A Maximum Mass-to-Size Ratio in Scalar-Tensor Theories of Gravity

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We derive a modified Buchdahl inequality for scalar-tensor theories of gravity. In general relativity, Buchdahl has shown that the maximum value of the mass-to-size ratio, $2M/R$, is $8/9$ for static and spherically symmetric stars under some physically reasonable assumptions. We formally apply Buchdahl's method to scalar-tensor theories and obtain theory-independent inequalities. After discussing the mass definition in scalar-tensor theories, these inequalities are related to a theory-dependent maximum mass-to-size ratio. We show that its value can exceed not only Buchdahl's limit, $8/9$, but also unity, which we call the black hole limit, in contrast to general relativity. Next, we numerically examine the validity of the assumptions made in deriving the inequalities and the applicability of our analytic results. We find that the assumptions are mostly satisfied and that the mass-to-size ratio exceeds both Buchdahl's limit and the black hole limit. However, we also find that this ratio never exceeds Buchdahl's limit when we impose the further condition, $\rho - 3p \geq 0$, on the density, $\rho$, and pressure, $p$, of the matter.

§1. Introduction

Einstein's general relativity postulates that gravitational interactions are mediated by a tensor field, $g_{\mu\nu}$. It is also well-known that electro-magnetic interactions are mediated by a vector field, $A_\mu$. One may therefore suspect that some unknown interactions may be mediated by scalar fields. Such theories have been suggested since before the appearance of general relativity. Moreover, it has been repeatedly pointed out over the years that unified theories that contain gravity as well as other interactions naturally give rise to scalar fields coupled to matter with gravitational strength. This motivation has led many theoretical physicists to study scalar-tensor theories of gravity (scalar-tensor theories). 1)–4) The scalar-tensor theories are natural alternatives to general relativity, and gravity is mediated not only by a tensor field but also by a scalar field in these theories. Recently, such theories have been of interest as effective theories of string theory at low energy scales. 5)

Many predictions of the scalar-tensor theories in strong gravitational fields are summarized in Refs. 4), 6) and 7). It has been found that a wide class of scalar-tensor theories can pass all experimental tests in weak gravitational fields. However, it has also been found that scalar-tensor theories exhibit different aspects of gravity in strong gravitational fields in contrast to general relativity. It has been shown numerically that nonperturbative effects in the scalar-tensor theories increase the...
maximum mass of an isolated system such as a neutron star.\textsuperscript{6,7)}

In general relativity, the mass-to-size ratio of a star has physical significance, especially for an isolated system. Buchdahl has obtained a maximum value of the mass-to-size ratio of a static and spherically symmetric star under the following physically reasonable assumptions.\textsuperscript{8)-10)}

- No black hole exists.
- The constitution of the star is a perfect fluid.
- The density at any point in the star is a positive and monotonously decreasing function of the radius.
- An interior solution of the star smoothly matches an exterior solution, i.e., Schwarzschild’s solution.

Buchdahl has obtained an upper limit of the mass-to-size ratio as $2M/R \leq 8/9$. We shall refer to this as the Buchdahl inequality.

Motivated by Buchdahl’s theorem, we shall derive a modified Buchdahl inequality to obtain the maximum mass-to-size ratio in scalar-tensor theories. We then numerically examine the validity of the assumptions made in deriving the inequality. The applicability of our analytic results is also examined. This paper is organized as follows. In §2, we summarize the basic equations in the scalar-tensor theories. In §3, we derive a modified Buchdahl inequality in the scalar-tensor theories, and the numerical results are compared with the analytic results in §4. A brief summary is given in §5.

§2. Basic equations

We shall consider the simplest scalar-tensor theory.\textsuperscript{1,4,11)} In this theory, gravitational interactions are mediated by a tensor field, $g_{\mu\nu}$, and a scalar field, $\phi$. The action of the theory is

$$S = \frac{1}{16\pi} \int \sqrt{-g} \left[ \phi R - \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} \right] d^4x + S_{\text{matter}}[\Psi_m, g_{\mu\nu}], \quad (2.1)$$

where $\omega(\phi)$ is a dimensionless arbitrary function of $\phi$, $\Psi_m$ represents matter fields, and $S_{\text{matter}}$ is the action of the matter fields. The scalar field, $\phi$, plays the role of an effective gravitational constant as $G \sim 1/\phi$. Varying the action by the tensor field, $g_{\mu\nu}$, and the scalar field, $\phi$, yields, respectively, the following field equations:

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2} \left( \phi_{,\mu} \phi_{,\nu} - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right) + \frac{1}{\phi} (\nabla_{\mu} \nabla_{\nu} \phi - g_{\mu\nu} \Box \phi), \quad (2.2)$$

$$\Box \phi = \frac{1}{3 + 2\omega(\phi)} \left( 8\pi T - \frac{d\omega}{d\phi} g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} \right). \quad (2.3)$$

Now we perform the conformal transformation,

$$g_{\mu\nu} = A^2(\phi) g_{\ast\mu\nu}, \quad (2.4)$$

such that

$$G_{\ast} A^2(\phi) = \frac{1}{\phi}, \quad (2.5)$$
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where \( G_* \) is a bare gravitational constant, and we call \( A(\varphi) \) a coupling function. Hereafter, the symbol, \(*\), denotes quantities or derivatives associated with \( g_*^{\mu\nu} \). Then the action can be rewritten as

\[
S = \frac{1}{16\pi G_*} \int \sqrt{-g_*} \left( R_* - 2g_*^{\mu\nu} \varphi_{,\mu}\varphi_{,\nu} \right) d^4x + S_{\text{matter}}[\Psi_m, A^2(\varphi)g_*^{\mu\nu}],
\]

where the scalar field, \( \varphi \), is defined by

\[
\alpha^2(\varphi) \equiv \left( \frac{d\ln A(\varphi)}{d\varphi} \right)^2 = \frac{1}{3 + 2\omega(\varphi)}. \tag{2.7}
\]

