Synchronization and Collective Behavior in Globally Coupled Logarithmic Maps

M. G. COSENZA and J. GONZÁLEZ

Centro de Astrofísica Teórica, Facultad de Ciencias, Universidad de Los Andes
A. Postal 26 La Hechicera, Mérida 5251, Venezuela

(Received October 27, 1997)

The collective phenomena arising in a system of globally coupled chaotic logarithmic maps are investigated by considering the properties of the mean field of the network. Several collective states are found in the phase diagram of the system: synchronized, collective periodic, collective chaotic, and fully turbulent states. In contrast with previously studied globally coupled systems, no splitting of the elements into different groups nor quasiperiodic collective states occur in this model. The organization of the observed nontrivial collective states is related to the presence of unstable periodic orbits in the local dynamics. The role that the properties of the local dynamics play in the emergence and characteristics of nontrivial collective behavior in globally coupled systems is discussed.

§1. Introduction

Coupled map lattices (CML) are discrete space, discrete time dynamical systems of interacting elements whose states vary continuously according to specific functions. Globally coupled maps constitute a class of CML where the coupling interaction is a function of all the elements. 1) Though CML models are idealized systems, they have proved capable of capturing much of the phenomenology observed in a variety of complex spatiotemporal processes, with the advantage of being computationally efficient and, in many cases, mathematically tractable. 2)

There has been recent interest in the use of CML models in the investigation of cooperative phenomena, such as synchronization or nontrivial collective behavior, which appear in many extended chaotic dynamical systems. 3) - 7) Synchronization consists of the complete coincidence in time of the states of the elements in a system, while nontrivial collective behavior is characterized by a well-defined temporal evolution of statistical quantities emerging out of local chaos.

An important category of systems with many degrees of freedom which can exhibit these collective effects is globally coupled nonlinear oscillators. Such systems arise naturally in the description of Josephson junctions arrays, charge density waves, multimode lasers, neural dynamics, and ecological and evolution models. 8) - 12) Globally coupled maps represent a useful approach to the study of many processes on this kind of systems, in particular to the search for the conditions leading to the occurrence of collective dynamics.

Studies in globally coupled chaotic maps have revealed interesting features such as: a) formation of clusters, i.e., differentiated subsets of synchronized elements within the network; 13) b) non-statistical properties in the fluctuations of the mean field of the ensemble; 13), 14) c) global quasiperiodic motion; 15), 16) d) different collec-
tive phases depending on the parameters of the system.\textsuperscript{15,17}) These works have treated mainly bounded or unimodal maps belonging to some universality class (quadratic, circle or tent maps) as the source of local chaos.

In this article, we investigate the phenomena of synchronization and nontrivial collective behavior in globally coupled systems by using unbounded chaotic elements whose properties differ from those of previously used maps in the same context. The local dynamics that we employ are given by the logarithmic map introduced by Kawabe and Kondo.\textsuperscript{18}) Logarithmic-type functions appear in some biological and physiological systems. Our model of globally coupled logarithmic maps provides a situation to examine the role that different properties of the local dynamics play on the emergence and universality of the various types of collective behavior in globally coupled systems. In addition, the unbounded character of the local functions places no restrictions on the range of the parameters of the system that can be explored. In §2 we define the model and describe the properties of the local logarithmic map. In §3 the synchronized states of the network are studied. The collective states arising from desynchronized local maps are investigated in the phase diagram of the system in §4. Some differences are found in comparison to previously studied globally coupled systems; for example, we find no splitting of the elements into different groups and no quasiperiodic collective behavior in our model. The influence of the size of the system on the emergence of collective states is calculated in §5. The results are discussed in §6.

\section{Globally coupled logarithmic maps}

We consider the globally coupled map system

\begin{equation}
 x_{t+1}(i) = (1 - \epsilon)f(x_t(i)) + \epsilon \sum_{j=1}^{N} f(x_t(j)),
\end{equation}

where $x_t(i)$ gives the state of the lattice element $i$ ($i = 1, \cdots, N; N =$ system size) at a discrete time step $t$, $\epsilon$ is the coupling parameter, and $f$ is a map defining the local dynamics.

