.10), we have
\[ \xi^1 = a/k \cdot \sinh k t, \quad \xi^2 = \xi^3 = 0, \quad \xi^4 = x/a \cdot \cosh k t. \] (8.9)

This (8.9) gives the infinitesimal form of \( L_1 \) in the coordinate system of (8.6).* When \( k \to 0 \) in (8.9), we have \( \xi^1 \to t \) and \( \xi^4 \to x \) and \( L_1 \) tends to the \( \tilde{L}_1 \) given by (4.4).**

Then we shall determine the finite form of (8.9). From (6.1), (6.4), (6.5) and (8.7), we have
\[ A_1 = y, \quad A_2 = z, \quad A_3 = 2x \cosh k t, \quad A_4 = 2(k x + \alpha \sinh k t), \] (8.10)
in the coordinate system of (8.6). Hence the finite form is (6.6) with (8.10). We can also obtain (8.9) from (8.10) by taking \( t \) as an infinitesimal quantity of the first order.

In this coordinate system we can make similar discussions as those made in §7 with respect to the relative motion of \( K \) and \( K' \). For example, the motion and the velocity of \( O \) relative to \( K' \) is given by
\[ k^2 x'^3(1 + V^2 \cdot \sinh^2 k t') - V^2 \sinh^2 k t' = 0, \] (8.11)
and
\[ v = V(1 - k^2 x'^2)^{3/2} \cosh k t' = V(1 - k^2 x'^2)^{3/2} \cosh k t' = V(1 - k^2 x'^2)^{3/2} \sqrt{1 + k^2 x'^2 (1/V^2 - 1)}, \] (8.12)
corresponding to (7.5) and (7.6) respectively.

Appendix

(1) An alternate method of introducing (7.6).

We may use the following method to determine the velocity \( v \) of the spatial origin \( O \) of \( K \) relative to \( K' \) : We define \( v \) by

*) From (8.9) we have the relation \( L_1 = (U_2 - U_3)/2k \) stated in the footnote concerning (4.10).

**) When \( k \to 0 \), the transformation (8.7) tends to the identical transformation.
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\[
v = \left[ \frac{dx'}{dt'} \right]_0 = \left[ \frac{\partial x'}{\partial t} + \frac{\partial x'}{\partial x} \frac{dx}{dt} \right] \left[ \frac{\partial t'}{\partial t} + \frac{\partial t'}{\partial x} \frac{dx}{dt} \right]^{-1} \bigg|_{0, \tau = 0} = \left[ \frac{\partial x'}{\partial t} \frac{dt'}{dt} \right]_0,
\]

(A.1)

where an index 0 means the value of the parenthesized quantity at \( x = y = z = 0 \). For \( L_1 \) defined by (5.5) and (5.8), we have

\[
\begin{align*}
[2kP]_0 &= 1, \\
[L]_0 &= \sinh kt, \\
[\partial/\partial \tau_0 \cdot 2L]_0 &= k \cosh kt, \\
[\partial/\partial \tau_0 \cdot kP]_0 &= -k \sinh kt,
\end{align*}
\]

using which, from (7.2) we have*

\[
\begin{align*}
[\partial' / \partial \tau]_0 &= \{ \cosh kt \cosh \tau + \sinh kt \} \{ \sinh kt \cosh \tau + \cosh kt \}^{-1}, \\
[k' x / \partial \tau]_0 &= \sinh kt \sinh \tau \{ \sinh kt \cosh \tau + \cosh kt \}^{-1}, \\
[k' x]_0 &= \sinh kt \sinh \tau \{ \sinh kt \cosh \tau + \cosh kt \}^{-1}.
\end{align*}
\]

(A.2)

From these relations and (7.4) we easily obtain (7.7) and hence (7.6).

(2) Finite form of \( U \).

Finite forms of all elements of \( \mathfrak{G} \) except \( U \) have been made clear, so we shall here give that of \( U \). Solving the differential equations:

\[
\begin{align*}
\frac{dx'}{d\tau} &= -kx', \\
\frac{dy'}{d\tau} &= -ky', \\
\frac{dz'}{d\tau} &= -ks', \\
\frac{dt'}{d\tau} &= 1,
\end{align*}
\]

(A.4)

with the initial condition (5.2), we have

\[
x' = e^{-\kappa \tau}, \quad y' = e^{-\kappa \tau}, \quad z' = e^{-\kappa \tau}, \quad t' = t + \tau,
\]

(A.5)

which is the finite form of \( U \) in the coordinate system of (4.2).

Further in the coordinate system of (8.6) its infinitesimal and finite forms are given by \( \partial_t \) and \( t' = t + \tau \) respectively. Hence it coincides with \( \tilde{U} \) in (4.3) in this coordinate system.

References

1) L. P. Eisenhart, Continuous Group of Transformations, Princeton (1933), 208.
3) M. Ikeda, Prog. Theor. Phys. 8 (1952), 382.
4) C. Møller, ibid., 323.
5) H. P. Robertson, Rev. Mod. Phys. 5 (1933), 62.

*) As to the ± sign of \( k' x \), the same circumstance as in § 7 holds.