Varying the action by \( g_*^{\mu\nu} \) and \( \varphi \) yields, respectively,

\[
G_*^{\mu\nu} = 8\pi G_* T_*^{\mu\nu} + 2 \left( \varphi_{,\mu}\varphi_{,\nu} - \frac{1}{2} g_*^{\mu\nu} g_*^{\alpha\beta} \varphi_{,\alpha}\varphi_{,\beta} \right), \tag{2.8}
\]

\[
\Box_* \varphi = -4\pi G_* \alpha(\varphi) T_*, \tag{2.9}
\]

where \( T_*^{\mu\nu} \) represents the energy-momentum tensor with respect to \( g_*^{\mu\nu} \) defined by

\[
T_*^{\mu\nu} \equiv \frac{2}{\sqrt{-g_*}} \frac{\delta S_{\text{matter}}[\Psi_m, A^2(\varphi)g_*^{\mu\nu}]}{\delta g_*^{\mu\nu}} = A^6(\varphi) T^{\mu\nu}. \tag{2.10}
\]

The conservation law for \( T_*^{\mu\nu} \) is given by

\[
\nabla_{\nu} T_*^{\nu\mu} = \alpha(\varphi) T_* \nabla_{\mu}\varphi. \tag{2.11}
\]

The field equations (2.9) and (2.11) tell us that the coupling strength, \( \alpha(\varphi) \), plays a role in mediating interactions between the scalar field, \( \varphi \), and the matter. General relativity is characterized by having a vanishing coupling strength: \( \alpha(\varphi) = 0 \), i.e., \( A(\varphi) = 1 \). The Jordan-Fierz-Brans-Dicke theory is characterized by having a \( \varphi \)-independent coupling strength: \( \alpha(\varphi) = \alpha_0 = \text{const} \), i.e., \( A(\varphi) = e^{\alpha_0 \varphi} \). Observational constraints on the coupling strength are summarized in Appendix A.

§3. Modified Buchdahl’s theorem in scalar-tensor theories

In this section, we consider a static and spherically symmetric space-time with a perfect fluid. First, we derive a modified Buchdahl inequality. Then the inequality is reformulated to obtain the maximum value of the mass-to-size ratio in the scalar-tensor theories. Hereafter, we refer to \((g_{\mu\nu}, \varphi)\) and \((g_*^{\mu\nu}, \varphi)\), respectively, as the physical frame and the Einstein frame.

3.1. A modified Buchdahl inequality in scalar-tensor theories

In the Einstein frame, a line element of the static and spherically symmetric space-time is written as

\[
ds^2 = -f_*(r)dt^2 + h_*(r)dr^2 + r^2d\Omega^2. \tag{3.1}
\]

The stress-energy tensor for the perfect fluid in the Einstein frame is given by

\[
T_*^{\mu\nu} = (\rho_* + p_*)u_*^\mu u_*^\nu + p_* g_*^{\mu\nu}, \quad u_{*\alpha} = -\sqrt{f_*(r)} \left( dt \right)_\alpha, \tag{3.2a}
\]
where \( u^\alpha \) is the four velocity of the matter. The stress-energy tensor for the perfect fluid in the physical frame is given by

\[
T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \quad u_\alpha = -A\sqrt{f_s(r)} \frac{dt}{(dt)_\alpha},
\]

(3.2b)

where \( u^\alpha \) is the four velocity of the matter. Fluid variables in the physical and Einstein frames are related according to

\[
u^\alpha = u^\alpha A^{-1}(\varphi), \quad \rho = \rho_\ast A^{-4}(\varphi), \quad p = p_\ast A^{-4}(\varphi).
\]

(3.3)

(3.4)

(3.5)

Now the field equations (2.8) and (2.9) are reduced to the following equations:

\[
\left( r(1 - h_\ast^{-1}) \right)' = 8\pi G_\ast \rho_\ast r^2 + \frac{r^2}{h_\ast} \varphi'^2,
\]

(3.6)

\[
-r^2 h_\ast(1 - h_\ast^{-1}) + r^{-1} f_s^{-1} f_s' = 8\pi G_\ast p_\ast h_\ast + \varphi'^2,
\]

(3.7)

\[
\left( \frac{f_s'}{2 f_s} \right)' + \frac{1}{r^2 f_s} + \frac{h_\ast'}{2 h_\ast} f_s' - \frac{1}{r} h_\ast' = 8\pi G_\ast p_\ast h_\ast - \varphi'^2,
\]

(3.8)

\[
r^{-2}(f_s h_\ast)^{-\frac{1}{2}} \left( \frac{f_s}{h_\ast} \right)^{\frac{1}{2}} r^2 \varphi'^2 = 4\pi G_\ast \alpha(\varphi)(\rho_\ast - 3p_\ast),
\]

(3.9)

where a prime denotes differentiation with respect to \( r \). As is often done in the cases of general relativity, we define a mass function, \( m_\ast(r) \), in the Einstein frame as follows:

\[
h_\ast(r) \equiv \left[ 1 - \frac{2m_\ast(r)}{r} \right]^{-1}.
\]

(3.10)

Then (3.6) is rewritten as

\[
m_\ast'(r) = 4\pi G_\ast \rho_{\text{eff}} r^2,
\]

(3.11)

where

\[
\rho_{\text{eff}} \equiv \rho_\ast + \frac{\varphi'^2}{8\pi G_\ast h_\ast}.
\]

(3.12)

That is, \( \rho_{\text{eff}} \) plays the role of an effective density in the Einstein frame.

