On the Transition Matrix and the Green Function in the Quantum Field Theory

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The J. Schwinger's formalism of the Green function in the quantum electrodynamics is applied to the transition problem of the state. It is shown that the many body kernel in the Heisenberg representation involves the information about the transition of the state and this is directly represented by the repeated use of the one body kernel $G_{\mu}$ and the vertex operator $\Gamma_{\mu}$ defined by J. Schwinger. Further, the renormalization is discussed without use of the usual perturbation theory, although there remains the difficulty associated with the $\delta$-divergence.

\section{Introduction}

The propagation of the interaction effects between the fields is usually represented by the Green functions in the present quantum field theory. In the interaction representation these Green functions are the well-known $A_{\mu}$, $S_{\mu}$, etc. The matrix describing the transition of the state is constructed by the repeated use of these Feynman’s functions. As was shown by Stückelberg\textsuperscript{3)}, introducing the Feynman’s functions and the usual Dyson’s $S$-matrix follows as a logical consequence of the requirements of the causality for the propagation of the interaction effects. This fact suggests that the Green function is one of the fundamental basis of the current quantum field theory.

The importance of these Green function becomes clearer in the Heisenberg representation, because there the state vector is time-independent and the temporal development of the system is described by the field operators and so it can be expected that the Green function involves the information about the transition of the state. On the other hand, many authors\textsuperscript{2),3)} have expected that the Green function of the $n$-body problem involves also the information about the stationary state of the $n$-body problem.

It is the aim of this paper to investigate the detailed property of the Green function in the Heisenberg representation. A crucial method to treat the Green function in the Heisenberg representation has been proposed by J. Schwinger.\textsuperscript{9)} In this paper we shall treat our problem along the same line as he has done and restrict ourselves only to the quantum electrodynamics.

In the usual perturbation theory, the Green function of the one body problem has a special importance. As Dyson\textsuperscript{9)} has shown in the quantum electrodynamics, the $S$-matrix element is obtained through the substitution of $S_{\mu}$, $D_{\mu}$, and $\Gamma$ for the electron line, photon line, and the vertex part in any possible irreducible graphs corresponding to the given transition process.
These $S'_\gamma$, $D'_\gamma$ correspond to the Green functions of the one body problem. In § 4, we show, without use of the usual perturbation theory, that the matrix element of any transition is written by an adequate graph which contains the Green function of the one body problem and $I_\mu$ as for the internal line and vertex part, respectively. Further, we will treat the renormalization problem in our method without use of the perturbation theory. However, on account of the difficulty associated with the $\delta$-divergence, the completion of the discussion is confined to be left in future.

§ 2. On the transition matrix and the Green function of the many body problem

We treat the problem in the quantum electrodynamics, whose Lagrange density is of the form

$$L = -\frac{1}{2} \cdot \bar{\psi} \left[ \gamma_\mu (\partial_\mu - ieA_\mu) + x \right] \psi + \frac{1}{2} \cdot \left[ \bar{\psi} \gamma^2 + \bar{\psi} \gamma \right] + h.c. \nonumber$$

$$- \frac{1}{4} \cdot F_{\mu\nu}^2 - \frac{1}{2} \cdot (\partial_\mu A_\mu)^2 - J_\mu A_\mu,$$  \hspace{1cm} (2.1)

where $\gamma$ is a spinor source which anticommutes with $\psi$ and $\bar{\psi}$, $\bar{\gamma}$ is given by $\gamma^* \gamma_4$ and $J_\mu$ is a c-number source current and further $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

From (2.1), the well-known equations of motion in the Heisenberg representation are obtained:

$$\{ \gamma_\mu (\partial_\mu - ieA_\mu) + x \} \psi = \eta,$$  \hspace{1cm} (2.2)

$$\Box A_\mu = -J_\mu - j_\mu,$$  \hspace{1cm} (2.3)

$$j_\mu \equiv -ie/2 \bar{\psi} \gamma_\mu \psi + h.c.$$

(2.4)

Now let us introduce the following notation for the operators in the Heisenberg representation, $A(x_1), B(x_2), \ldots,$

$$\langle A(x_1), B(x_2), \ldots \rangle = \langle \Psi_0, P(A(x_1), B(x_2), \ldots) \Psi_0 \rangle,$$  \hspace{1cm} (2.5)

where $\Psi_0$ is a vacuum state vector defined as the minimum energy state in this representation. Further, we use the interaction representation which coincides with the Heisenberg representation at $\sigma(x) = 0$ and denote the minimum energy state in this representation as $\Phi_0$. If the quantities in the Heisenberg representation are denoted by $A(x_1), B(x_2), \ldots$, corresponding to $A(x_1), B(x_2), \ldots$, respectively, then there are following relations between both quantities:

$$A(x_1) = U(\sigma(x_1), 0) A(x_1) U^{-1}(\sigma(x_1), 0), \text{ etc.}$$  \hspace{1cm} (2.6)

As Gell-Mann and Low\cite{Gell-Mann and Low} has shown, $\Psi_0$ may be written as

$$\Psi_0 = \frac{1}{c} U^{-1}(\pm \infty, 0) \Phi_0/(\Phi_0, U^{-1}(\pm \infty, 0) \Phi_0),$$  \hspace{1cm} (2.7)

where $c$ is a normalization constant.

