Hamiltonian Formalism in Non-local Field Theories

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(Received September 30, 1953)

The Yang-Feldman's method is generalized in order to obtain a covariant Hamiltonian formulation of quantized fields with non-local interactions. The interaction Hamiltonian is constructed according to the perturbation theory in such a way that the equation of motion in the interaction representation is integrable. The calculations are carried out actually up to the fourth order approximation.

§ 1. Introduction and summary

It has been generally believed that for fields with non-local interactions the Hamiltonian formalism is impossible and only the S-matrix exists which connects free fields in the infinite past and in the infinite future\(^{11,21}\). With the S-matrix alone, however, there remain questions how to treat unambiguously the problems involving bound states such as energy levels, line-breadths and mutual scatterings of composite particles. The aim of this paper is to find a generalization of the current covariant Hamiltonian formalism which will offer a field-theoretical basis for these problems and, at the same time, will give a S-matrix which is obviously unitary.

Recently, Umezawa-Takahashi\(^5\) and Katayama\(^6\) have shown that fields with interaction Lagrangians containing higher derivatives of field quantities can be quantized in a covariant way by generalizing the method of Yang-Feldman\(^5\). Their method can be applied, but with modifications, to the fields with non-local interactions developed by Kristensen-Møller\(^3\) and Bloch\(^9\). We shall confine ourselves to the solutions of field equations which are described by the same number of independent variables as the free fields and agree with those of the free fields in the limit of vanishing coupling constant. These solutions will be described well by the integral equations of Yang-Feldman's type which are solved by iteration. These equations, however, are not unique in that any homogeneous terms, which satisfy the free field equations and vanish as the coupling constant does, can be added to them. We can determine in a covariant way these additive terms together with the interaction Hamiltonians by successive approximations in the coupling constant such that these Hamiltonians satisfy the integrability condition and there exist unitary operators \(U(\sigma, \sigma')\) which connect states on surfaces \(\sigma\) and \(\sigma'\). Hamiltonians thus obtained are expressed as infinite power series of the coupling constant, each term of which is an integral of products of field operators over the whole space-time and has an explicit dependence on the entire form of surface \(\sigma\) on which their argument lies. These Hamiltonians are not unique, but the different systems are connected by unitary transformations in a way similar to the local theories.
The calculations have been carried out actually up to the fourth order approximation. The S-matrix constructed from these Hamiltonians is obviously in agreement up to the third order with those obtained according to the theories of Kristensen-Møller and Bloch since the in- and out-fields are defined by the same integral equations. In the fourth order, however, our equations differ by one special term, which will show that S-matrix defined by them is in general not unitary.

We shall not investigate the convergence of the Hamiltonians as infinite series, which seems to have the same character as that of the S-matrix. Then, in our formalism the integrability condition for the equation of motion is satisfied for any two points which lie not only on a space-like surface but also on a surface which can be time-like in the finite domain of space-time. It is a characteristic of the fields with non-local interactions that the Hamiltonians can not be defined without such an extension. Indeed, in the local theories we have interaction Hamiltonians which consist of a finite number of terms and satisfy the integrability condition if surfaces \( \sigma \) are restricted to the space-like ones, but if Hamiltonians are allowed to be infinite series \( \sigma \) will not always have to be space-like.

If we choose flat surfaces we can obtain easily Hamiltonians in the Schrödinger representation. In the first approximation the results agree with those of recent Pauli’s theory\(^6\). It seems to be possible that the degree of singularities of these Hamiltonians is low enough if we choose the form factor appropriately. This can be verified with the form factor of Kristensen-Møller for the first order Hamiltonian. In this way, it will be possible to treat the problems involving bound states without divergence difficulties, for instance, according to the Tamm-Dancoff’s theory\(^7\). In another way, the Hamiltonian formalism will also offer a basis to extend the Bethe-Salpeter equation\(^8,\)\(^9\) after Gell-Mann and Low\(^10\), such that its kernel is free from singularities due to the effect of the form factor.

§ 2. Method of quantization

For the sake of simplicity, we shall consider charged scalar fields \( \psi, \psi^* \) and a neutral scalar field \( u \) with non-local interactions of Kristensen-Møller’s type.\(^3\) The similar results are obtained for spinor fields \( \psi \) and \( \psi^* \). In what follows the same notations as Yang-Feldman’s\(^5\) and Kristensen-Møller’s will be used if not otherwise stated. The Lagrangian is given by

\[
\begin{align*}
L dx &= \left\{ L_0(x) dx + g \right\} \phi^* (1) \phi (1, 2, 3) u (2) \phi (3) d1 d2 d3, \\
L_0 (x) &= - \left( \partial_\mu \psi^* \cdot \partial_\mu \psi + M^2 \psi^* \psi \right) - 1/2 \left( \partial_\mu u \cdot \partial_\mu u + m^2 u^2 \right),
\end{align*}
\]

where \( \phi (1, 2, 3) \) is a form factor satisfying the Hermitian condition \( \phi (1, 2, 3) = \phi^* (3, 2, 1) \). Numbers 1, 2, 3; 1', 2', 3'; ... which denote points in space-time will always be used as variables of integration, and to simplify the expressions we shall write

\[
\phi (1, 2, 3) d1 d2 d3 \equiv d(1, 2, 3).
\]

Field equations are given by
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\[ \Box - m^2 \phi(x) = -g \{ u(2) \phi(3) \delta(x-1) d(1 2 3), \]
\[ \Box - m^2 u(x) = -g \{ \phi^*(1) \phi(3) \delta(x-2) d(1 2 3), \]
\[ \Box - m^2 \phi^*(x) = -g \{ \phi^*(1) u(2) \delta(x-3) d(123). \] \tag{4}

The solutions of (4), which agree with those of the free fields in the limit \( g \to 0 \), will be described by the integral equations

\[ \phi(x) = \phi(x, \sigma) - g \int \frac{\delta(x-1) - \delta(\sigma, 1)}{2} \Delta(x-1) u(2) \phi(3) d(123) + \sum_{n=1}^{\infty} g^n \varphi_n(x, \sigma), \]
\[ u(x) = u(x, \sigma) - g \int \frac{\phi^*(1) \delta(x-2) - \delta(\sigma, 2)}{2} \Delta(x-2) \phi(3) d(123) + \sum_{n=1}^{\infty} g^n v_n(x, \sigma), \]
\[ \phi^*(x) = \phi^*(x, \sigma) - g \int \frac{\phi^*(1) u(2) \delta(x-3) - \delta(\sigma, 3)}{2} \Delta(x-3) d(123) + \sum_{n=1}^{\infty} g^n \phi^*_n(x, \sigma). \] \tag{5}