In order to derive a modified Buchdahl inequality, we assume

\[
h_\ast(r) \geq 0, \quad f_s(r) \geq 0, \quad \rho_{\text{eff}}(r) \geq 0, \quad \rho_{\text{eff}}'(r) \leq 0.
\]

(3.13)

(3.14)

(3.15)

(3.16)

These assumptions imply the following:

- No black hole exists in the Einstein frame.
- The effective density, \( \rho_{\text{eff}} \), is a positive and monotonously decreasing function of the radius.
Moreover, we assume that an interior solution of the above field equations smoothly matches the corresponding exterior one. Note that these assumptions are concerned with the unphysical variables and that their validity should be examined. This will be done later.

Using the assumption (3.16), it is easy to verify the following inequality:

\[
\left( \frac{m_\star}{r^3} \right)' \leq 0. \tag{3.17}
\]

Moreover, with (3.7), (3.8) and (3.11), we obtain

\[
-(\frac{(\sqrt{f_\star})'}{r\sqrt{h_\star}})' = \sqrt{f_\star h_\star} \left( -\left( \frac{m_\star}{r^3} \right)' + 2\frac{\varphi'^2}{r h_\star} \right) \geq 0. \tag{3.18}
\]

Accordingly, we have

\[
\left( \frac{\sqrt{f_\star(r_1)}}{r_1\sqrt{h_\star(r_1)}} \right)' \geq \left( \frac{\sqrt{f_\star(r_2)}}{r_2\sqrt{h_\star(r_2)}} \right)' , \quad r_1 \leq r_2. \tag{3.19}
\]

Now let the inequality (3.19) be reformulated in terms of variables of the exterior solution. The exterior solution, whose derivation is given in Appendix B, is

\[
d_{s\ast}^2 = -e^{\gamma(x)}dt^2 + e^{-\gamma(x)}d\chi^2 + e^{\lambda(x)-\gamma(x)}d\Omega^2 , \tag{3.20}
\]

where

\[
e^{\lambda(x)} = \chi^2 - a\chi , \tag{3.21}
\]

\[
e^{\gamma(x)} = \left( 1 - \frac{a}{\chi} \right)^{\frac{b}{a}} , \tag{3.22}
\]

\[
\varphi(x) = \varphi_0 + \frac{c}{a} \ln \left( 1 - \frac{a}{\chi} \right) . \tag{3.23}
\]

Here \( a, b, c \) and \( \varphi_0 \) are constants of integration, and \( \varphi_0 \) is the asymptotic value of \( \varphi \) at infinity. Moreover, the constants, \( a, b \) and \( c \), must satisfy the following relation (Appendix B):

\[
a^2 - b^2 = 4c^2 . \tag{3.24}
\]

One may expect that \( \chi = a \) is an event horizon. However, this is not the case in generic scalar-tensor theories, where the null surface, \( \chi = a \), is a curvature singularity in the Einstein frame. The singular nature of the unphysical space-time at \( \chi = a \) can also be seen when transformation to the Schwarzschild coordinate is made. The Schwarzschild coordinate, \( r \), and the Just coordinate, \( \chi \), are related by the following relation:

\[
r = \chi \left( 1 - \frac{a}{\chi} \right)^{\frac{a-b}{2a}} . \tag{3.25}
\]

One finds that, when \( a \neq b \), \( \chi = a \) in the Just coordinate corresponds to \( r = 0 \) in the Schwarzschild coordinate.
By matching the interior solution to the exterior solution, we obtain the following relations:

\[ r_s = \chi_s \left( 1 - \frac{a}{\chi_s} \right)^{\frac{a-b}{2a}} , \quad (3.26) \]

\[ f_*(r_s) = \left( 1 - \frac{a}{\chi_s} \right)^{\frac{b}{a}} , \quad (3.27) \]

\[ h_*(r_s) = \left( 1 - \frac{a}{\chi_s} \right) \left( 1 - \frac{a+b}{2\chi_s} \right)^{-2} , \quad (3.28) \]

where the subscript, s, refers to \( \chi \) evaluated at the surface, \( r = r_s \). Note that

\[ \chi_s > a \geq b . \quad (3.29) \]

Since \( h_*(r)^{-1} = 1 - 2m_*(r)/r \), (3.28) becomes

\[ 2m_*(r_s) = \left( b - \frac{(a+b)^2}{4\chi_s} \right) \left( 1 - \frac{a}{\chi_s} \right)^{-\frac{(a+b)}{2a}} . \quad (3.30) \]

Accordingly, by virtue of the positivity of \( m_*(r) \), we obtain the additional inequality

\[ b - \frac{(a+b)^2}{4\chi_s} > 0 . \quad (3.31) \]

With (3.19) and (3.25) \sim (3.28), we obtain the following relation for \( r \leq r_s \):

\[ \frac{(\sqrt{f_*}')}{r \sqrt{h_*}} \geq \frac{(\sqrt{f_*(r_s)})'}{r_s \sqrt{h_*(r_s)}} = \frac{b}{2r_s^3} . \quad (3.32) \]