The collective behavior of globally coupled maps can be described through the instantaneous mean field of the system, defined as

\begin{equation}
 h_t = \frac{1}{N} \sum_{j=1}^{N} f(x_t(j)).
\end{equation}

The local dynamics correspond to the logarithmic map,\textsuperscript{18})

\begin{equation}
 x_{t+1} = f(x_t) = b + \ln |x_t|,
\end{equation}

where $b$ is a real parameter. This map does not belong to a standard class of universality of unimodal or bounded maps. It possesses no maximum or minimum, and its Schwarzian derivative is always positive. Figure 1 shows the bifurcation diagram of the iterates of the logarithmic map as a function of the parameter $b$. Two stable
Fig. 1. Bifurcation diagram of the logarithmic map Eq. (3) as a function of the parameter $b$. For each value of $b$, 100 iterates were plotted after discarding 1000 points representing transient behavior.

Fig. 2. Some unstable periodic orbits of the logarithmic map, indicated by dotted lines, as a function of the parameter $b$. The stable fixed points $x_1^*$ and $x_2^*$ are plotted with solid lines. $\bar{x}_1$ is an unstable fixed point arising from $x_1^*$; $\bar{x}_a$ and $\bar{x}_b$ form an unstable period two orbit, $f(\bar{x}_a) = f(\bar{x}_b)$. 
fixed points satisfying $f(x^*) = x^*$ and $|f'(x^*)| < 1$ exist for this map: $x_1^* < -1$, for $b < -1$; and $x_2^* > 1$, for $b > 1$; both are indicated in Fig. 1. Chaos occurs in the parameter interval $b \in [-1, 1]$. Figure 1 reveals the absence of separated chaotic bands at any given value of $b \in [-1, 1]$. The fixed point $x_1^*$ becomes unstable at $b = -1$, giving rise to chaos via type III intermittency associated with an inverse period doubling bifurcation. The stable fixed point $x_2^*$ originates from a tangent bifurcation at $b = 1$, and the transition to chaos at this value of $b$ takes place through type I intermittency. There exist several unstable period-$m$ orbits $\bar{x}$ satisfying $f^{(m)}(\bar{x}) = \bar{x}$ in the chaotic range $b \in [-1, 1]$. Figure 2 shows some unstable periodic orbits of the logarithmic map as a function of the parameter $b$.

Figure 3 displays the Lyapunov exponent $\lambda$ of the logarithmic map as a function of the parameter $b$. Note that $\lambda$ is positive in the entire region $b \in [-1, 1]$, and thus no periodic windows appear in any subinterval of $b$ in the chaotic region. The properties of the logarithmic map are similar to those found in the class of singular maps $x_{t+1} = 1 - a|x_t|^z$, with $|z| < 1$. This family of maps is characterized by a positive Schwarzian derivative, by unbounded dynamics (for $z \in (-1, 0)$), by the absence of periodic windows in a single chaotic interval of the parameter $a$, and by types I and III intermittency transitions to chaos at the endpoints of this chaotic interval.\footnote{The properties of the logarithmic map are similar to those found in the class of singular maps $x_{t+1} = 1 - a|x_t|^z$, with $|z| < 1$. This family of maps is characterized by a positive Schwarzian derivative, by unbounded dynamics (for $z \in (-1, 0)$), by the absence of periodic windows in a single chaotic interval of the parameter $a$, and by types I and III intermittency transitions to chaos at the endpoints of this chaotic interval.}

§3. Synchronization

The coupled map system (1) can be expressed in vector form as
Synchronization and Collective Behavior in Globally Coupled Logarithmic Maps

4.

Fig. 4. Phase diagram of the system Eq. (1). The synchronization boundaries are indicated by thick lines. The ±1 labels on each curve identify the corresponding sign in Eqs. (6) and (7). The regions of parameters where the different collective states occur are identified. CPB stands for “collective periodic behavior”, and CCB for “collective chaotic bands”. Fully turbulent states are also present.

\[ x_{t+1} = \left[ (1 - \epsilon) I + \frac{\epsilon}{N} M \right] f(x_t), \]

where the \( N \)-dimensional vectors \( x_t \) and \( f(x_t) \) have components \( [x_t]_i = x_t(i) \) and \( [f(x_t)]_i = f(x_t(i)) \), respectively, \( M \) is an \( N \times N \) matrix with all its components being equal to 1, and \( I \) is the \( N \times N \) identity matrix.

The simplest kind of global attractor in the system Eq. (4) is the synchronized state, with \( x_t(i) = x_t(j) \quad \forall \ i, j \), in which case the dynamics is described by the single logarithmic map \( x_{t+1} = f(x_t) \).