Here we assume that \( \phi(x, \sigma) \) and \( u(x, \sigma) \), being the independent variables which describe the motion of fields, satisfy the free field equations and commutation relations

\[ \Box - m^2 \phi(x, \sigma) = 0, \quad \Box - m^2 u(x, \sigma) = 0, \quad [\phi(x, \sigma), \phi^*(x', \sigma)] = 2i \delta(x-x'), \quad [u(x, \sigma), u(x', \sigma)] = 2i \delta(x-x'), \]
\[ [\phi(x, \sigma), \phi(x', \sigma)] = [\phi^*(x, \sigma), u(x', \sigma)] = 0, \] \tag{6}
\[ [\phi(x, \sigma), \phi^*(x', \sigma)] = \frac{\delta(x-3)}{2} \Delta(x-3) d(123) + \sum_{n=1}^{\infty} g^n \phi^*_n(x, \sigma). \] \tag{7}

and are connected by a unitary operator \( U(\sigma, \sigma') \) for any two surfaces \( \sigma \) and \( \sigma' \)

\[ \phi(x, \sigma) = U^{-1}(\sigma, \sigma') \phi(x, \sigma') U(\sigma, \sigma'), \quad u(x, \sigma) = U^{-1}(\sigma, \sigma') u(x, \sigma') U(\sigma, \sigma'). \] \tag{8}

The additive terms \( \varphi_n(x, \sigma) \) and \( v_n(x, \sigma) \) should satisfy the free field equations

\[ \Box - m^2 \varphi_n(x, \sigma) = 0, \quad \Box - m^2 v_n(x, \sigma) = 0, \quad (n = 1, 2, \ldots), \] \tag{9}

and should be determined successively such that there exist the above unitary operator \( U(\sigma, \sigma') \) and consequently the interaction Hamiltonian. It will be prescribed that they are expressed as integrals over space-time of products of Heisenberg operators \( \phi^*(1), u(2), \phi(3), \phi^*(1'), \ldots \) and depend on the form of \( \sigma \) explicitly. If they all vanish for both \( \sigma = -\infty \) and \( +\infty \)

\[ \varphi_n(x, \sigma = \pm \infty) = v_n(x, \sigma = \pm \infty) = 0, \quad (n = 1, 2, \ldots), \] \tag{10}

the equations (5) for \( \sigma = \pm \infty \) agree with those of Kristensen-Møller\(^1\) and Bloch\(^2\) which define the incoming and outgoing fields, with an obvious consequence that the same \( S \)-matrix as theirs is obtained from the Hamiltonian. As shown in § 3, the conditions (10) can be satisfied up to the third order, but a particular term appears in \( \varphi_4(x, \sigma) \) which can not be made to vanish in general for both \( \sigma = -\infty \) and \( +\infty \).

If the difference between \( \sigma \) and \( \sigma' \) is infinitesimal, \( U(\sigma, \sigma') \) can be written with some Hermitian operator \( H(x'/\sigma) \) as

\[ U(\sigma, \sigma') = 1 - i \int_{\sigma'}^\sigma H(x'/\sigma) dx'. \] \tag{11}

Then, we have from (8) and (5) for a point \( x' \) on \( \sigma \)
where \( \partial / \partial \sigma(x') \) means a functional derivative with respect to \( \sigma \) which appears explicitly.

Now, we write the solutions of (5) which are obtained by iteration as

\[
\begin{align*}
\psi(x) &= \psi(x, \sigma) + \sum_{n=1}^{\infty} \epsilon_n \psi_n(x, \sigma), \\
u(x) &= \nu(x, \sigma) + \sum_{n=1}^{\infty} \epsilon_n \nu_n(x, \sigma),
\end{align*}
\]

and define \( H(x/\sigma) \) as a power series

\[
H(x/\sigma) = \sum_{n=1}^{\infty} \epsilon_n H_n(x/\sigma),
\]

where \( \psi_n(x, \sigma), \nu_n(x, \sigma) \) and \( H_n(x/\sigma) \) are all expressed as integrals of products of operators in the interaction representation \( \psi^*(1, \sigma), \nu(2, \sigma), \psi(3, \sigma), \psi^*(1', \sigma), \ldots \) and factors depending on \( \sigma \) explicitly. If the additive terms \( \phi_n(x, \sigma) \) and \( \nu_n(x, \sigma) \) are known up to \( n=i \), we have \( \psi_n(x, \sigma) \) and \( \nu_n(x, \sigma) \) up to \( n=i \) from (5) and we can write down the equations (12) in the \( i+1 \)-th approximation. If we obtain \( H_{i+1}(x/\sigma) \) and \( \phi_{i+1}(x, \sigma) \) and \( \nu_{i+1}(x, \sigma) \) by solving these equations and this procedure proceeds to any higher orders, it can be shown as follows that the Hamiltonian thus obtained satisfies the integrability condition which is extended from that in the local theories in order to have a Hamiltonian formalism in the case of non-local interactions.

As seen from the form of equation (5), \( \phi_n(x, \sigma), \nu_n(x, \sigma) \) and \( H_n(x/\sigma) \) will all depend on \( \sigma \) explicitly through the sign-functions \( \epsilon(\sigma, 1), \epsilon(\sigma, 2), \epsilon(\sigma, 3), \epsilon(\sigma, 1'), \ldots \). Only if surfaces \( \sigma \) divide the whole space-time into two simply connected regions and are space-like at an infinite distance, these sign-functions have a well-defined meaning for \( \sigma \) which is not always space-like. Accordingly, as made clear by considering equations (12) as a limit of coupled difference equations when the space-time is divided into small elements, the integrability conditions of (12) for \( \phi(x, \sigma) \) and \( \nu(x, \sigma) \), that they should be determined uniquely when their initial values \( \phi(x, \sigma_0) \) and \( \nu(x, \sigma_0) \) are given for a specified surface \( \sigma_0 \), are for any two points \( x' \) and \( x'' \) lying on \( \sigma \) (we can not confine ourselves to the case where the distance between \( x' \) and \( x'' \) is infinitesimal even if \( \sigma \) is space-like)

\[
\begin{align*}
\left\{ \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} \psi(x, \sigma) &= 0, \\
\left\{ \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} \nu(x, \sigma) &= 0,
\end{align*}
\]
or noting that \( \phi_n(x, \sigma) \) and \( \varphi_n(x, \sigma) \) are written in Heisenberg operators

\[
\left\{ \frac{\partial^2}{\partial \sigma'(x') \partial \sigma'(x'')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} \varphi_n(x, \sigma) = 0,
\]

\[
\left\{ \frac{\partial^2}{\partial \sigma(x'') \partial \sigma(x')} - \frac{\partial^2}{\partial \sigma(x') \partial \sigma(x'')} \right\} \varphi_n(x, \sigma) = 0, \quad (n = 1, 2, \ldots)
\]

If \( \phi_n(x, \sigma) \) and \( \varphi_n(x, \sigma) \) are obtained as one-valued functions of \( \sigma \) on which they depend explicitly, the condition (16) is satisfied. In other words, we should determine \( H_n(x/\sigma) \) such that \( \partial \phi_n(x, \sigma) / \partial \sigma(x') \) and \( \partial \varphi_n(x, \sigma) / \partial \sigma(x') \) appearing in (12) are actually integrable.