* The star expresses the hermitian conjugate.
Using (2.7), (2.5) is rewritten as

\[
\langle A(x_1), B(x_2), \ldots \rangle = (\Phi_0, U(\infty, 0) P(A(x_1), B(x_2), \ldots) \times U^{-1}(-\infty, 0) \Phi_0) / (\Phi_0, U(\infty, -\infty) \Phi_0),
\]

where we used the following relation;

\[
U(a, 0)U^{-1}(a', 0) = U(a, a').
\]

Since the annihilation and creation of \(\alpha\)-particle are described by the positive and negative frequency parts \(Q^+_{\alpha}(x)\) of \(\alpha\)-field operator \(Q_{\alpha}(x)\), respectively, we have from (2.8) the next theorem. Namely, the transition matrix \(S_{\alpha}\) between the initial state, where \(\alpha\)-particle with the energy-momentum \(k_{\mu}^1\) and spin state etc. exists, and the final state, where \(\alpha\)-particle with the energy-momentum \(k_{\mu}^2\) and spin state etc. exists, is

\[
S_{\alpha} = a \Phi^f (k_{\mu}^1, r^1; \ldots) \langle Q_{\alpha}(x_1), \ldots, Q_{\alpha}(x_1), \ldots \rangle, \tag{2.10}\]

where \(F^f(\ldots)\) is the operator which takes out the Fourier's amplitudes referring to \(-k_{\mu}^1, r^1\) and the negative frequency part at the world point \(x_1\) and \(-k_{\mu}^1, r^1\) and the positive frequency part at the world point \(x_1\). a is a normalization constant and \(x_1\) and \(x_1\) are the coordinates of the particles in the initial and final state, respectively, and so their time components are \(-\infty\) and \(+\infty\), respectively. If we bring out the quantities of \(x_1\) in front of the one of \(x_f\), we have the following relation:

\[
\langle Q_{\alpha}(x_1), \ldots, Q_{\alpha}(x) \rangle = (\Phi_0, Q_{\alpha}(x_f) \ldots, U(\infty, -\infty) Q_{\alpha}(x_1) \ldots \Phi_0) \times 1 / (\Phi_0, U(\infty, -\infty) \Phi_0),
\]

which gives the proof of (2.10).

Using the creation and annihilation operators \(q^+\), the Fourier transform of \(Q_{\alpha}\) is expressed in the form

\[
Q_{\alpha}(x) = \lim_{r \to \infty} \sum_k V^{-1/2} [d_{\alpha \nu}(k) q^+_{\nu}(k) \exp i(kr - \xi_0) + d_{\alpha \nu}^*(k) q^-_{\nu}(k) \exp -i(kr - \xi_0)]. \tag{2.11}
\]

The normalization constant in (2.10) is expressed by the reciprocal of the product of

\[
\ast\ \text{If we denote the interaction Hamiltonian density as } H(x) \text{ and apply the usual perturbation theory, we have}
\]

\[
\langle A(x_1), B(x_2), \ldots \rangle = (\Phi_0, \sum_{n=0}^{\infty} (-i)^n \int \cdots \int d\eta_1 \ldots d\eta_n \times P(A(x_1), B(x_2), \ldots H(y_1) \ldots H(y_n) \Phi_0) / (\Phi_0, U(\infty, -\infty) \Phi_0). \tag{2.8}'
\]

The appearance of the denominator in (2.8)' corresponds to the procedure in the usual perturbation which leaves out of consideration the isolated diagram, whose initial and final states are the vacuum states.

\[
\ast\ast \text{(2.10) has been applied to the problem of the multiple production of meson; H. Umezawa et al., Phys. Rev. 86 (1952), 505.}
\]
Now we define the Green functions for many body problems as follows:

electron Green functions

\[
G_N(x_1, x_1') = \frac{\partial}{\partial \eta} \langle \psi(x_1) \rangle \langle \overline{\psi}(x_1') \rangle + \epsilon,
\]

\[
G_N(x_1, x_2, x_1') = \frac{\partial}{\partial \eta} \langle \psi(x_1) \rangle \langle \overline{\psi}(x_1') \rangle + \epsilon,
\]

\[
G_N(x_1, x_1', x_2) = \frac{\partial}{\partial \eta} \langle \psi(x_1) \rangle \langle \overline{\psi}(x_1') \rangle + \epsilon,
\]

\[
G_N(x_1, x_2, x_1', x_2') = \frac{\partial}{\partial \eta} \langle \psi(x_1) \rangle \langle \overline{\psi}(x_1') \rangle \langle \psi(x_2) \rangle + \epsilon \text{ etc.}
\]

photon Green functions

\[
\Theta_j^{\mu \nu}(\xi_1, \xi_2) = \frac{\partial}{\partial j_\mu}(\xi_2) \langle A_\mu(\xi_2) \rangle,
\]