From the linear stability analysis of synchronized states in coupled map lattices, it can be shown that these states are stable if the following condition

\[ \left| \left(1 - \epsilon + \frac{\epsilon}{N} \mu_k \right) e^\lambda \right| < 1 \]

is satisfied,\(^1,20\) where \( \{\mu_k : k = 1, \ldots, N\} \) is the set of eigenvalues of the coupling matrix \( M \), and \( \lambda \) is the Lyapunov exponent of the single logarithmic map (Fig. 3). In the globally coupled case, the eigenvalues are \( \mu_k = 0 (k = 1, 2, \ldots, N - 1) \), which has \((N-1)\)-fold degeneracy, and \( \mu_N = N \). Because of these eigenvalues, the synchronization condition is independent of \( N \); i.e., it can be achieved with any
number of globally coupled maps. The eigenvectors of the matrix $M$ constitute a complete basis in terms of which any state $x_t$ of the system can be expressed as a linear combination. The eigenvector corresponding to the eigenvalue $\mu_N = N$ is homogeneous. Thus perturbations of the state $x_t$ along this eigenvector do not destroy the coherence, and the stability condition associated with $\mu_N = N$ is irrelevant for a synchronized state. The other $(N - 1)$ eigenvectors associated with $\mu_k = 0$ are not homogeneous. Thus, condition (5) with $\mu_k = 0$ defines a region in the space of parameters $(b, \varepsilon)$ of the system (4) where all the stable synchronized states can be observed.

Two types of stable synchronized states satisfying condition (5) are found in the system of globally coupled logarithmic maps:  

1) **synchronized stationary states** for which $x_t(i) = x_1^*$ for $b < -1$, and $x_t(i) = x_2^*$ for $b > 1$, $\forall i$, corresponding to constant values of the mean field. The boundaries of the stability regions of the synchronized stationary states are given by

$$
(1 - \varepsilon + \frac{\varepsilon}{N} \mu_k) |f'(x_{1,2}^*)| = \pm 1. \quad (6)
$$

Equation (6) yields curves in the parameter plane $(b, \varepsilon)$ which determine where each coherent stationary state exists. Figure 4 displays the stability boundaries of each of these states, with $b < -1$ and $b > 1$, respectively, corresponding to the eigenvalues $\mu_k = 0$ and $\mu_N = N$. Inside these regions of stable synchronized stationary states, the mean field takes the values $h_t = x_1^*$, for $b < -1$, and $h_t = x_2^*$, for $b > 1$. Note that synchronization can occur for all values of the coupling in the phase diagram of the system, since the local dynamics is unbounded.

2) **synchronized chaotic state** for which $x_t(i) = f(x_t)$, $\forall i$, in the parameter range $b \in [-1, 1]$. The stability region of this state is bounded by the curves

$$
(1 - \varepsilon + \frac{\varepsilon}{N} \mu_k) e^{\lambda} = \pm 1, \quad (7)
$$

with $\lambda(b)$ calculated in Fig. 2. Figure 4 exhibits the boundaries given by Eq. (7) with $\mu_k = 0$. The curves corresponding to the eigenvalue $\mu_n = N$ reduce to $b = -1$ and $b = 1$ in both Eqs. (6) and (7). These values of $b$ separate the synchronized chaotic state $h_t = f(x_t)$ from the synchronized stationary states $h_t = x_1^*$ and $h_t = x_2^*$. In either region of synchronized states, the globally coupled system, Eq. (4), forms a stable single cluster. This has been verified by direct simulations with random initial conditions.

The synchronization boundaries, given by Eqs. (6) and (7) with $\mu_k = 0$, are symmetric about the value $\varepsilon = 1$ in the parameter plane $(b, \varepsilon)$. This symmetry is related to the assumed form of the globally coupled map system, Eq. (1). In this case, the unbounded property of the local dynamics permits the exploration of the states of the system for any values of its parameters, allowing the observation of possible symmetries or structures in the entire parameter plane.