\( H_n(x/\sigma), \; \phi_n(x, \sigma) \) and \( \varphi_n(x, \sigma) \) are not determined uniquely at each stage of the approximation, corresponding to the fact that the following unitary transformations are always possible. If the equation

\[
\frac{\partial \psi(x, \sigma)}{\partial \sigma(x')} = \frac{1}{i} [\psi(x, \sigma), H(x'/\sigma)]
\]

holds, \( \psi'(x, \sigma) \) obtained by the unitary transformation with some Hermitian operator \( G(\sigma) \) depending on \( \sigma \),

\[
\psi'(x, \sigma) = e^{iG(\sigma)} \psi(x, \sigma) e^{-iG(\sigma)}
\]

satisfies the following equation in the lowest order approximation in the expansion of the parameter appearing in \( G(\sigma) \)

\[
\frac{\partial \psi'(x, \sigma)}{\partial \sigma(x')} = \frac{1}{i} [\psi'(x, \sigma), H'(x'/\sigma) + \partial G(\sigma) / \partial \sigma(x')]
\]

where \( H'(x/\sigma) \) is the one transformed from \( H(x/\sigma) \) according to (18).

The only restriction imposed on the form factor \( \Phi(1, 2, 3) \) to perform the above program is that the integrals containing the form factor have a well-defined meaning, which will be realized if we introduce a suitable damping factor. The essential effect of the form factor will appear in the convergence of the series. This problem will not be considered here. It can be said safely that our method gives at least the asymptotic solutions correctly which agree with those of free fields in the limit \( g \to 0 \).

\section*{§ 3. Derivation of the interaction Hamiltonian}

Following the above method we shall derive \( H_n(x/\sigma), \; \phi_n(x, \sigma) \) and \( \varphi_n(x, \sigma) \) successively up to the fourth order approximation. In this section we shall omit the notation (3) for simplicity, noting that \( 1, 2, 3; \; 1', 2', 3'; \ldots \) are integral variables multiplied always by form factors \( \Phi(1, 2, 3), \; \Phi(1', 2', 3'), \ldots \).

(1) \textit{First order}

The first order equations of (12) are
\[
(\frac{\partial \phi_1(x, \sigma)}{\partial \sigma(x')} )_I = -\int A(x-1)u(2, \sigma) \psi(3, \sigma) \delta(x'-1) + \epsilon[\phi(x, \sigma), H_I(x'/\sigma)],
\]
\[
(\frac{\partial \psi_1(x, \sigma)}{\partial \sigma(x')} )_I = -\int \phi^*(1, \sigma) D(x-2) \psi(3, \sigma) \delta(x'-2) + \epsilon[u(x, \sigma), H_I(x'/\sigma)],
\]
where the suffix \( I \) means to take the lowest order approximation of the term in the bracket, that is, to replace the Heisenberg operators \( \phi^*(1), \psi(2), \cdots \) by the operators in the interaction representation \( \phi^*(1, \sigma), \psi(2, \sigma), \cdots \). In order that the second terms in (20) may be of the same form as the first terms, \( H_I(x'/\sigma) \) must be of a form
\[
H_I(x'/\sigma) = -\{ \phi^*(1, \sigma) u(2, \sigma) \psi(3, \sigma) [a \delta(x'-1) + b \delta(x'-2) + \alpha \delta(x'-3)] \},
\]
where \( a \) and \( b \) are real constants. Inserting (21) into (20) we have for the first equation
\[
(\frac{\partial \phi_1(x, \sigma)}{\partial \sigma(x')} )_I = \int A(x-1)u(2, \sigma) \psi(3, \sigma) \{(a-1) \delta(x'-1) + b \delta(x'-2) + \alpha \delta(x'-3)}.
\]
Returning \( u(2, \sigma) \) and \( \psi(3, \sigma) \) back to the Heisenberg operators \( u(2) \) and \( \psi(3) \), where their orders are prescribed such that \( \phi^*, \psi \) and \( \phi \) always stand from the left in this order, and using an integral
\[
\int^a \delta(x'-1)dx' = \frac{\epsilon(a+1)}{2} + \text{const.},
\]
we can integrate (22) and obtain
\[
\phi_1(x, \sigma) = \int A(x-1)u(2) \psi(3) \left\{ (a-1) \frac{\epsilon(a+1)}{2} + b \frac{\epsilon(a+2)}{2} + \alpha \frac{\epsilon(a+3)}{2} \right\} + C,
\]
where \( C \) is a term independent of \( \sigma \). The condition (10) can be satisfied if we choose \( C = 0 \) and
\[
2a + b = 1.
\]
These restrictions are also necessary to have a correspondence to the local case where (24a) should vanish for any \( \sigma \). In the same way, under the condition (10) we have (25) and
\[
\psi_1(x, \sigma) = \int \phi^*(1) D(x-2) \phi(3) \left\{ (a-1) \frac{\epsilon(a+1)}{2} + (b-1) \frac{\epsilon(a+2)}{2} + \alpha \frac{\epsilon(a+3)}{2} \right\}.
\]
The arbitrariness of (21) within the limit of (25) is explained by the unitary transformation (18). If we choose
\[
G(\sigma) = -g \int \phi^*(1, \sigma) u'(2, \sigma) \psi'(3, \sigma) \left\{ (a-a') \frac{\epsilon(a+1)}{2} + (b-b') \frac{\epsilon(a+2)}{2} + \alpha \frac{\epsilon(a+3)}{2} \right\},
\]
with \( 2a' + b' = 1 \), we have the interaction Hamiltonian (21) with new constants \( a' \) and \( b' \).
in place of $a$ and $b$.

(ii) Second order

It will be convenient to add $\varphi_i(x, \sigma)$ and $\upsilon_i(x, \sigma)$ obtained above to the second terms in (5) and write

\[
\begin{align*}
\psi(x) &= \psi(x, \sigma) - \frac{1}{2} \int \frac{\varepsilon(x-1) - \varepsilon(\sigma, 123)}{\varepsilon(x-1)} D(x-1) \upsilon(2) \psi(3) + \sum_{n=2}^{\infty} g^n \varphi_n(x, \sigma), \\
\upsilon(x) &= \upsilon(x, \sigma) - \frac{1}{2} \int \varphi^*(1) \frac{\varepsilon(x-2) - \varepsilon(\sigma, 123)}{\varepsilon(x-2)} D(x-2) \psi(3) + \sum_{n=2}^{\infty} g^n \upsilon_n(x, \sigma),
\end{align*}
\]

(27)

where we have written

\[
\varepsilon(\sigma, 123) = a \varepsilon(\sigma, 1) + b \varepsilon(\sigma, 2) + c \varepsilon(\sigma, 3).
\]

(28)

Then, the second order equations of (12) are

\[
\begin{align*}
\left( \frac{\partial \varphi_i}{\partial \sigma(x')} \right)_t &= - \int D(x-1) \{ \upsilon_1(2, \sigma) \varphi_i(3, \sigma) + \upsilon_i(2, \sigma) \varphi_1(3, \sigma) \} \delta(x', 123) \\
&\quad + i \left[ \varphi(x, \sigma), H_2(x'/\sigma) \right], \\
\left( \frac{\partial \upsilon_i}{\partial \sigma(x')} \right)_t &= - \int D(x-2) \{ \varphi^*_i(1, \sigma) \upsilon_i(3, \sigma) + \varphi^*_i(1, \sigma) \upsilon_i(3, \sigma) \} \delta(x', 123) \\
&\quad + i \left[ \upsilon(x, \sigma), H_2(x'/\sigma) \right],
\end{align*}
\]