Integrating (3.32) from the center, \( r = 0 \), to the surface, \( r = r_s \), we obtain

\[
0 \leq \sqrt{f_*(0)}
\leq \sqrt{f_*(r_s)} - \frac{b}{2r_s^3} \int_0^{r_s} r \sqrt{h_*(r)} dr
\leq \sqrt{f_*(r_s)} - \frac{b}{2r_s^3} \int_0^{r_s} r \left( 1 - \frac{2m_*(r_s)}{r_s^3} r^2 \right)^{-\frac{1}{2}} dr
= \left( 1 - \frac{a}{\chi_s} \right)^{\frac{b}{2a}} + \frac{b}{4m_*(r_s)} \left( \sqrt{1 - \frac{2m_*(r_s)}{r_s}} - 1 \right) , \quad (3.33)
\]

where the inequality (3.17) has been used.

The inequalities obtained to this point can be simplified in terms of new parameters defined by

\[ a_s \equiv \frac{a}{\chi_s} , \quad b_s \equiv \frac{b}{\chi_s} , \quad c_s \equiv \frac{c}{\chi_s} . \quad (3.34) \]

Substituting (3.30) into (3.33), we obtain the following inequality:
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\[ 0 \leq \sqrt{f_s(0)} \]

\[ \leq \frac{1}{2} (1 - a_s) \frac{b_s}{2a_s} \left( b_s - \frac{(a_s + b_s)^2}{4} \right)^{-1} \]

\[ \times \left( 2b_s - \frac{(a_s + b_s)^2}{2} - b_s \sqrt{1 - a_s} \left( 1 - \frac{4b_s - (a_s + b_s)^2}{4(1 - a_s)} \right) \right). \] (3.35)

The above inequality can be further simplified, and we finally obtain a modified Buchdahl inequality in the scalar-tensor theories as

\[ F(a_s, b_s) \equiv 3b_s - \frac{1}{2} (a_s + b_s)(a_s + 2b_s) - b_s \sqrt{1 - a_s} \geq 0, \] (3.36)

supplemented with (3.29) and (3.31).

The modified Buchdahl inequality can be solved to yield

\[
\begin{align*}
    b_s &\leq a_s \leq 2\sqrt{b_s} - b_s & \quad \text{for } 0 \leq b_s \leq 4(3 - 2\sqrt{2}), \\
    b_s &\leq a_s \leq 2\sqrt{2b_s} - 2b_s & \quad \text{for } 4(3 - 2\sqrt{2}) \leq b_s \leq \frac{8}{9} \\
    \text{Forbidden} & & \quad \text{for } b_s > \frac{8}{9}.
\end{align*}
\] (3.37)

Figure 1 displays the allowed region, \( D \), of \((a_s, b_s)\).

In addition, we can obtain, with (3.24), the following upper limit on \(|c_s|\):

\[ |c_s| \leq \frac{2\sqrt{3}}{9}. \] (3.38)

The inequality (3.37) is significant. The third inequality of (3.37) gives us a necessary condition for a spherical star to exist and is reduced to Buchdahl's theorem.

Fig. 1. The allowed region, \( D \), is depicted, where horizontal and vertical axes denote \( b_s \) and \( a_s \) respectively. The characteristic points, \( P = (4(3 - 2\sqrt{2}), 4(3\sqrt{2} - 4)), Q = (8/9, 8/9) \) and \( K = (4/9, 8/9) \), are shown. In general relativity, \( a_s = b_s \). Buchdahl's limit is denoted by \( Q \). On \( K \), \(|c_s|\) takes the maximum value, \(|c_s|_{\text{MAX}} = 2\sqrt{3}/9.\)
in general relativity when \( c = 0 \), i.e., \( a = b \), and, accordingly, \( \chi = r \). In this case, we have (\( R = r_s \))

\[
c_s = 0 \iff a_s = b_s = \frac{2M}{R} \leq \frac{8}{9},
\]

(3.39)

where \( M \) is the ADM mass defined at spatial infinity, and \( R \) is related to the surface area, \( S \), as \( S = 4\pi R^2 \).

The new and important inequality (3.38) is characteristic of the scalar-tensor theories and does not have a general relativistic counterpart. It has been found that nonperturbative behavior of the scalar field appears in the previous numerical studies.\(^6\), \(^7\) Our result implies that, even in a strong gravitational field, the characteristic amplitude of the scalar field, \(|c_s|\), is bounded.

It is important to note that we have not used any assumption regarding the coupling function, \( A(\varphi) \), in deriving the inequalities. In particular, the inequalities (3.37) and (3.38) give theory-independent constraints on the parameters, \( b_s \) and \( c_s \).