The global coupling induces different degrees of synchronization in the network, depending on the proximity of the parameter $\varepsilon$ to its values on the synchronization
Fig. 5. The mean standard deviation $\langle \sigma \rangle$ vs the coupling parameter $\epsilon$, with fixed $b = -0.7$. The system size is $N = 10^5$. $\langle \sigma \rangle$ was calculated over 5000 iterations of the system Eq. (1), after discarding 2000 points representing transient behavior. The zones corresponding to the different collective states in this diagram are indicated with a notation similar to Fig. 4.

boundaries. The synchronization of the lattice can be characterized by the time-average $\langle \sigma \rangle$ of the instantaneous standard deviations $\sigma_t$ of the distribution of site variables $x_t(i)$, defined as

$$\sigma_t = \left( \frac{1}{N} \sum_{i=1}^{N} [f(x_t(i)) - h_t]^2 \right)^{1/2}.$$  \hspace{1cm} (8)

Figure 5 shows the quantity $\langle \sigma \rangle$ as a function of the coupling parameter $\epsilon$, with fixed $b = -0.7$, calculated after discarding the transients and starting from random initial conditions on the local maps for each value of $\epsilon$. Within the synchronization region, $\langle \sigma \rangle$ reaches its minimum value very close to zero. Perfect synchronization (i.e. $\langle \sigma \rangle \equiv 0$) is limited by the finite precision of the calculations. The “amount” of synchronization decreases with increasing distance from the synchronization boundaries. Figure 5 also shows the location of the collective states associated with desynchronization in the system, which are discussed in the next section.

§4. Desynchronized states

The crossing of the synchronization boundaries with $\mu_k = 0$ in Fig. 4 marks the appearance of desynchronized states for which $x_t(i) \neq x_t(j)$. In order to investigate
the collective dynamics of the system (4) in different regions of its parameter space 
\((b, \epsilon)\) outside the synchronization zone, we have constructed bifurcation diagrams of 
the asymptotic mean field \(h_t\) as functions of the parameters. Note that this system 
allows any real values of \(b\) and \(\epsilon\). Negative values of \(\epsilon\) may correspond to some 
physical situations. These have also been studied with CML models.\(^{21,22}\)

We have observed desynchronized states with different types of collective temporal 
manifestations, existing in different regions of the phase diagram, Fig. 4. These 
states consist of: a) collective periodic behavior, b) collective chaotic bands, and c) 
full turbulence.

4.1. Collective periodic behavior

When the parameter \(b\) is in the range \([-1,1]\), the elements \(x_t(i)\) are chaotic 
and desynchronized. However, the mean field of the system reveals the existence of 
global periodic attractors. Figures 6(a) and (b) display the bifurcation diagrams \(h_t\) 
vs \(b\) for two different fixed values of the coupling parameter \(\epsilon\). For each value of \(b\), 
the mean field was calculated at each time step during a run starting from random 
initial conditions on the local maps, uniformly distributed on the interval \([-8,4]\), 
after discarding the transients. In this representation, collective periodic states at a 
given value of the parameter \(b\) appear as sets of vertical segments which correspond 
to intrinsic fluctuations of the periodic orbits of the mean field.

For small values of the coupling \(\epsilon\), Fig. 6(a) shows simple periodic collective states 
occurring in the chaotic range of the local dynamics: a pitchfork bifurcation takes 
place from a collective fixed point (a state for which the time series of \(h_t\) statistically 
fluctuates around a single value) to a collective period-two state (a state for which 
the time series of \(h_t\) alternatingly moves between the corresponding neighborhoods 
of two separated values). Increasing the coupling induces the emergence of collective 
states of higher periodicity. Figure 6(b) shows that global attractors of period 2, 4 
and 8 are possible in this globally coupled map system.

Figure 4 indicates the region in the phase diagram of the system where collective 
periodic behavior occurs. No quasiperiodic or more complex collective dynamics have 
been found in this region of parameters. This has been verified by looking at the 
time series of \(h_t\) and its return map. We have also checked system-size effects: when 
the lattice size \(N\) is increased, the segments in the bifurcation diagrams such as 
Fig. 6 shrink, indicating that the global periodic attractors become better defined 
in the large system limit. Collective quasiperiodic orbits, instead, reach a finite size 
with increasing \(N\).\(^{6}\) As a comparison, global quasiperiodic behavior is a common 
feature in globally coupled systems of unimodal or bounded maps.\(^{13-16}\)