(29)

with

\[
\begin{align*}
\varphi_i(x, \sigma) &= - \int \frac{\varepsilon(x-1) - \varepsilon(\sigma, 123)}{\varepsilon(x-1)} D(x-1) \upsilon(2, \sigma) \varphi_1(3, \sigma), \\
\upsilon_i(x, \sigma) &= - \int \varphi^*_i(1, \sigma) \frac{\varepsilon(x-2) - \varepsilon(\sigma, 123)}{\varepsilon(x-2)} D(x-2) \psi(3, \sigma),
\end{align*}
\]

(30)

\[
\delta(x', 123) = - \frac{\partial}{\partial \sigma(x')} \frac{\varepsilon(\sigma, 123)}{2} = a \delta(x'-1) + b \delta(x'-2) + c \delta(x'-3).
\]

(31)

In order that the second terms in (29) may be of the same form as the first terms, we must put

\[
H_2(x'/\sigma) = - \int \{ c \varphi^*_i(1, \sigma) \upsilon_i(2, \sigma) \psi_1(3, \sigma) + d \varphi^*_i(1, \sigma) \upsilon_1(2, \sigma) \psi(3, \sigma) \\
+ c \varphi^*_i(1, \sigma) \upsilon_1(2, \sigma) \psi_i(3, \sigma) \} \delta(x', 123),
\]

(32)

where $c$ and $d$ are real constants. Inserting (32) into (29) and using (30) we have for the first equation

\[
\begin{align*}
\left( \frac{\partial \varphi_i}{\partial \sigma(x')} \right)_t &= \int D(x-1) \varphi^*_i(1, \sigma) D(2-2') \psi(3', \sigma) \psi(3, \sigma) \left[ (1-d) \\
&\times \frac{\varepsilon(2-2') - \varepsilon(\sigma, 1'2'3')}{2} \delta(x', 123) + d \frac{\varepsilon(2'-2) - \varepsilon(\sigma, 123)}{2} \delta(x', 1'2'3') \right] \\
&\quad + \int D(x-1) \upsilon(2, \sigma) D(3-1') \upsilon(2', \sigma) \psi(3', \sigma) \left[ (1-c) \frac{\varepsilon(3-1') - \varepsilon(\sigma, 1'2'3')}{2} \right]
\end{align*}
\]

\[
(32)
\]
In order that the expression in the bracket on the right of (33) may have a form
\[ \frac{\partial}{\partial \sigma}(x') \], we must choose
\[ c = d = 1/2. \] (34)

Returning \( \phi^*(1, \sigma), \phi^*(1', \sigma), \ldots \) back to the Heisenberg operators we have
\[
\varphi_s(x, \sigma) = -\frac{1}{2} \int d(x-1) D(2-2') \phi^*(1') \psi(3') \psi(3) \frac{\varepsilon(2-2') - \varepsilon(\sigma, 1'2'3')}{2}
\times \frac{\varepsilon(2' - 2) - \varepsilon(\sigma, 123)}{2}
- \frac{1}{2} \int d(x-1) D(3 - 1') u(2') u(2) \psi(3') \frac{\varepsilon(3-1') - \varepsilon(\sigma, 1'2'3')}{2}
\times \frac{\varepsilon(1' - 3) - \varepsilon(\sigma, 123)}{2}, \] (35a)

where the integration constant is chosen such that \( \varphi_s(x, \sigma) \) vanishes for \( \sigma = -\infty \) and
\(+ \infty \) under the restriction (25). In deriving (35a) the order of \( \psi(3') \) and \( \psi(3) \) is
not determined. If we choose the reverse order in place of (35a), \( \varphi_s(x, \sigma) \) and all terms
higher than the fourth are affected. This alteration will correspond to making some unitary
transformation. In the same way we have with (34)
\[
\varphi_s(x, \sigma) = -\frac{1}{2} \int d(1 - 3') D(x - 2) \phi^*(1') u(2') \psi(3) \frac{\varepsilon(1 - 3') - \varepsilon(\sigma, 1'2'3')}{2}
\times \frac{\varepsilon(3' - 1) - \varepsilon(\sigma, 123)}{2} + \text{Herm. conj.} \] (35b)

(iii) Third order

Using (35a) and (35b), we have the third order equation of (12) for \( \varphi_s(x, \sigma) \)
\[
\left( \frac{\partial \varphi_s(x, \sigma)}{\partial \sigma(x')} \right) = -\int d(x-1) \{ u_0(2, \sigma) \psi(3, \sigma) + u_1(2, \sigma) \psi_1(3, \sigma)
+ u(2, \sigma) \psi_3(3, \sigma) \} \delta(x', 123) + \frac{1}{2} \int d(x-1) D(2 - 2') \{ \psi^*(1', \sigma) \psi(3', \sigma) \psi(3, \sigma)
+ \phi^*(1', \sigma) \psi_1(3', \sigma) \psi(3', \sigma) + \phi^*(1', \sigma) \psi(3', \sigma) \psi(3, \sigma) \psi_1(3, \sigma) \}
\frac{\partial}{\partial \sigma(x')} \zeta(2, 2') \zeta(2', 2)
+ \frac{1}{2} \int d(x-1) D(3 - 1') \{ u_1(2, \sigma) u(2', \sigma) \psi(3', \sigma) + u(2, \sigma) u_1(2', \sigma) \psi(3', \sigma)
+ u(2, \sigma) u(2', \sigma) \psi_3(3, \sigma) \}
\frac{\partial}{\partial \sigma(x')} \zeta(3, 1') \zeta(1', 3) + i \{ \psi(x, \sigma), H_3(x'/\sigma) \}, \] (36)

where we have introduced a new notation.
In the approximations higher than the third, it will be convenient to proceed as follows. Terms on the right of (36) except the last one are all developed and graphically equivalent terms are grouped together by the interchanges of the integration variables. Each group is expressed as an integral of the products of definite propagation functions such as $D(3-1')$, $D(2'-2'')$, ..., definite operators such as $\phi^*(1, \sigma)$, $u(2, \sigma)$, ..., and a factor consisting of sign-functions and $\delta$-functions. This factor is divided into two parts as simply as possible, one part having the form $\partial/\partial \sigma(x')$ and the other having not. The Hamiltonian is chosen in such a way that the latter part is canceled by the last term in (36). For instance, we have as one of five graphically independent groups appearing in (36)

\[-\frac{1}{2} \int d(x-1) d(3-1') d(3'-1'') u(2, \sigma) u(2', \sigma) u(2'', \sigma) (3', \sigma)\]

\[\times \{\zeta(3, 1') \zeta(3', 1'') \delta(x', 123) + \zeta(1', 3) \zeta(1'', 3') \delta(x', 1' 2'' 3'')\}.