\[
\Theta_j^{\mu \nu}(\xi_1, \xi_2) = \frac{\partial}{\partial j_\mu}(\xi_2) \langle A_\mu(\xi_2), A_\nu(\xi_2) \rangle + \epsilon,
\]

\[
\Theta_j^{\mu \nu \rho}(\xi_1, \xi_2, \xi_3) = \frac{\partial}{\partial j_\mu}(\xi_2) \langle A_\mu(\xi_2), A_\rho(\xi_3), A_\nu(\xi_3) \rangle + \epsilon \text{ etc.}
\]

mixed Green functions

\[
K_N(x_1, x_1') = \frac{\partial}{\partial j_\mu}(\xi_1) \langle \psi(x_1), \overline{\psi}(x_1') \rangle + \epsilon,
\]

\[
K_N(x_1, x_1', x_2) = \frac{\partial}{\partial j_\mu}(\xi_1) \langle \psi(x_1), \overline{\psi}(x_1') \rangle, A_\mu(\xi_1) + \epsilon,
\]

\[
K_N(x_1, x_1', x_2, x_2') = \frac{\partial}{\partial j_\mu}(\xi_1) \langle \psi(x_1), \overline{\psi}(x_1'), A_\mu(\xi_1) \rangle + \epsilon \text{ etc.}
\]

According to the calculation rule given by J. Schwinger, we have

\[
G_N(x_1, x_1') = i \langle \psi(x_1), \overline{\psi}(x_1') \rangle + \epsilon - i \langle \psi(x_1') \rangle \langle \overline{\psi}(x_1) \rangle + \epsilon
\]

\[
G_N(x_1, x_2, x_1') = i \langle \psi(x_1), \overline{\psi}(x_1') \rangle, \psi(x_2) + \epsilon
\]

\[
- i \langle \psi(x_1), \overline{\psi}(x_1') \rangle \langle \psi(x_2) \rangle + \epsilon
\]

\[
\Theta_j^{\mu \nu}(\xi_1, \xi_2) = i \langle A(\xi_2), A(\xi_3) \rangle - i \langle A(\xi_1) \rangle \langle A(\xi_2) \rangle,
\]

\[
K_N(x_1, x_1') = i \langle \psi(x_1), \overline{\psi}(x_1') \rangle, A(\xi_1) + \epsilon
\]

\[
- i \langle A(\xi_1) \rangle \langle \psi(x_1), \overline{\psi}(x_1') \rangle + \epsilon
\]

\[
\text{etc.}
\]

** (2.11) for the case of the field with the arbitrary spin has been discussed by Y. Takahashi and H. Umezawa; Prog. Theor. Phys. 9 (1953), 14.

** \( \epsilon \) in the expression \langle \ldots \rangle_+ \epsilon \ means the sign function as to the coordinates appearing in \langle \ldots \rangle.

For instance, for the case of \langle \psi(x_1), \psi(x_2), \overline{\psi}(x_1'), \overline{\psi}(x_2') \rangle + \epsilon, \ \epsilon \ is given by

\[
\epsilon = \epsilon (x_1, x_2) \epsilon (x_1', x_2') \epsilon (x_3, x_2') \epsilon (x_3, x_1') \epsilon (x_5, x_2') \epsilon (x_3, x_1') \epsilon (x_5, x_1'),
\]

where

\[
\epsilon (x_1, x_2) = 1 \quad \text{for} \quad x_1 < x_2,
\]

\[
= -1 \quad \text{for} \quad x_1 > x_2.
\]

In the following discussion we shall eliminate this symbol as far as it does not give rise to mistake.
Since \( U(\sigma, \sigma') \) has non vanishing matrix element only for the transition satisfying the charge conservation law, we have the following relation for \( F \), in which the numbers of the \( \phi \) and \( \bar{\phi} \) are different each other:

\[
\langle F \rangle_{\eta \to 0} = 0. \tag{2.16}
\]

For example, we have

\[
\langle \psi \rangle_{\eta \to 0} = \langle \bar{\psi} \rangle_{\eta \to 0} = 0. \tag{2.17}
\]

From the invariance of the theory under the charge conjugation, we have the Furry's theorem,

\[
\langle A_{\mu} \rangle_{J \to 0} = 0.
\]

From these relation and \((2.10)\), \((2.15)\), we find that the many body Green function at the limit \((J \to 0, \eta \to 0)\) corresponds to the transition matrix element. Hereafter, we denote these quantities with \((J \to 0, \eta \to 0)\) by those with the super-suffix 0.* For example, \(G_\eta^0(x_1, x_2, x_3, x_4)\), \(\mathcal{G}_\eta^0(\xi_1, \xi_2, \xi_3, \xi_4)\), and \(K_\eta^0(x_1, \xi_1, \xi_2, \xi_3)\) correspond to the Möller scattering of the two electrons, photon-photon scattering, and Compton-scattering, respectively.