The amplitudes of the collective periodic motions manifested in the mean field 
\(h_t\) do not decrease with an increase of the system size \(N\). As a consequence, the 
variance of the fluctuations of \(h_t\) itself does not decrease as \(N^{-1}\) with increasing 
\(N\), but rather it saturates at some constant value related to the amplitude of the 
collective period. This is a phenomenon of nontrivial collective behavior, where 
macroscopic quantities in a system of nonlinear elements exhibit regular temporal 
evolution in spite of the presence of local chaos.\(^{5}\)

The dynamics of the collective periodic states is related to the existence of unsta-
Fig. 6. Bifurcation diagram of the mean field $h_t$ as a function of the parameter $b$ for two different values of the coupling. $N = 10^5$. (a) $\epsilon = 0.2$; (b) $\epsilon = 0.25$.

ble periodic orbits and to the dynamics of the iterates of the single logarithmic map in its chaotic range. Figure 7 shows the superposition of successive instantaneous distributions $P_t(x)$ of site variables $x_i(t)$ of the globally coupled system with parameters yielding a collective period 4. The locations of the unstable orbits of periods 1 and 2 of the logarithmic map are also indicated. The distribution $P_t(x)$ varies periodically in time, with a period of approximately 4. The chaotic elements tend to move together and periodically around the unstable periodic orbits at successive time steps. The individual values may be different within each zone separated by the unstable periodic orbits. The resulting mean field oscillates with a period of approximately 4 around the unstable fixed point $\bar{x}_1$ and the unstable period 2 orbits $\bar{x}_a$, $\bar{x}_b$ (Fig. 2). These establish "reference" lines for the organization of the collective periodic behavior. In the collective period 2 state, oscillations of the distribution $P_t(x)$ and of the mean field $h_t$ occur alternatively about the unstable fixed point $\bar{x}_1$. 
Similarly, the observed collective period 8 arises from oscillations of the distributions $P_t(x)$ with a period of approximately 8 around the unstable orbits of periods 1, 2, and 4 of the local map in the region $b \in [-1, 1]$.

The collective periodic states reflect the dynamics of the iterates of the single logarithmic map in the given region of the local parameter $b$. These iterates tend to move in a certain sequence between the intervals limited by the unstable periodic orbits of the logarithmic map, but chaotically within each of these intervals, even though there are no gaps separating chaotic bands. In the region of the phase diagram corresponding to collective periodic behavior, coupling induces a weak synchronization in the network, in the sense that the individual chaotic sites tend to move together around the local unstable periodic orbits as a single map dose, but keeping some dispersion between them.

4.2. Collective chaotic bands

As the parameters of the system approach their values on the synchronization boundary, the collective periodic attractors are destroyed, giving way to collective chaotic bands. Figure 8(a) shows the bifurcation diagram of $h_t$ as a function of the coupling $\epsilon$, with a fixed value of the local parameter $b$ in the chaotic range of the local maps. Random initial conditions are used for each value of $\epsilon$ in Figs. 8(a) and (b).
Synchronization and Collective Behavior in Globally Coupled Logarithmic Maps

Fig. 8. (a) Bifurcation diagram of the mean field $h_t$ as a function of the coupling $\epsilon$, with fixed $b = -0.7$ and $N = 10^5$. (b) Magnification of the left part of (a). The horizontal lines indicate the value of the unstable fixed point $\bar{x}_1$ and the values of the unstable period 2 orbit on each side of $\bar{x}_1$, for the logarithmic map with $b = -0.7$. The vertical lines correspond to the approximate boundaries of different collective states indicated by S (synchronized state), CCB (collective chaotic bands), and CPB (collective periodic behavior).

Near the synchronization boundaries, the collective states described by the mean field take the form of chaotic bands which successively merge until complete synchronization is achieved. Figure 8(b) shows a magnification of the bifurcation diagram of (a) close to the lower synchronization boundary for the fixed value $b = -0.7$. There can be distinguished eight-band, four-band, two-band and one-band chaotic collective states. Adjacent to the synchronization boundaries in $b \in [-1, 1]$ lies the collective chaotic one-band state, corresponding to a swarm of chaotic elements which maintain some coherence. Similar chaotic band structures are present along the synchronization boundary in the region $b \in [-1, 1]$. Figure 4 displays the approximate location of the collective chaotic band states in the phase diagram of the
system. The collective chaotic bands, as well as the collective periodic states, are manifestations of nontrivial collective behavior. In both cases, the fluctuations of the mean field $h_t$ do not decrease as $N^{-1}$ when increasing the system size $N$.