The second and the third terms in the bracket are canceled if we choose as one part of $H_s(x'/\sigma)$

\[-\frac{1}{2} \int d(3-1') d(3'-1'') \phi^*(1, \sigma) u(2, \sigma) u(2', \sigma) u(2'', \sigma) (3', \sigma)\]

\[\times \{\zeta(3, 1') \zeta(3', 1'') \delta(x', 123) + \zeta(1', 3) \zeta(1'', 3') \delta(x', 1' 2'' 3'')\}.

We shall write the results alone since the actual calculations are lengthy and tedious. In what follows some simplifications of expressions will be made. $\delta(x', 123)$, $\delta(x', 1'2'3')$, $\delta(x', 1''2''3'')$, ... be written as $\delta$, $\delta'$, $\delta''$, ...; products of operators in the interaction representation such as $\phi^*(1', \sigma) u(2, \sigma) \phi(3', \sigma) \phi(3, \sigma)$ and products of Heisenberg operators such as $\phi^*(1) u(2) u(3') \phi(3'')$ be written as $[1'2'3'3]$ and $[1 2 2' 3'']$, respectively, noting that 1, 2 and 3 always represent the arguments of $\phi^*$, $u$ and $\phi$, respectively. The results are

\[H_s(x'/\sigma) = -\frac{1}{2} \int d(3-1') d(3'-1'') [1 2 2' 2'' 3'3'] \zeta(3, 1') \zeta(3', 1'') \delta\]

\[+ \zeta(1', 3) \zeta(1'', 3') \delta''\]  

(38)

\[-\frac{1}{2} \int d(1-3') D(2'-2'') [1' 1'' 2 3' 3'] \zeta(1, 3') \zeta(2', 2'') \delta + \zeta(3', 1) \zeta(2'', 2') \delta''\]

\[-\frac{1}{2} \int d(3-1') D(2'-2'') [1' 1'' 2 3' 3'] \zeta(3, 1') \zeta(2', 2'') \delta + \zeta(1', 3) \zeta(2'', 2') \delta''\],

$\varphi_s(x, \sigma) = -\frac{1}{2} \int d(x-1) d(3-1') d(3'-1'') [2 2' 2'' 3'3'] \zeta(1', 3) \zeta(3', 1'') \zeta(1'', 3')$
\[ -\frac{1}{2} \int d(x-1) D(2-2') d(1'-3'') \{1'' 2'' 3' 3'' \} \zeta(2', 2) \zeta(1', 3'') \zeta(3'', 1') \]
\[ -\frac{1}{2} \int d(x-1) d(3-1') D(2'-2'') \{1'' 2'' 3' 3'' \} \zeta(1', 3 \zeta(2', 2') \zeta(2'', 2') \]
\[ -\frac{1}{2} \int d(x-1) D(2-2') d(3'-1'') \{1'' 2'' 3' 3'' \} \zeta(2', 2) \zeta(3', 1') \zeta(1'', 3'), \]
\[ v_3(x, \sigma) = -\frac{1}{2} \int D(x-2) d(1-3') D(2'-2'') \{1'' 2'' 3' 3'' \} \zeta(3', 1) \zeta(2', 2'') \zeta(2'', 2') \]
\[ -\frac{1}{2} \int D(x-2) d(1-3') d(1'-3'') \{1'' 2'' 3' 3'' \} \zeta(3', 1) \zeta(1', 3'') \zeta(3'', 1') \]
+ Herm. conj.,

where we choose the integration constants such that \( \varphi_3(x, \sigma) \) and \( v_3(x, \sigma) \) vanish for \( \sigma = -\infty \) and \( +\infty \) with (25), noting that \( \zeta(x, y') \zeta(y', x) \) vanishes for \( \sigma = -\infty \) and \( +\infty \).

Our choice of \( H_3(x'/\sigma) \) as (38) has reduced the numbers of independent terms from five to four in (39a) and (39b).

(iv) Fourth order

We have fourteen and sixteen graphically independent groups in the fourth order equations of (12). In the equation for \( \varphi_4(x, \sigma) \) some groups consist of two members in which the orders of operators are different, and as the result of contraction we have new terms, the last ones in (40) and (41a) below. Such a term does not appear in \( v_4(x, \sigma) \) and in the lower orders. The results are, writing simply \( D(x-2), d(1-3'), D(2'-2''), \ldots, \) as \( (x-2), (1-3'), (2'-2''), \ldots, \) respectively, \( \zeta(x, y') \) as \( (x, y') \) and \( \zeta(x, y') \zeta(y', x) \) as \( ((x, y')) \),

\[ H_4(x'/\sigma) = \frac{1}{2} \int \{ (3-1') (3'-1'') (3''-1''') [1 2 2' 2'' 2''' 3'''] (3'', 1'') \} \{(3, 1') \]
\[ \times (3', 1'') \delta - \frac{1}{4} (3, 1') (3'', 1'') \delta - \frac{1}{4} (3', 1') (3'', 1'') \delta \}
\[ + \frac{1}{2} \int \{ (2-2') (1'-3') (2''-2''') [1 1'' 1''' 1'''' 3' 3'] (2''', 2'') \} \{(2, 2') (1', 3'') \delta \]
\[ - \frac{1}{4} (2', 2') (2''', 2'') \delta - \frac{1}{4} (2', 2') (2''', 2'') \delta \}
\[ + \frac{1}{2} \int \{ (3-1') (2'-2'') (3'-1''') [1 1' 2 2'' 3'' 3''''] (3', 1''') \]
\[ \times \{(3, 1') (2', 2'') - ((2', 2'')) \} \delta \]
\[ + \frac{1}{2} \int \{ (3-1''') (2-2') (3'-1''') [1 1' 2' 2''' 3'' 3''''] (1', 3') \} \{(2', 2) (3, 1''') \} \delta \]
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\[
+ \frac{1}{4} (3, 1'''') (3', 1'') \delta + \frac{1}{4} (1'''', 3) (3', 1'') \delta'''
\]

\[
+ \frac{1}{2} \left( (3 - 1') (3' - 1'') (2 - 2''') [1 1''' 2' 2'' 3' 3'''] \left\{ (1', 3) (1'', 3') (2, 2''') \delta''
\right. \right. \\
- (2'''', 2) (3, 1') (3', 1'') \delta'' + \frac{1}{2} (2, 2''') ((3', 1'')) \delta + \frac{1}{2} (2'''', 2) ((3', 1'')) \delta'''
\]

\[
+ \frac{1}{2} \left( (2 - 2') (1' - 3''') (3 - 1'') [1 1''' 2' 2'' 3' 3'''] (3'', 1') \left\{ (2', 2) (3, 1'''') \delta'''
\right. \right. \\
+ \frac{1}{4} (3, 1'''') (1', 3'') \delta + \frac{1}{4} (1''', 3) (1', 3'') \delta'''
\]

\[
+ \frac{i}{4} \left( (3 - 1') (2' - 2'') (1' - 3''') (2'' - 2) [1 1''' 2' 2'' 3' 3'''] \left\{ \frac{1}{2} ((2', 2'')) (2, 2''')
\right. \right. \\
- (2'', 2') (3'', 1'') (2', 2'') - (2', 2') (3, 1') (1'', 3'') \right\} \delta
\]