One body Green function is connected with the current \( J_\mu(x) \) as follows:

\[
\langle j_\mu(x) \rangle = -Tr\{\gamma_\mu G_\eta(x, x)\}, \tag{2.18}
\]

where

\[
G_\eta(x, x) = \lim_{x' \to x} \left[ G_\eta(x, x') + G_\eta(x', x) \right]/2. \tag{2.19}
\]

Hereafter, the matrix representation for the coordinates of electron and photon is used and so the matrices \( \partial_\mu, \phi, \bar{\phi}, A_\mu, G_\eta \) and \( \mathcal{G}_\eta^\mu(x, x') \) are \( \delta(x-x') \), \( \delta(x-x') \partial_\mu \), \( \phi(x) \delta(x-x') \), \( \bar{\phi}(x) \delta(x-x') \), \( A_\mu(x) \delta(x-x') \), \( G_\eta(x, x') \) and \( \mathcal{G}_\eta^\mu(x, x') \), respectively.

From the equations of motion \((2.2), (2.3)\), we have for \( G_\eta, \mathcal{G}_\eta^\mu \) the equations,

\[
\{\gamma_\mu (\partial_\mu - ieA_\mu) + m\} G_\eta = 1 + i\epsilon_\mu \langle \psi \rangle \partial / \partial J^\mu \langle \bar{\psi} \rangle, \tag{2.20}**
\]

\[
\{\delta - P\} \mathcal{G}_\eta^\mu = \partial_\mu 1. \tag{2.21}
\]

In the above expression the mass operator \( \tilde{M} \) and polarization operator \( \tilde{P} \) are the matrices whose elements are given by \( \tilde{M}(x, x') \), \( \tilde{P}(\xi, \xi') \) defined as follows:

\[
\int dx'' \tilde{M}(x, x'') G_\eta(x'', x') \equiv \langle x-\epsilon \gamma_\mu \delta / \delta J^\mu(x) \rangle G_\eta(x, x'), \tag{2.22}
\]

\[
\int d\xi'' \tilde{P}(\xi, \xi'') \mathcal{G}_\eta^\mu(\xi'', \xi') \equiv \epsilon Tr\{\gamma_\mu \delta / \delta J^\mu(\xi') \} G_\eta(\xi, \xi'). \tag{2.23}
\]

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* The limiting process \((J \to 0, \eta \to 0)\) should be taken after the variational operation of \((2.12), (2.13)\) and \((2.14)\).

** The product of the two matrices \( A \) and \( B \) is defined by

\[
\langle x | AB | x' \rangle \equiv \int dy | A | y \langle y | B | x' \rangle.
\]

Further the symbol \( \cdot \) in the expression \( A' \) or \( \cdot A \) denotes the right or left coordinate of the matrix \( A \).
As was shown by J. Schwinger, we have

\[ \frac{\partial}{\partial \Gamma_{\mu}(\xi)} G_{\eta}(x, x') \bigg|_{x, x'} = i e \int d\xi' \Gamma_{\mu}(\xi') G_{\eta}(x, x') \Omega^{\mu}_{x', \xi}(\xi', \xi). \]  

where the vertex operator \( \Gamma_{\mu}(\xi) \) is defined as follows:

\[ \frac{\partial}{\partial \Gamma_{\mu}(\xi)} G_{\eta}(x, x') \bigg|_{x, x'} = i e \int d\xi' \Gamma(\xi'; x, x') \Omega^{\mu}_{x', \xi}(\xi', \xi). \]  

(2.24)

(2.25)

From (2.22) and (2.24), we have

\[ M = x + i e \gamma_{\mu} \langle A_{\mu} \rangle - e \int d\xi' G_{\eta}(x, x') \Omega^{\mu}_{x', \xi}(\xi', \xi). \]  

(2.26)

\[ M^0 = M \bigg|_{x, x'} = x + i e \gamma_{\mu} \langle A_{\mu} \rangle - e \int d\xi' G_{\eta}(x, x') \Omega^{\mu}_{x', \xi}(\xi', \xi). \]  

(2.27)

§ 3. Note on the renormalization theory

In this section a short note on the relation of the above theory with Dyson's one is given.

The differential equation (2.21) can be transformed into the integral form;

\[ G_{\eta}(x, x') = \Sigma_{\eta}'(x - x') \delta x \int dy S_{\eta}(x - y) G_{\eta}(y, x') - \int dy' S_{\eta}(x - y) \Sigma_{\eta}'(y', x') G_{\eta}(y', x'), \]  

(3.1)

where

\[ \Sigma_{\eta}' = M^0 - x', \quad x' = x + \delta x. \]  

Using (2.27) we have

\[ G_{\eta}(x, x') = S_{\eta}(x - x') - \delta x \int dy S_{\eta}(x - y) G_{\eta}(y, x') - e^2 \int dy dy' dy'' d\xi' S_{\eta}(x - y) G_{\eta}(y, y') \times \Gamma_{\eta}(\xi'; y', x') G_{\eta}(y'', x') \Omega^{\mu}_{x', \xi}(\xi', \xi). \]  

(3.2)

If we replace \( G_{\eta} \) by \( S_{\eta} \) in (3.1), this equation corresponds to the integral equation given by Dyson and \( \Sigma_{\eta}' \) amounts to the total contribution from the self-energy graph. (3.2) agrees with the final integration of the self-energy graph given by Dyson and corresponds to Fig. 1, in which the electron line, photon line and vertex \( \delta \) correspond to \( G_{\eta} \), \( \Omega^{\mu}_{x', \xi} \) and \( \Gamma_{\mu} \), respectively.
The fact that \( r \) of the point \( a \) is not replaced by \( r' \) coresponds to the Dyson's argument on the \( b \)-divergence, because we must take into account only one of the equivalent graphs (i), (ii) in Fig. 2.