The dynamics of the chaotic band states, as those of the collective periodic behavior manifested in $h_t$, are related to the presence of unstable periodic orbits in the chaotic interval of the local map (3). The values of the unstable fixed point and the unstable period 2 orbit corresponding to $b = -0.7$ are indicated in Fig. 8(b) by horizontal lines. Figure 9 shows the successive instantaneous distributions $P_t(x)$ associated with the collective chaotic two-band state. The distribution $P_t(x)$ moves alternatively as a single group on each side of the unstable fixed point $\bar{x}_1$, but its motion is not periodic. In general, the elements of the globally coupled system Eq. (1) do not split into separated groups in order to produce either the observed collective chaotic band states or the collective periodic states. We attribute this behavior to the total absence of separated chaotic bands in the dynamics of the local maps.

Figure 10(a) displays the return map of $h_t$ at parameter values corresponding to the four-band state. The collective behavior comes from a global low-dimensional chaotic attractor. Figure 10(b) shows the structure of the lower portion of this global chaotic attractor. No quasiperiodic collective motion appears to take place. In fact, no form of quasiperiodic or more complex collective dynamics have been found in the
Fig. 10. (a) Return map of the mean field $h_t$ corresponding to a collective chaotic four-band state, $b = -0.7, \epsilon = 0.265; N = 10^5$. 8000 iterates are shown, after transients. (b) Magnification of the lower part of (a).
phase diagram of this system of globally coupled logarithmic maps. Additionally, we have verified that there are no appreciable changes in the chaotic band attractors, such as in Fig. 10, when different realizations of random initial conditions are used in the simulations. Thus, the existence of multiple collective band attractors depending on initial conditions does not seem to occur in the present globally coupled system.

4.3. Full turbulence

In Figs. 6 and 8, we can see the presence of collective fixed point states in the region $b \in [-1, 1]$. Actually, these states exist for parameter values outside the synchronization boundaries, as Fig. 11 shows. In those ranges of parameters, the mean field $h_t$ behaves as the superposition of $N$ completely desynchronized and uncorrelated chaotic elements.

The time series of the mean field corresponding to these states fluctuates about a mean value. The variance (mean square deviation) of the time series appears to decrease as $N^{-1}$ with increasing system size $N$, obeying normal statistical behavior (the law of large numbers). We have verified this behavior up to size $N = 10^7$.

The collective fixed points correspond to fully turbulent states of the system. It should be noticed that similar turbulent phases were reported by Kaneko in globally coupled tent maps. However, recent studies have shown that this fully turbulent state is unstable; i.e., small amplitude collective motion always occurs in globally coupled tent or piecewise linear maps. Because of this smallness, the nontrivial collective dynamics was not detected numerically in the globally coupled tent maps. In our case, in addition to the above described statistical study of the fluctuations of $h_t$, we have constructed detailed return maps of $h_t$ in this turbulent region, for several parameter values and for long asymptotic times (after discarding $10^6$ iterations) without finding clear numerical evidence of nontrivial collective behavior. Although this does not constitute a proof that such dynamics does not exist,

Fig. 11. Bifurcation diagram of $h_t$ vs $b$ for two different values of $\epsilon$, corresponding to fully turbulent states.
it reveals that underlying collective motion is not easily observed in simulations; it could be very tiny, or much longer times may be needed to observe a decay of the possibly unstable fully turbulent state.

Note also that the results of Refs. 23) ~ 25) are based on the assumptions that the local dynamics \( f(x) \) is everywhere expanding, i.e., \( |f'(x)| > 1 \), and that \( f(x) \) is bounded on an interval. This is not the case for the logarithmic map. Thus, there might be differences between the logarithmic and the tent globally coupled systems regarding this collective turbulent phase. However, the question regarding the possible inevitability of nontrivial collective behavior for any globally coupled map system is an important issue which requires further investigation.

Figure 4 indicates the region on the space of parameters of the globally coupled system Eq. (1) where the collective turbulent states have been found.