### 3.2. The mass-to-size ratio

Now we reformulate the inequalities derived in the previous section, which are given in terms of variables in the Einstein frame, in order to obtain the mass-to-size ratio in the physical frame. To do this, the coupling function, \( A(\varphi) \), should be specified. In this paper, we assume as an example of this coupling function the simple form

\[
A(\varphi) = e^{\frac{1}{2} \beta \varphi^2},
\]

(3.40)

where \( \beta \) is a constant.\(^6\), \(^7\) Then the coupling strength, \( \alpha(\varphi) \), becomes

\[
\alpha(\varphi) = \beta \varphi.
\]

(3.41)

A natural definition of the radius of a spherical star is obtained by using its (physical) surface area as follows. In the physical frame, the surface area, \( S \), is given by

\[
S = 4\pi A^2(\varphi_s) e^{\chi(\varphi_s) - \gamma(\varphi_s)}
\]

\[
= 4\pi \chi_s^2(1 - a_s)^{1 - \frac{b_s}{a_s}} \exp \left[ \beta \left( \frac{c_s}{a_s} \ln(1 - a_s) \right)^2 \right],
\]

(3.42)

where we take the asymptotic value of the scalar field as \( \varphi_0 = 0 \), and, accordingly, we have \( A(\varphi_0) = 1 \) and \( \alpha(\varphi_0) = 0 \). This surface area defines the physical radius, \( R \), of the star in a similar manner as in general relativity:

\[
R \equiv \sqrt{\frac{S}{4\pi}}.
\]

(3.43)

When \( \varphi_0 = 0 \), the effective gravitational constant, \( G \), defined in Appendix A, is equal to \( G_* \). If \( \varphi_0 \neq 0 \), contributions of the scalar field appear in the above expression of \( R \) in terms of \( c \beta \varphi_0 \).

The definition of the mass in the Brans-Dicke theory is found in Ref. 12), and the mass in the scalar-tensor theories is defined in the same manner. In defining the
mass, the metric should be expressed in the isotropic coordinate, $\tilde{r}$, which is related to $\chi$ as

$$\chi = \tilde{r} \left(1 + \frac{a}{4\tilde{r}}\right)^2.$$  

(3.44)

In our model, the exterior solution is then rewritten in terms of $\tilde{r}$ as

$$ds^2 = A^2(\varphi) \left[- \left(1 - \frac{a}{4\tilde{r}}\right) \frac{2b}{1 + \frac{a}{4\tilde{r}}} \frac{2(a+b)}{a} \left(1 - \frac{a}{4\tilde{r}}\right)^{\frac{2(a-b)}{a}} \left(d\tilde{r}^2 + \tilde{r}^2 d\Omega^2\right)\right],$$

$$G_\ast \phi = A^{-2}(\varphi) = \exp \left[- \frac{4\beta c^2}{a^2} \left(\ln \frac{1}{1 + \frac{a}{4\tilde{r}}}\right)^2\right].$$  

(3.45)

By introducing the asymptotic Cartesian coordinates such that $\tilde{r} = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2}$, the asymptotic form of the solution is easily found to be

$$G_\ast \phi \sim 1 + \frac{0}{\tilde{r}} \equiv 1 + \frac{2M_S}{\tilde{r}}, \quad g_{00} \sim -1 + \frac{b}{\tilde{r}} \equiv -1 + \frac{2(M_T + M_S)}{\tilde{r}},$$

$$g_{ij} \sim \left(1 + \frac{b}{\tilde{r}}\right) \delta_{ij} \equiv \left(1 + \frac{2(M_T - M_S)}{\tilde{r}}\right) \delta_{ij},$$  

(3.46)

where the quantities, $M_S$ and $M_T$, are called, respectively, the scalar mass and the tensor mass. At Newtonian order, their sum, $M = M_T + M_S$, plays the role of the mass and is called the active gravitational mass. In our model, $M_S = 0$, and, accordingly, $M_T = M = b/2$. Hereafter, we call $M$ the mass for simplicity.

Now we are ready to calculate the mass-to-size ratio in the scalar-tensor theory as a function of $a_s$, $b_s$, $\chi_s$ and a specific parameter of our model, $\beta$. We obtain

$$H(a_s, b_s; \beta) \equiv \frac{b}{R} = b_s \left(1 - a_s\right)^{\frac{b_s - a_s}{2a_s}} \exp \left[-\frac{1}{2}\beta \left(\frac{c_s}{a_s} \ln(1 - a_s)\right)^2\right].$$  

(3.47)

Fig. 2. On each line, $H(a_s, b_s; \beta) = 8/9$ for various values of $\beta$: 0, -3, -5, -10 and -100. Horizontal and vertical axes denote $b_s$ and $a_s$, respectively.
In Fig. 2, in the allowed region of \((a_s, b_s)\), we display lines on which \(H(a_s, b_s; \beta)\) is equal to \(8/9\) for various values of \(\beta\). For a fixed value of \(\beta\), the region above the line corresponds to the case that the mass-to-size ratio, \(2M/R\), exceeds Buchdahl’s limit, \(8/9\). Moreover, in some cases, it may be greater than unity. We refer to this case as the black hole limit. The maximum values of \(H(a_s, b_s; \beta)\) for various values of \(\beta\) are shown in Fig. 3. Indeed, the maximum mass-to-size ratio can sometimes become larger than the black hole limit. However, the physical exterior solution generically does not have an event horizon in scalar-tensor theories, in contrast to general relativity, and, therefore, the condition, \(2M/R > 1\), does not imply the existence of a black hole.

Now suppose that a space rocket approaches a star for which \(2M/R > 1\) and goes into its Schwarzschild radius defined by \(2M\). A spaceman in the rocket would be resigned to his fate to die, but we know that he still has a chance to return alive from a false black hole.

§4. Numerical results

Equations (3·6) \sim (3·9) are numerically solved to obtain an interior solution. This solution is then matched to the exterior one, and numerical values of the parameters \(a_s, b_s\) and \(c_s\) are calculated. Some details of the numerical methods are summarized in Appendix C. Since we take \(\varphi_0 = 0\), \(G*_s\) is equal to \(G\) (Appendix A). Hereafter, we use units in which \(G*_s = G = 1\).