In the quantum electrodynamics we can normalize \( \phi \) and \( A_\mu \) so that in the high energy region all possible quantities with dimension of the length are the momenta \( p_\mu \) of particles. In this case the dimensions of \( G_\eta^0 \) and \( \Theta^0_\eta \) agree with those of \( S_\mu \) and \( D_\mu \), respectively, and \( \Gamma^0_\mu \) and \( \gamma_\mu \) are the dimensionless quantities.

Let us separate the infinite constants from \( G_\eta^0 \), \( \Theta^0_\eta \) and \( \Gamma^0_\mu \) as follows:

\[
G_\eta^0 = Z_2 G_\eta^1, \\
\Theta^0_\eta = Z_3 \Theta^1_\eta, \\
\Gamma^0_\mu = Z_1^{-1} \Gamma^1_\mu, \\
(3.3)
\]

where \( G_\eta^1 \), \( \Theta^1_\eta \) and \( \Gamma^1_\mu \) are free from infinity. Substituting (3.3) into (3.2), we have

\[
G_\eta^1(x, x') = \frac{1}{Z_2} S_\mu(x-x') - \partial x \int dy S_\mu(x-y) G_\eta^1(y, x') \\
- \partial \xi G_\mu^1(x-y) G_\mu^1(y, x') \\
\times \Gamma^0_\mu(\xi; x') G_\eta^1(y, x') \Theta^0_\eta(\xi, x) \\
(3.4)
\]

In the above equation, while the integrand of the third term is finite, its integration may be divergent. Since this divergence comes from the contribution of the high energy region, the following discussion shows that this integral is at most linearly or logarithmically divergent. \( Z_1 \), \( Z_2 \), and \( Z_3 \) are the function of the upper limit \( p \to \infty \) of the integration concerning the internal momentum and so the dimension of the divergent \( Z_1 \), \( Z_2 \), and \( Z_3 \) should be zero or negative power of the length. Therefore, the dimension of \( G_\eta^1 \), \( \Theta^0_\eta \), and \( \Gamma^1_\mu \) are \([L^{-n}]\) \((n \leq 3, n \leq 2, n \leq 0)\), respectively. Since the integral (3.4) is most strongly divergent in the case of \( n=3, 2, 0 \), it is sufficient to treat this case for the consideration of the highest degree of divergence. Then the integral has a dimension \([L^{3-3-3-2}] = [L]\), and can be written as follows:

\[
\int d\xi' d\xi'' d\xi' \mu S_\mu(x-y) G_\eta^0(y, y') \Gamma^0_\mu(\xi'; y', y'') G_\eta^0(y'', x') \Theta^0_\eta(\xi', x) \\
= Z_1^{-1} [A + B(\gamma_\mu \varphi_\mu + x) + C(\gamma_\mu \varphi_\mu + x)^2 + \ldots] \\
\times \int dy S_\mu(x-y) G_\eta^0(y, x') \\
(3.5)
\]

where \( A \) and \( B \) are linearly and logarithmically divergent and \( C \) is a finite quantity.

As Fig. 1 is symmetric in association with the two vertices \( a \) and \( b \), it is expected that a infinite constant factor \( Z_1^{-1} \) appears from the vertex \( a \) as well as the vertex \( b \) after

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* In the interaction of the second kind, the situation is not so simple as in this case, because the coupling constant has the dimension of the length.
the integration is analogous to Dyson’s argument. However, the consistent proof of this situation is not yet verified in our method. This defect which is due to the asymmetrical treatment of the two vertices makes it difficult to compare our method with Dyson’s one also in the discussion of the skeleton approximation in the next section.*

Substituting (3.5) into (3.4), we obtain the following relations as a necessary condition for the convergence of the right side of (3.4):

\[ \partial_x = -i\epsilon_1^{\alpha} A Z^{-1}_2, \]
\[ Z_2 = 1 - i\epsilon_1^{\beta} B, \]
\[ \epsilon_1 = Z^{-1}_1 Z_2 Z_3^{1/2} \epsilon. \]  

(3.6)

Here \( \epsilon_1 \) and \( x' \) should be considered as a finite and observable electric charge and mass, respectively. Then (3.4) becomes as follows:

\[ G_\eta^0(x, x') = S_p(x - x') - \epsilon_1^{\gamma} C(\gamma_\mu \partial_\mu + x) G_\eta^0(x, x') + \ldots. \]  

(3.7)

(3.6) is in agreement with the condition as to the renormalization constant given by Dyson. Further it is easily found by the same dimensional argument as in the above (3.5) that the divergent quantities are restricted only to self-energy parts \( G_\eta^0 \), \( \Theta^0 \), and vertex part \( \Gamma^0_\mu \).