+ Herm. conj.,

\[
\varphi(x, \sigma) = - \frac{1}{2} \int (x - 1) (3 - 1') (3' - 1'') (3'' - 1''') [2 2' 2'' 2''' 3' 3'''] (1', 3) ((3'', 1'''))
\]

\[
\times \left\{ (1'', 3') - \frac{1}{4} (3, 1') \right\} - \frac{1}{2} \int (x - 1) (2 - 2') (1' - 3''') (2'' - 2') [1 1''' 3''' 3''] \\
(2', 2) ((2'', 2'')) \left\{ (3''', 1') - \frac{1}{4} (2, 2') \right\} - \frac{1}{2} \int (x - 1) (2 - 2') (1' - 3'') (3' - 1''')
\]

\[
\times \left\{ (1'' 2'' 2''' 3''' 3'') ((2, 2'')) (1', 3'') (3', 1''') \right\} - \left\{ (2', 2'') (1', 3'') (2', 2''') (1' - 3''') (2'' - 2) [1 1''' 3'''] \right\}
\]

\[
\times \left\{ (2, 2'') (1', 3) ((2', 2'')) - (1'', 3''') (1', 3) ((2', 2'')) - \frac{1}{4} ((2, 2'')) ((2', 2''))
\right. \right. \\
+ \frac{1}{2} (3, 1') (2', 2'') (1'', 3''') (2''', 2) - \frac{1}{2} (2, 2'') (3'', 1) (2'', 2') (1', 3) + C, \right. \right. \\
\]

\[
v(x, \sigma) = - \frac{1}{2} \int (x - 2) (3 - 1') (3' - 1''') (3'' - 1''') [1 2' 2'' 2''' 3' 3'''] (1', 3) ((3'', 1''))
\]

\[
\times \left\{ (1'', 3') - \frac{1}{4} (3, 1') \right\} - \frac{1}{2} \int (x - 2) (3 - 1') (2 - 2') (3' - 1''') [1 1'' 2'' 3''' 3''']
\]

\[
\times (1', 3) (3', 1''') ((2', 2'')) \right\} - \left\{ (2', 2'') (1', 3'') (2', 2''') (1' - 3''') (2'' - 2) [1 1''' 3'''] \right\}
\]

\[
\times (1', 3) (3', 1''') ((2', 2'')) \right\} - \left\{ (2', 2'') (1', 3'') (2', 2''') (1' - 3''') (2'' - 2) [1 1''' 3'''] \right\}
\]

+ Herm. conj.,

where (41a) except the last term and (41b) vanish for \( \sigma = - \infty \) and \( + \infty \), but the last
term in (41a) can not be made to vanish for both \( \sigma = -\infty \) and \( +\infty \) by choosing the integration constant \( C \) appropriately. If we choose \( C \) such that \( \varphi_i(x, \sigma) \) vanishes for \( \sigma = -\infty \), we have

\[
\varphi_i(x, \sigma = +\infty) = \frac{i}{2} \left\{ \delta(\sigma - 1) \delta(3 - 1') \delta(2' - 2'') \delta(1'' - 3'') \delta(2'' - 2) \{ 1'' 3'' 3' \} 
\times \left\{ \theta(2 - 2'') \theta(3'' - 1'') \theta(2'' - 2') \theta(1' - 3) 
- \theta(3 - 1') \theta(2' - 2'') \theta(1'' - 3''') \theta(2'' - 2) \right\},
\]

where

\[
\theta(x) = \frac{\varepsilon(x) + 1}{2}.
\]

In the local limit

\[
\Phi(1, 2, 3) \rightarrow \delta(1 - 2) \delta(2 - 3)
\]

(42) vanishes, but in general it will not. From the examination of (42) in the form of Fourier integrals in momentum space it seems unlikely that there exist form factors other than (44) which are analytic and make (42) vanish. Bloch has proved on a basis of the equations containing no additive terms

\[
\begin{align*}
\varphi'(x) &= \varphi'(x, -\infty) - \varepsilon \int \frac{\varepsilon(x - 1) + 1}{2} \delta(x - 1) u(2) \varphi(3), \\
u(x) &= u(x, -\infty) - \varepsilon \int \varphi(1) \frac{\varepsilon(x - 2) + 1}{2} D(x - 2) \varphi(3), \\
\varphi'(x) &= \varphi'(x, +\infty) - \varepsilon \int \frac{\varepsilon(x - 1) - 1}{2} \delta(x - 1) u(2) \varphi(3), \\
u(x) &= u(x, +\infty) - \varepsilon \int \varphi(1) \frac{\varepsilon(x - 2) - 1}{2} D(x - 2) \varphi(3),
\end{align*}
\]

that if the in-fields \( \varphi'(x, -\infty) \) and \( u(x, -\infty) \) satisfy the commutation relations (7) the out-fields \( \varphi'(x, +\infty) \) and \( u(x, +\infty) \) also satisfy the same relations. His proof seems to be doubtful. Indeed, if we calculate directly, for instance, \([\varphi(x', +\infty), \varphi(x', +\infty)]\) from (45) and (46) with the commutation relations of the in-fields, a term similar to (42) remains in the fourth order

\[
[\varphi(x', +\infty), \varphi(x', +\infty)] = \varepsilon^4 \left\{ \delta(x - 1) \delta(x' - 1''') \delta(3 - 1') \delta(2' - 2'') 
\times \delta(1'' - 3''') \delta(2'' - 2) \{ \theta(2 - 2'') \theta(3'' - 1'') \theta(2'' - 2') \theta(1' - 3) 
- \theta(3 - 1') \theta(2' - 2'') \theta(1'' - 3''') \theta(2'' - 2) \right\}.
\]

That is to say, in-fields and out-fields defined by (45) and (46) will not be connected by a unitary \( S \)-matrix. If we retain, however, the term (42), (47) is canceled by it.

It will be apparent from the results in this section that \( H_{n+1}(x', \sigma) \), \( \varphi_n(x, \sigma) \) and \( \psi_n(x, \sigma) \) depend on \( \sigma \) explicitly through the products of \( n \) \( \zeta \)-functions defined by (37).
In the local limit (44), $H_t(x/\sigma)$ agrees with the usual one and $H_n(x/\sigma)$ ($n=2, 3, \ldots$) together with all additive terms $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ vanish for space-like $\sigma$. This can be seen easily from that $\varphi_t(x, \sigma), v_t(x, \sigma), \phi_t(x/\sigma), u_t(x/\sigma)$ and terms of a form

$$\Delta \frac{e(1-1')-e(\sigma, 1')}{2} \frac{e(1'-1)-e(\sigma, 1)}{2}$$

vanish. On the other hand, if $\sigma$ contains some portions which are time-like, $H_t(x/\sigma)$, $\varphi_n(x, \sigma)$ and $v_n(x, \sigma)$ ($n=2, 3, \ldots$) all survive and guarantee the integrability condition which is not satisfied by $H_t(x/\sigma)$ alone.