As for the matter, we assume the following polytropic equation of state:\(^{13}\)

\[
\rho = m_b n + \frac{K n_0 m_b}{\Gamma - 1} \left( \frac{n}{n_0} \right)^\Gamma,
\]  

\(4·1\)
Table I. We summarize our numerical results. In the 1st column, $\beta$ is given. In the 2nd column, we give a range of $n_c$ from our numerical studies. In the 3rd and 4th columns, we indicate, respectively, whether the assumption, $\rho'_{\text{eff}} \leq 0$, and the condition, $\rho - 3p \geq 0$, are satisfied. In the 5th column, the maximum mass-to-size ratio is shown for each $\beta$.

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$n_c/n_0$</th>
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<th>$\rho - 3p \geq 0$</th>
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$$p = K m_b n_0 \left( \frac{n}{n_0} \right)^\Gamma,$$  

$(4.2)$

$$m_b = 1.66 \times 10^{-24} \text{g},$$  

$(4.3)$

$$n_0 = 0.1 \text{fm}^{-3}.$$  

$(4.4)$

We take the parameters values, $\Gamma = 2.34$ and $K = 0.0195$, which fit a realistic equation of state of high density nuclear matter, and probably also that of a neutron star quite well. Our numerical solutions are therefore parametrized by $\beta$ and $n_c/n_0 = 10$. Horizontal and vertical axes denote, respectively, the radial coordinate, $r$, in the unit of 10km, and the effective density, $\rho_{\text{eff}}(r)/(m_b n_0)$. It is seen that the assumption, $\rho'_{\text{eff}}(r) \leq 0$, is satisfied.
$n_c \equiv n(0)$. It has been shown numerically that significant effects of $\varphi$ appear when $\beta \leq -4.35,6,7)$ and we are mostly interested in cases of negative values of $\beta$. In cases of positive values of $\beta$, we cannot numerically find any significantly different behavior of the solutions compared with those in general relativity, and any further discussion in these cases is no longer done.

First, we examine whether our assumption, $\rho'_{\text{eff}} \leq 0$, is satisfied. In Fig. 4, we give an example of numerical behavior of the effective density for $\beta = -5$ and $n_c/n_0 = 10$. Including this case, we find that the assumption, $\rho'_{\text{eff}}(r) \leq 0$, is mostly satisfied, as summarized in the 3rd column of Table I. Differentiating (3·12), we

Fig. 5. Each term of (4·5) in $\rho'_{\text{eff}}(r)$ is shown for $\beta = -5$ and $n_c/n_0 = 10$. Horizontal and vertical axes denote, respectively, the radial coordinate, $r$, in the unit of 10 km, and the 1st, 2nd and 3rd terms in (4·5).

Fig. 6. We compare $\phi(r)$, $\rho(r)$ and $\varphi(r)$ in the case that $\beta = -5$ and $n_c/n_0 = 7.9$. Horizontal and vertical axes denote, respectively, the radial coordinate, $r$, in the unit of 10 km, and $\rho(r)/(m_0 n_0)$, $\phi(r)$ and $\varphi(r)$. On a thin dotted line, $G \equiv 1/\phi_0 = 1$, i.e., general relativity.
obtain

$$\rho'_{\text{eff}} = A^4(\varphi)\rho' + 4A^4(\varphi)\alpha(\varphi)\varphi'\rho + \left(\frac{\varphi'^2}{8\pi G_s h_s}\right)'.$$

(4:5)

In Fig. 5, we show the 1st, 2nd and 3rd terms in (4:5) with the same parameters as Fig. 4. The first term is indeed dominant and always negative. In Fig. 6, we compare the corresponding physical quantities, $\rho(r)$ and $\phi(r)$, and the unphysical scalar field, $\varphi(r)$. It is found that the local gravitational constant, $G(\phi) \equiv 1/\phi$, increases as $\rho$ decreases toward the surface. However, this behavior is strongly dependent on the

Fig. 7. The effective density, $\rho_{\text{eff}}(r)$, is shown in the case that $\beta = -30$ and $n_c/n_0 = 10$. Horizontal and vertical axes denote, respectively, the radial coordinate, $r$, in the unit of 10 km, and the effective density, $\rho_{\text{eff}}(r)/(m_b n_0)$. It is seen that the assumption, $\rho'_{\text{eff}}(r) \leq 0$, is partially violated between two rectangles.

Fig. 8. Each term of (4:5) in $\rho'_{\text{eff}}(r)$ is shown for $\beta = -30$ and $n_c/n_0 = 10$. Horizontal and vertical axes denote, respectively, the radial coordinate, $r$, in the unit of 10 km, and the 1st, 2nd and 3rd terms in (4:5).
coupling function, and the sign of \( \beta \) is crucial in the present case.