Therefore, the theory is entirely free from divergence, provided that it is shown that any Feynman diagram of \( S \)-matrix is expressed by the irreducible skeleton in which internal lines, vertex, and charge correspond to \( G_\eta^0 \), \( \Theta^0 \), \( \Gamma^0_\mu \), and \( \epsilon_1 \), respectively. In this paper the method in which any transition matrix is represented entirely by the \( G_\eta^0 \), \( \Theta^0 \), and \( \Gamma^0_\mu \) is called the skeleton approximation. If we have the formulation of the skeleton approximation, any vertex in Fig. 3 is of the form \( \epsilon(G_\eta^0 \Theta^0 G_\eta^0)^{1/2} \Gamma^0_\mu \) which can be rewritten as follows:

\[ \epsilon Z_1^{-1} Z_2 Z_3^{1/2} (G_\eta^0 \Theta^0 G_\eta^0)^{1/2} \Gamma^0_\mu = \epsilon_1 (G_\eta^0 \Theta^0 G_\eta^0) \Gamma^0_\mu, \]  

(3.8)

so that it turns out that there exists no more any infinite quantity in the theory.

Thus it is necessary for completion of our procedure only to investigate the possibility of the above skeleton approximation and this is the aim of the next section. As will be shown there, unfortunately, we can not yet find the complete formulation of the skeleton approximation and this problem is left to be investigated in future.

§ 4. The skeleton approximation

In this section we shall prove that the transition matrix element is obtained through substituting the Green function of the one body problem and \( \Gamma^0_\mu \) into any internal lines and the vertex part of the adequate graph corresponding to its process. However this graph is not completely equivalent to the skeleton and we must often use the uncorrected

* If this defects are get rid of, one could set up a non-singular theory by using the Lagrangian given by G. Takeda^7 and applying the variational method in § 2.
vertex $\gamma_\mu$, although it is needless to use $S_\mathbb{P}$ and $D_\mathbb{P}$.

According to (2.10), it turns out that the problem is how to express all the many body Green function by the one body Green function and the vertex function $I_\mathbb{P}_\mu$; i.e., the corrected function approximation. In the following we discuss on this problem in some examples, i.e., the Möller scattering of two electrons and the photon-photon scattering.

(i) Möller scattering

According to (2.10), the transition matrix element for the Möller scattering corresponds to $G_\eta^0(x_1, x_2, x'_1, x'_2)$. Using (2.12), we have

$$G^0_\eta(x_1, x_2, x'_1, x'_2) = -\frac{\delta^a}{\delta \eta(x'_2)} \frac{\delta}{\delta \eta(x_2)} G_\eta(x_1, x'_1) \big|_{x_\eta = 0} + G^0_\eta (x_1, x_2, x'_1, x'_2), \tag{4.1}$$

where

$$G^0_\eta (x_1, x_2, x'_1, x'_2) = i G^0_\eta (x_1, x'_2) G^0_\eta (x_2, x'_1) - i G^0_\eta (x_1, x'_1) G^0_\eta (x_2, x'_2). \tag{4.2}$$

This is represented by the diagrams denoted in Fig. 3.

In this diagram and hereafter it should be noted that the straight and waved line, the vertex and vertex with circle correspond to $G^0_\eta$, $\gamma_\mu$, and $I^0_\mu$, respectively.

The second term of (4.1) expresses two independent electrons scattering without the real interaction. The effect of the true Möller scattering due to the real interaction of two electrons is involved in the first terms of (4.1). From (2.21), we have

$$\frac{\delta}{\delta \eta(x_2)} G_\eta = -ie \gamma_\mu \langle \psi' | \frac{\delta}{\delta \eta(x_2)} \langle \psi' \rangle \big| G_\eta,$$

$$+ G_\eta \left( ie \gamma_\mu \frac{\delta}{\delta \eta(x_2)} \langle A_\mu \rangle - \frac{\delta}{\delta \eta(x_2)} \frac{\delta}{\delta \eta(x_2)} \langle \psi' \rangle \big| \frac{\delta}{\delta \eta(x_2)} \langle \psi' \rangle \big| G_\eta \right), \tag{4.3}$$

Further using (2.17) and the Furry's theorem, we obtain

$$\frac{\delta^2}{\delta \eta(x'_2) \delta \eta(x_2)} G_\eta \big|_{x_\eta = 0} = G^0_\eta \frac{\delta^2}{\delta \eta(x'_2) \delta \eta(x_2)} \left( ie \gamma_\mu \langle A_\mu \rangle - \frac{\delta}{\delta \eta(x_2)} \frac{\delta}{\delta \eta(x_2)} \langle \psi' \rangle \big| \frac{\delta}{\delta \eta(x_2)} \langle \psi' \rangle \big| G_\eta \right), \tag{4.4}$$