§5. System size effects

Finally, we investigate the influence of the size of the network on the emergence of the different types of collective states observed in the globally coupled system, Eq. (1). Figures 12(a) and (b) show the asymptotic mean field \( \langle h_t \rangle \) as a function of the system size \( N \), for different parameter values corresponding to collective period 4 behavior and a collective chaotic two-band state, respectively. In each case, there is a different critical size of the system \( N_c \) at which the corresponding nontrivial collective behavior distinctively emerges, i.e., \( N_c \approx 25 \) for the two-band state, and \( N_c \approx 450 \) for the period 4 state. The smaller value of \( N_c \) for the two-band state is related to the greater synchronization of this state in comparison with the period 4 state (Fig. 5). Note that the synchronized states are independent of \( N \); i.e., \( N_c \to 0 \). The critical system size for the onset of collective states tends to increase as the parameter values move away from the synchronization boundary.

§6. Discussion

The collective behavior of a system of globally coupled logarithmic maps has been investigated in its space of parameters. We have found two kinds of collective states in the phase diagram of the system, corresponding to synchronized and to desynchronized states. The latter correspond to: i) nontrivial collective behavior, consisting of collective periodicity and banded chaos, and ii) full turbulence, which display normal statistical behavior. The global states are reminiscent of thermodynamic phases, characterized by statistical quantities, such as the mean field and the mean standard deviation, which may act as order parameters. The critical size of the system for the onset of nontrivial collective behavior varies for the different states.

The existence of unstable periodic orbits in the local maps is a relevant factor for the organization of nontrivial collective behavior. We have noted that the dynamics of both the collective periodic states and the collective chaotic bands are related to the unstable periodic orbits and to the dynamics of the isolated logarithmic map.
Fig. 12. Bifurcation diagram of $h_t$ as a function of the system size $N$, corresponding to different collective states. (a) collective period 4, $b = -0.7$, $\epsilon = 0.24$; (b) collective chaotic two-band, $b = -0.7$, $\epsilon = 0.276$. 
Coupling can induce global ordered motion around the unstable periodic orbits of the local map. On the other hand, the presence of periodic windows in the local dynamics does not seem essential for the emergence of ordered collective behavior in systems of globally coupled maps. This contrasts with previous conjectures about the need for window structures in the elements for the appearance of collective motion.\textsuperscript{26}

The collective states of the globally coupled logarithmic maps look simpler than those associated with other systems with global couplings. It appears that there is no clustering, no splitting of elements into distinct groups, and no quasiperiodicity nor more complex collective behavior in our model. Instead, those features are commonly observed in the collective dynamics of globally coupled maps belonging to standard universality classes (quadratic, unimodal or circle maps). Additionally, the coexistence of multiple collective chaotic band attractors does not occur in globally coupled logarithmic maps. Clustering does not exist in globally coupled tent maps either, but there is splitting of the elements into separated groups, quasiperiodicity, and coexistence of several attractors in the corresponding collective motion of such system.\textsuperscript{13} The collective behavior of globally coupled tent maps shares some characteristics with the corresponding logarithmic case (no clustering, for example) and with the quadratic case (quasiperiodicity, coexistence of multiple global attractors).

The tent map possesses a zero Schwarzian derivative, no periodic windows, and chaotic band splittings. On the other hand, quadratic or unimodal maps have a negative Schwarzian derivative, periodic windows, and chaotic band splittings. The properties of the tent map appear to be intermediate between those of the logarithmic and quadratic maps. Inspection of the collective states found in previously studied globally coupled maps together with our results on the globally coupled logarithmic maps suggest that different local properties may be related to the different types of collective behavior observed in the corresponding globally coupled elements. Clustering seems to be associated with the existence of periodic windows in the local dynamics, whereas simple collective periodic motion of the elements as a single group appears related to the properties of the logarithmic map. Recent results on globally coupled “similar maps” (a class to which the logarithmic map belongs) have shown that the corresponding collective states are similar to those found in the present system.\textsuperscript{19} A systematic study of the relationship between the general properties of the local maps, besides their universality class, and the characteristics of the collective behavior that emerges in globally coupled systems is an interesting problem for future work.

Much effort has been dedicated to establishing the necessary conditions for the emergence of nontrivial collective behavior in lattices of nonlinear coupled elements, mostly involving numerical simulations. The observation of ordered collective behavior in the present system, where the local dynamics have very specific properties, suggests that this kind of collective behavior should be a rather common phenomenon in deterministic systems of globally coupled chaotic elements.
Acknowledgements

We thank Professor A. Parravano for useful discussions. This work was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico, and calculations were performed at the Centro de Cálculo Científico, both institutions of the Universidad de Los Andes, Mérida, Venezuela.

References

1) K. Kaneko, Physica D41 (1990), 137.