§ 4. Hamiltonian in the Schrödinger representation

If we specify a surface $\sigma_0$, which will have to be space-like in order that field operators may be defined initially on it, and divide the space-time region between $\sigma_0$ and any $\sigma$ by a number of surfaces denoted by $\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \sigma_n=\sigma$, we have for the unitary operator defined by (8) and (11)

$$U(\sigma, \sigma_0) = U(\sigma_1, \sigma_0)U(\sigma_2, \sigma_1)\cdots U(\sigma, \sigma_{n-1}).$$

(48)

Then, noting that

$$U(\sigma, \sigma_0)\psi(x, \sigma) = \psi(x, \sigma_0)U(\sigma, \sigma_0), U(\sigma, \sigma_0)u(x, \sigma) = u(x, \sigma_0)U(\sigma, \sigma_0),$$

we have

$$\frac{\partial U(\sigma, \sigma_0)}{\partial \sigma(x)} = U(\sigma, \sigma_0)H(x/\sigma) = H_t(x/\sigma, \sigma_0)U(\sigma, \sigma_0),$$

(49)

where $H_t(x/\sigma, \sigma_0)$ is the one obtained from $H(x/\sigma)$ by replacing the operators $\psi^*(1, \sigma), u(2, \sigma), \ldots$ contained in it by $\psi^*(1, \sigma_0), u(2, \sigma_0), \ldots$, $\sigma$ explicitly contained being unchanged. If we put, with any state vector $\Psi_0$ in the Hilbert space,

$$\Psi(\sigma) = U(\sigma, \sigma_0)\Psi_0,$$

(50)

we have a Tomonaga-Schwinger equation

$$\frac{\partial \Psi(\sigma)}{\partial \sigma(x)} = H_t(x/\sigma, \sigma_0)\Psi(\sigma).$$

(51)

If the integrability condition (15) for the equations (12), which is expressed in terms of $H(x/\sigma)$ as

$$\frac{1}{i}[H(x/\sigma), H(x'/\sigma)] + \frac{\partial H(x/\sigma)}{\partial \sigma(x')} - \frac{\partial H(x'/\sigma)}{\partial \sigma(x)} = 0,$$

(52)

holds, the condition for (51) is also satisfied for $\sigma$ which is not always space-like since this condition is nothing other than the unitary transform of (52) with $U(\sigma, \sigma_0)$. The transformation to the Schrödinger representation will be performed most easily by making all $\sigma$ flat. Writing $\sigma=t$ and $\sigma_0=t_0$ we have from (51)
Expressing the free Hamiltonian, denoted by $\mathcal{H}_0(t_0)$, in terms of $\psi^*(1, t_0), u(2, t_0), \ldots$, and putting

$$i \frac{\partial \Phi(t)}{\partial t} = (\mathcal{H}_0(t_0) + \mathcal{H}_e(t_0)) \Phi(t),$$

we have for $\Phi(t)$ a Schrödinger equation

$$\mathcal{H}_e(t_0) = e^{-i\mathcal{H}_0(t_0)\cdot(t-t_0)} \mathcal{H}_e(t_0) e^{i\mathcal{H}_0(t_0)\cdot(t-t_0)}.$$

It can be seen from the invariance of the form factor for time displacement that $\mathcal{H}_e(t_0)$ does not depend on $t$. Actually, in the first order we have

$$\mathcal{H}_e(t_0) = -g \int \psi^*(1, t_0) u(2, t_0) \psi(3, t_0) \{a\delta(t_0-t_1) + b\delta(t_0-t_2) + a\delta(t_0-t_3)\} d^3r.$$  

If we choose $a=0$ and $b=1$, (57) agrees with that obtained by Pauli. The $\psi(x, t_0)$ and $u(x, t_0)$ can be expressed as surface integrals

$$\psi(x, t_0) = -\int \{\hat{J}(x-3) \psi(3, t_0) + \hat{J}(x-3) \psi(3, t_0) \} \delta(t_0-t) d^3r,$$

$$u(x, t_0) = -\int \{\hat{J}(x-2) u(2, t_0) + \hat{J}(x-2) u(2, t_0) \} \delta(t_0-t) d^3r,$$

where $\psi(3, t_0) \mid_{t_3=to}, \psi(3, t_0) \mid_{t_3=to}, u(2, t_0) \mid_{t_2=to}, \ldots$, which will be denoted later on as $\psi(x_3), \hat{J}(x_3), u(x_3), \ldots$, respectively, are the canonical variables in the Schrödinger representation satisfying the well-known commutation relations. Using (58), (57) becomes

$$\mathcal{H}_e(t_0) = -g \int F_1(x_1, x_2, x_3) \psi(x_1) \psi(x_2) \psi(x_3) d^3x_1 d^3x_2 d^3x_3$$

$$-g \int F_2(x_1, x_2, x_3) \psi(x_1) \psi(x_2) \psi(x_3) d^3x_1 d^3x_2 d^3x_3$$

$$-g \int F_3(x_1, x_2, x_3) \psi(x_1) \psi(x_2) \psi(x_3) d^3x_1 d^3x_2 d^3x_3$$

where the form factors in the three-dimensional space

$$F_1(x_1, x_2, x_3) = -\int \hat{J}(t_1, x_1, x_2, x_3) \hat{D}(t_2, x_2, x_3) \hat{J}(t_3, x_3) d^3t_1 d^3t_2 d^3t_3$$

$$F_2(x_1, x_2, x_3) = -\int \hat{J}(t_1, x_1, x_2, x_3) \hat{D}(t_2, x_2, x_3) \hat{J}(t_3, x_3) d^3t_1 d^3t_2 d^3t_3$$

$$F_3(x_1, x_2, x_3) = -\int \hat{J}(t_1, x_1, x_2, x_3) \hat{D}(t_2, x_2, x_3) \hat{J}(t_3, x_3) d^3t_1 d^3t_2 d^3t_3$$

$$F_4(x_1, x_2, x_3) = -\int \hat{J}(t_1, x_1, x_2, x_3) \hat{D}(t_2, x_2, x_3) \hat{J}(t_3, x_3) d^3t_1 d^3t_2 d^3t_3$$

$$F_5(x_1, x_2, x_3) = -\int \hat{J}(t_1, x_1, x_2, x_3) \hat{D}(t_2, x_2, x_3) \hat{J}(t_3, x_3) d^3t_1 d^3t_2 d^3t_3$$

do not depend on $t_0$. Thus, the Schrödinger representation is defined independently of the
choice of \( t_0 \) if this is finite. (59) shows that derivative couplings appear in the first order. In the local limit, however, (60) all vanish except \( F_1(\alpha_1, \alpha_2, \alpha_3) \) which becomes
\[
F_1(\alpha_1, \alpha_2, \alpha_3) \rightarrow \delta(\alpha_1 - \alpha_2) \delta(\alpha_2 - \alpha_3).
\]