Substituting the following relation

$$\frac{\delta^2}{\delta \eta(x'_2) \delta \eta(x_2)} \langle A_\mu(x) \rangle = \frac{\delta}{\delta x_2} \langle A_\mu(x) \rangle, \tag{4.5}$$
into (4.4), we have

\[
\frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} G_{\eta|\eta,\eta^0} = G_{\eta|\eta,\eta^0}^{(2)}(x_2, x_2') - G_{\eta|\eta,\eta^0}^{(2)} \left\{ \frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2')} \bar{M} \right\} G_{\eta|\eta,\eta^0}^{(0)}, \tag{4.6}
\]

where

\[
G_{\eta|\eta,\eta^0}^{(2)}(x_2, x_2') = -\varepsilon^2 \left[ \frac{\partial}{\partial \eta(x_2)} \mathcal{G}_{\eta}^{(2)}(x_2) \right] \left\{ I'_{\mu}(\xi') \mathcal{G}_{\eta}(x_2) \Theta_j(\xi', \cdot) \mathcal{G}_{\eta} \right\} \\
+ \mathcal{G}_{\eta} \frac{\partial}{\partial \eta(x_2')} \mathcal{G}_{\eta}(\cdot, x_2') \mathcal{G}_{\eta}(x_2) \left\{ I'(\xi') \Theta_j(\xi', \cdot) \mathcal{G}_{\eta} \right\}|_{\eta, \eta^0}. \tag{4.7}
\]

This is represented by the diagrams in Fig. 4, and this diagrams correspond to \(\varepsilon^2\)-skeleton.

It can be shown that the second term of (4.6) contributes to the skeleton higher than the fourth order. Using (2.22), we have

\[
\frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} \bar{M} = \frac{\partial^2}{\partial \eta(x_2) \partial \eta(x_2)} (\bar{M} \mathcal{G}_{\eta} \mathcal{G}_{\eta^{-1}}) \\
= \frac{\partial}{\partial \eta(x_2') \partial \eta(x_2)} \left[ \left( x - c_{\mu} \frac{\partial}{\partial x_{\mu}} \mathcal{G}_{\eta} \right) \mathcal{G}_{\eta^{-1}} \right]. \tag{4.8}
\]

After the tedious calculation, (4.8) is rewritten in the form;

\[
\frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} \bar{M} |_{\eta, \eta^0} = -\varepsilon^2 \mathcal{G}_{\eta} \frac{\partial}{\partial x_{\mu}} \left[ \mathcal{G}_{\eta}^{-1} \frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} \mathcal{G}_{\eta} \mathcal{G}_{\eta} \right] G_{\eta^{-1}} |_{\eta, \eta^0}. \tag{4.9}
\]

Therefore, from (4.6) we have

\[
\frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} G_{\eta|\eta,\eta^0} = G_{\eta|\eta,\eta^0}^{(2)}(x_2, x_2') + \varepsilon^2 \mathcal{G}_{\eta} \mathcal{G}_{\eta} \frac{\partial}{\partial x_{\mu}} \left[ \mathcal{G}_{\eta}^{-1} \frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} \mathcal{G}_{\eta} \mathcal{G}_{\eta} \right] G_{\eta^{-1}} |_{\eta, \eta^0}. \tag{4.10}
\]

From (4.3), (2.17), (4.5), we can obtain the relation

\[
\frac{\partial}{\partial f(\xi)} \frac{\partial}{\partial \eta(x_2') \partial \eta(x_2)} G_{\eta|\eta,\eta^0} = \frac{\partial}{\partial f(\xi)} \left[ G_{\eta|\eta,\eta^0}^{(2)}(x_2, x_2') \right] |_{\eta, \eta^0} \\
- \frac{\partial}{\partial f(\xi)} \left[ \mathcal{G}_{\eta} \left\{ \frac{\partial}{\partial \eta(x_2') \partial \eta(x_2)} \bar{M} \right\} \mathcal{G}_{\eta} \right] |_{\eta, \eta^0}, \tag{4.11}
\]

where \(G_{\eta|\eta,\eta^0}^{(2)}\) is defined by the left side of (4.7) without superscript 0. The second term of (4.4) gives to (4.10) the skeleton higher than the \(\varepsilon^0\)-approximation, and we write this part as \(O(\varepsilon^0)\). Using (4.10), we can rewrite (4.10) as follows:

\[
\frac{\partial^2}{\partial \eta(x_2') \partial \eta(x_2)} G_{\eta|\eta,\eta^0} = G_{\eta|\eta,\eta^0}^{(2)}(x_2, x_2') + G_{\eta|\eta,\eta^0}^{(0)}(x_2, x_2') + O(\varepsilon^0). \tag{4.12}
\]
The second term of (4.12) corresponds to $\epsilon^4$-skeleton, and is given by

$$
G_{\eta}(x_2, x_2') = -ie^4 \gamma G_{\eta} \left[ \gamma G_{\eta}(x_2) \int \Gamma(\xi'') G_{\eta}(\xi') \int \gamma G_{\eta}(x_2) \Gamma(\xi'') G_{\eta}(\xi') \right] J_{\eta, \eta, 0}^{(4)}
$$

This is represented by the diagrams in Fig. 5.