Next, we examine an extreme example in which the assumption, \( \rho'_{\text{eff}} \leq 0 \), is violated. Figure 7 displays the effective density, \( \rho_{\text{eff}} \), for \( \beta = -30 \) and \( n_c/n_0 = 10 \). It is seen that \( \rho_{\text{eff}} \) remains constant in the central part and then increases between the two rectangles in Fig. 7. In Fig. 8, we show the three terms in (4·5) and find that the positive second term becomes partially dominant. Again, this behavior is strongly dependent on the coupling function. In the present case, we have \( \alpha(\varphi) = \beta \varphi \), where \( \beta < 0 \). Therefore, when \( \varphi' < 0 \) and \(|\beta|\) is large, such that the second term in
(4.5) is dominant, $\rho'_{\text{eff}}$ becomes positive, and the assumption is violated. In Fig. 9, we compare $\rho(r)$, $\phi(r)$ and $\varphi(r)$. It is found that, despite small values of $\varphi$, $\phi$ can be large due to a large value of $|\beta|$. However, it should be noted that this assumption is concerned with the unphysical quantity, $\rho_{\text{eff}}$, and that its violation does not necessarily mean that this extreme case is unreal. In Fig. 10, we show the energy density, $\rho(r)$, and the pressure, $p(r)$, in the physical frame in the extreme case: $\beta = -30$, $n_c/n_0 = 10$. The behavior of these quantities seems ordinary, that is, they are monotonously decreasing functions of $r$. Accordingly, one may think that this can be a physically acceptable equilibrium solution despite the violation of the assumption, $\rho'_{\text{eff}} \leq 0$. However, we are forbidden to take $\beta$ smaller than $-5$ because of experimental constraints (Appendix A7,14).

For each value of $\beta$, the mass-to-size ratio, $2M/R = H(a_s, b_s; \beta)$, can be numerically calculated as a function of $n_c$. By changing $n_c$, we search for a maximum value of $H(a_s, b_s; \beta)$ for each $\beta$. In the 5th column of Table I, we summarize our results for the maximum mass-to-size ratio, where the parameters are chosen such that the assumption, $\rho'_{\text{eff}} \leq 0$, is satisfied. For $\beta < -12.07$, we find numerically that the assumption is always violated. The first interesting example is found in the case, $\beta = -12.07$, in which the maximal mass-to-size ratio is obtained as $H_{\text{MAX}} = 1.018$ when $n_c/n_0 = 11.2$. This is a case in which $H_{\text{MAX}}$ exceeds the black hole limit, $H = 1$. Another interesting example is found in the case, $\beta = -11$, in which $H_{\text{MAX}} = 0.919$ when $n_c/n_0 = 11.3$. This is a case in which $H_{\text{MAX}}$ exceeds Buchdahl’s limit, $H = 8/9 \approx 0.889$. These examples have academic importance in the sense that our analytic results in the previous section are partially realized also in the numerical solutions.

To this point, the stability of our numerical solutions has not been taken into account. We have found that $\rho - 3p$ may be a good estimate of the stability, as described below. The baryonic mass of a star is defined as

$$m_b = m \int_0^{r_s} 4\pi n A^3(\varphi)r^2 \left(1 - \frac{2m_s}{r}\right)^{-\frac{1}{2}} dr.$$  \hspace{1em} (4.6)

We have numerically examined how $m$ depends on $n_c$ and find a significant correlation between the signature of $d\bar{m}/dn_c$ and that of $\rho - 3p$. That is, the cases, $\rho - 3p < 0$ and $\rho - 3p > 0$, approximately correspond to the cases, $d\bar{m}/dn_c < 0$ and $d\bar{m}/dn_c > 0$, respectively. Accordingly, we shall interpret the sign of $\rho - 3p$ as a measure of the onset of the instability in our numerical calculations. When we impose the condition, $\rho - 3p \geq 0$, we cannot numerically find cases in which $H_{\text{MAX}}$ exceeds Buchdahl’s limit, as is seen in the 4th and 5th columns in Table I. Note that the condition, $\beta < -5$, also excludes all the interesting cases in which $H_{\text{MAX}}$ exceeds Buchdahl’s limit.

Let us find a critical value, $\beta_c < 0$, of $\beta$, such that, for $\beta < \beta_c$, nonlinear behavior of the scalar field begins to appear. We numerically calculated $a_s$, $b_s$ and $c_s$ as functions of $n_c$ for $\beta = -4, -5$ and $-6$ under the conditions, $\rho'_{\text{eff}} \leq 0$ and $\rho - 3p \geq 0$. We show $(a_s, b_s)$ and $c_s$ in Figs. 11 and 12, respectively. For $\beta \geq -4$, almost no deviation from general relativity appears. In the cases that $\beta = -5$ and $-6$, these parameters show deviations from general relativity in which $a_s = b_s$ and $c_s = 0$. Our results are consistent with the previous works,6,7) in which $\beta_c$ is found
to be $-4.35$. Note that, even when $\beta < \beta_c$, our inequality, $|c_\phi| \leq 2\sqrt{3}/9$, is surely satisfied. This reconfirms our assertion that the nonlinear effects are always bounded in this sense.

Now we shall briefly compare our numerical results with those in previous works.\textsuperscript{6), 7) The maximum baryonic mass of a star is defined as the peak of the $\bar{m}-n_c$ relation. We numerically found that the maximum baryonic mass increases from the general relativistic value, 2.23 $M_\odot$, to 2.38 $M_\odot$ and 2.96 $M_\odot$ for $\beta = -5$ and $-6$, respectively. The corresponding radius defined by (3·43) also increases from the general relativistic value, 11.0 km, to 12.0 km and 12.9 km for $\beta = -5$ and $-6$, respectively. In Ref. 6), a fractional binding energy, $f_{BE} = 2\bar{m}/b - 1$, is used as a measure of the scalar field contribution to the mass. We numerically found that the maximum value of $f_{BE}$ increases from the general relativistic value, $f_{BE} = 0.14$, to $f_{BE} = 0.16$ and 0.22 for $\beta = -5$ and $-6$, respectively. Though a slightly different asymptotic value of $\varphi_0$ has been adopted in Ref. 6), these results are consistent with the previous results.