We shall examine the degrees of singularity in the form factors (60). For illustration we shall choose \( F_1(\alpha_1, \alpha_2, \alpha_3) \) which has the highest singularity. Using the form factor given by Kristensen-Møller
\[
\Phi(x_1, x_2, x_3) = (2\pi)^{-3} \left\{ G(l_1, l_2) e^{i(l_1 x_1 + l_2 x_2 - (l_1 + l_2) x_3)} dl_1 dl_2 \right\},
\]
where \( G(l_1, l_2) \) is a function of \( l^2 \) only
\[
\Pi^2 = \left( \frac{l_1 - l_2}{2} \right)^2 - \frac{[(l_1 + l_2)(l_1 - l_2)]^2}{4(l_1 + l_2)^2},
\]
and expressing \( \beta \)-functions in Fourier integrals, we have in the case \( a=0 \) and \( b=1 \)
\[
F_1(\alpha_1, \alpha_2, \alpha_3) = (2\pi)^{-3} \left\{ G(2 \Pi^2 = \sqrt{l_1^2 + M^2}\sqrt{l_2^2 + M^2} - M^2, l_1, l_2) + G(2 \Pi^2 = -\sqrt{l_1^2 + M^2}\sqrt{l_2^2 + M^2} - M^2, l_1, l_2) \right\} e^{i(l_1(\alpha_1 - \alpha_2) + l_2(\alpha_3 - \alpha_2))} dl_1 dl_2.
\]

We can see by estimating the integral with respect to relative angle between \( l_1 \) and \( l_2 \) that (64) is less singular than (61). Especially, a value of (64) for \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) turns out to be \( P^3/2 \) as compared with \( P^3 \) of (61), where \( P \) is the upper limit of \( |l_1| \) and \( |l_2| \), and \( 1/2 \) is the cut-off value of \( \Pi^2 \). These differences will be made more distinct if we consider space integrals of \( F_1(\alpha_1, \alpha_2, \alpha_3) \) over one of variables \( \alpha_1, \alpha_2 \) or \( \alpha_3 \). We have
\[
\int F(\alpha_1, \alpha_2, \alpha_3) d\alpha_1 = (2\pi)^{-3} \left\{ 1/2 \cdot G(2 \Pi^2 = M \sqrt{P^2 + M^2} - M^2) + G(2 \Pi^2 = -M \sqrt{P^2 + M^2} - M^2) \right\} e^{iH(\alpha_3 - \alpha_2)} dl_1
\]
\[
\int F(\alpha_1, \alpha_2, \alpha_3) d\alpha_2 = (2\pi)^{-3} \left\{ 1/2 \cdot G(\Pi^2 = P^2) + G(\Pi^2 = -M^2) \right\} e^{iH(\alpha_1 - \alpha_3)} dl_1.
\]

Then, if \( G(\Pi^2) \) vanishes for \( |\Pi^2| \geq M^2 \), (65) are finite for any values of \( \alpha_1 - \alpha_2 \) and \( \alpha_1 - \alpha_3 \). In this way, we shall have a possibility to treat the problems involving bound states without divergence difficulties according to the Tamm-Dancoff’s theory.

In conclusion, the author would like to express his appreciations to Professor H. Yukawa and Dr. Y. Katayama for many valuable discussions.
Note added in proof: We can show as follows that, starting from the field equations (4), it is always possible to find the interaction Hamiltonian $H(x/a)$ which satisfies (12). For a specified space-like surface $\sigma^*$, we have directly from (4)

$$
\phi(x) = \phi(x, \sigma^*) - g\left\{\epsilon(x-1) - \epsilon(\sigma^*, 1)\right\}/2 \cdot \Delta (x-1) u(2) \phi(3)d(123),
$$

$$(N \cdot 1)$$

$$
\phi(x, \sigma^*) = \int dq\mu'\left\{\Delta (x-x') - \frac{1}{2} \cdot \phi(x') - \phi(x') - \phi(x-x')\right\} u_{\mu}(q).
$$

$$(N \cdot 2)$$

We assume that, for given operators $\psi(x, \sigma^*)$ and $u(x, \sigma^*)$ which satisfy (6) and (7), equation (N·1) and the corresponding one for $u(x)$ have a unique solution, that is, the form factor belongs to the Pauli's "normal class" $\mu_0$. If not unique, however, it will be possible to define by iteration an asymptotic solution which agrees with that of the free fields in the limit $g \to 0$. With these known operators $\psi(x)$ and $u(x)$, we write for general surface $\sigma$ the equations (5) as

$$
\psi(x, \sigma) = \psi_0(x, \sigma) - \psi(x, \sigma), \quad u(x, \sigma) = u_0(x, \sigma) - v(x, \sigma),
$$

$$(N \cdot 3)$$

$$
\psi_0(x, \sigma) = \psi_0_0(x, \sigma) + g\left\{\epsilon(x-1) - \epsilon(\sigma, 1)\right\}/2 \cdot \Delta (x-1) u(2) \psi(3)d(123), \quad u_0(x, \sigma) = \cdots,
$$

$$(N \cdot 4)$$

where additive terms $\psi(x, \sigma)$ and $v(x, \sigma)$ should vanish for $\sigma = \sigma^*$. The known operators $\psi_0(x, \sigma)$ and $u_0(x, \sigma)$ defined by (N·4) are not in general connected by unitary transformations for fixed $x$ and varying $\sigma$. By subtracting from them suitable operators $\psi(x, \sigma)$ and $v(x, \sigma)$ which are functionals of known Heisenberg operators and depend on $\sigma$ explicitly, we obtain operators $\psi(x, \sigma)$ and $u(x, \sigma)$ which satisfy (8). The way of subtraction will not be unique, but $\psi(x, \sigma)$ and another solution $\psi'(x, \sigma)$ are both connected by unitary transformations with $\psi(x, \sigma^*)$, and they are also connected with each other by some unitary transformation as (18). Thus, the existence of $\psi(x, \sigma)$ and $v(x, \sigma)$ as one-valued functions of $\sigma$ assures the possibility of finding $H(x/a)$ which satisfies (12) and the integrability condition.

The results in § 3 are given for $\sigma^* = -\infty$. In this case we can define energy-momentum four-vectors $P_\mu(x)$, which are constant and form the infinitesimal generators of the coordinate transformation for finite $\sigma$, $\sigma_0$, and $\sigma^*$.

$$
P_\mu(x) = U^{-1}(\sigma_0, \sigma_0) \left[P_\mu^{(0)}(\sigma_0) - \int dF_x(x'|\sigma, \sigma_0) d\sigma_\mu U(x, \sigma_0)\right] U(x, \sigma_0),
$$

$$(N \cdot 5)$$

where $P_\mu^{(0)}(\sigma_0)$ are free energy-momentum operators written in $\psi(x, \sigma_0)$ and $u(x, \sigma_0)$, and $H_\mu(x/\sigma, \sigma_0)$ is defined by (49). It is shown from the results given in § 3 that, if the form factor contains a suitable damping factor such as $\exp[-\epsilon(\sigma_1 + \sigma_2 + \sigma_3)]$, the following equations hold except terms of order $\epsilon$

$$
\delta P_\mu(x)/\delta \sigma(x) = 0, \quad \delta \psi(x)/\delta x_\mu = i[\psi(x), P_\mu], \quad \delta u(x)/\delta x_\mu = i[u(x), P_\mu].
$$

$$(N \cdot 6)$$