As stated in the preceding section, it should be noted here that only some part of all vertices are replaced by $\Gamma_\mu$. For instance, in Fig. 4, only one vertex among two is replaced by $\Gamma_\mu$. Thus the contribution which corresponds to Fig. 6 in the usual perturbation theory is not involved in this diagram, but in Fig. 5 which belongs to $\epsilon^4$-skeleton. Of course, if we take into account infinitely higher order terms in the present approximation, then the both vertices will be completely corrected and so by $\Gamma_\mu$, and then the skeleton approximation may be obtained. However such a procedure is of a perturbation theoretical concept. This unfavourable situation of our method is due to the fact that, in the course of obtaining the higher order skelton by applying (2.24) to one body Green function, only one vertex among two of the self-energy part is replaced by $\Gamma_\mu$.

(ii) Photon-photon scattering

Now we discuss briefly on the photon-photon scattering. This transition matrix ele-
ment corresponds to
\begin{equation}
\mathcal{G}_0^\gamma(\xi_1, \xi_2, \xi_3, \xi_4) = -\frac{\partial^2}{\partial \xi_1 \partial \xi_2} \mathcal{G}_0^\gamma(\xi_1, \xi_2) \bigg|_{J \to 0} + \mathcal{G}_0^\gamma(\xi_1, \xi_2, \xi_3, \xi_4),
\end{equation}
where
\begin{equation}
\mathcal{G}_0^\gamma(\xi_1, \xi_2, \xi_3, \xi_4) = -i \left[ \mathcal{G}_0^\gamma(\xi_1, \xi_2) \mathcal{G}_0^\gamma(\xi_3, \xi_4) + \mathcal{G}_0^\gamma(\xi_1, \xi_3) \mathcal{G}_0^\gamma(\xi_2, \xi_4) 
+ \mathcal{G}_0^\gamma(\xi_1, \xi_4) \mathcal{G}_0^\gamma(\xi_2, \xi_3) \right].
\end{equation}
This is represented by the diagrams in Fig. 7. After the analogous calculation as for \( \frac{\partial^2 G}{\partial \eta \partial \eta} \), the first term is transformed into the following form:
\begin{equation}
\frac{\partial^2 \mathcal{P}}{\partial \xi_1 \partial \xi_2} \bigg|_{J \to 0} = -\mathcal{G}_0^\gamma \frac{\partial^2 \mathcal{P}}{\partial \xi_1 \partial \xi_2} \bigg|_{J \to 0},
\end{equation}
Using (2.23), we can rewrite \( \frac{\partial^2 \mathcal{P}}{\partial J} \) as follows:
\begin{equation}
\frac{\partial^2 \mathcal{P}}{\partial J} \bigg|_{J \to 0} = -eT \left[ \frac{\partial \mathcal{G}}{\partial J} \frac{\partial \mathcal{G}}{\partial J} \bigg|_{J \to 0} \right] \mathcal{G}_0 \bigg|_{J \to 0} + eT \left[ \frac{\partial \mathcal{G}}{\partial J} \frac{\partial \mathcal{G}}{\partial J} \bigg|_{J \to 0} \right] \mathcal{G}_0 \bigg|_{J \to 0}. \end{equation}
Further, we have
\begin{equation}
\frac{\partial G}{\partial J} \bigg|_{J \to 0} = \frac{\partial G}{\partial J} \bigg|_{J \to 0} \left( i e \frac{\partial \mathcal{G}}{\partial J} \bigg|_{J \to 0} \right) + \mathcal{G}_0(\xi_1 \leftrightarrow \xi_2),
\end{equation}
where the symbol \((x \leftrightarrow y)\) means the term obtained by the exchange of \(x\) and \(y\) in the preceding term. Thus the final result is given by
\begin{equation}
\frac{\partial \mathcal{G}}{\partial J} \bigg|_{J \to 0} = \mathcal{G}_0(\xi_1, \xi_2, \xi_3) + O(\epsilon^0),
\end{equation}
where

\[
\mathcal{G}_J^{(4)}(\xi_0, \xi_2, \xi_4) = -ie^4 \mathcal{G}_J^T \left[ \gamma G_\eta \int \Gamma^{(4)}(\xi', \xi, \xi_3, \xi_4) G_\eta \int \Gamma^{(4)}(\xi'', \xi_3, \xi_4) G_\eta \right]
\]

\[
+ \gamma G_\eta \int \Gamma^{(4)}(\xi', \xi_3, \xi_4) G_\eta \int \Gamma^{(4)}(\xi'', \xi_3, \xi_4) G_\eta \int \Gamma^{(4)}(\xi'''', \xi_3, \xi_4) G_\eta
\]

These correspond to \(\epsilon^4\)-skelton approximation and are represented by diagrams in Fig. 8.

Fig. 8

All other processes be treated in the similar method as to the above two examples.

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