The mass-size relation of neutron stars has been thoroughly studied in general relativity by solving the Oppenheimer-Volkoff equation, and it has been found numerically that, as the equation of state becomes softer, the mass-to-size ratio becomes larger when its mass is fixed.\textsuperscript{13) In Fig. 13, we compare, under the condition, $\rho - 3p \geq 0$, the relations between the mass-to-size ratio and the mass in general relativity and in scalar-tensor theories. It is seen in Fig. 13 that the deviation from general relativity due to the scalar field begins appearing for $M > 1.2 M_\odot$. When $M < 1.7 M_\odot$ ($M > 1.7 M_\odot$), our numerical solutions in the scalar-tensor theory correspond to the solutions in general relativity with the softer (stiffer) equation of state. The mass of PSR1913+16 has been evaluated as 1.4 $M_\odot$,\textsuperscript{13} and, if the adopted equation of state is adequate, the scalar field contribution to the mass-to-size ratio is negligibly small when $\beta > -5$. Further discussion on the equation of state is left as a future work.

Finally, we shall derive a redshift formula in the scalar-tensor theories. A null vector, $k^\mu$, tangent to the radial null geodesic, and a four-velocity, $U^\mu$, of a static observer are, respectively, given by

$$k^\mu = A^{-2}(e^{-\gamma}, 1, 0, 0), \quad U^\mu = (A^{-1}e^{-\gamma/2}, 0, 0, 0), \quad U^\mu U_\mu = -1.$$  \hspace{1cm} (4·7)

The frequency, $\omega$, of a light ray is given by

$$\omega = -g_{\mu\nu}k^\mu U^\nu = A^{-1}e^{-\gamma/2}.$$  \hspace{1cm} (4·8)

The redshift, $z$, is then obtained as

$$1 + z = \frac{\omega_{\text{source}}}{\omega_{\text{observer}}} = A^{-1}e^{-\gamma/2}.$$  \hspace{1cm} (4·9)

where the observer is assumed to be at the spatial infinity, $a/\chi \to 0$. By using (3·22), (3·23) and (3·40), we obtain the redshift formula in the present specific scalar-
tensor theory as

$$1 + z = (1 - a_s) \frac{b_s}{2a_s} \exp \left\{ -\frac{1}{2} \beta \left[ \frac{c_s}{a_s} \ln(1 - a_s) \right]^2 \right\} = \frac{1}{b_s} (1 - a_s)^{\frac{a_s - 2b_s}{2a_s}} \left( \frac{2M}{R} \right).$$

The maximum value of $z$ depends on $\beta$ and the parameters, $a_s$ and $b_s$, in the allowed region, $D$, in Fig. 1. Theoretically, the possible maximum value of $z$, $z_{\text{max}}$, is obtained as $z_{\text{max}} = 2$ in general relativity and $z_{\text{max}} = 164$ and $356$ for $\beta = -5$ and

---

**Fig. 11.** We show the parameters, $(a_s, b_s)$, in each equilibrium solution for $n_c/n_0 = 2.5 \sim 10.3$. We take $\beta = -4, -5$ and $-6$, and impose the conditions, $\rho(r) - 3p(r) \geq 0$ and $\rho_{\text{eff}}(r) \leq 0$. Horizontal and vertical axes denote $b_s$ and $a_s$, respectively.

---

**Fig. 12.** We show the parameter, $c_s$, in each equilibrium solution for $n_c/n_0 = 1.0 \sim 10.3$. We take $\beta = -4, -5$ and $-6$, and impose the conditions, $\rho(r) - 3p(r) \geq 0$ and $\rho_{\text{eff}}(r) \leq 0$. Horizontal and vertical axes denote $n_c/n_0$ and $c_s$, respectively. A horizontal thin dotted line denotes the limit on $c_s$, i.e., $c_s = -2\sqrt{3}/9$. 
We have derived a modified Buchdahl inequality in scalar-tensor theories of gravity. As a result, we have obtained two theory-independent inequalities, \( b_s \leq 8/9 \) and \( |c_s| \leq 2\sqrt{3}/9 \). The first inequality corresponds to the Buchdahl inequality in general relativity. The second inequality is characteristic of scalar-tensor theories. Consequently, even if the scalar field is locally amplified due to non-perturbative effects in a strong gravitational field, the characteristic amplitude of the scalar field, \( |c_s| \), is bounded in this sense.

The modified Buchdahl inequality is then reformulated to obtain a theory-dependent mass-to-size ratio, \( 2M/R \), with an example of the coupling function, \( A(\varphi) \), in a simple form. If we take \( \varphi_0 = 0 \), the mass in the physical frame is the same as that in general relativity, \( M = b/2 \). However, the physical radius, \( R \), of the star can be smaller than the general relativistic one. As a result, the mass-to-size ratio can exceed not only Buchdahl’s limit but also the black hole limit in contrast to general relativity.
Our analytic results have been numerically confirmed when we assume a polytropic equation of state for the matter. In particular, we have found numerical solutions in which the mass-to-size ratio exceeds both Buchdahl’s limit and the black hole limit. However, these theoretically interesting stars could not be found numerically under the condition, $\rho - 3\rho \geq 0$, which is interpreted as a numerical measure of the onset of the instability of a star. Moreover, under this condition, we find numerically that any quantitative deviation from general relativity due to the scalar field remains comparatively small in contrast to our analytic results, where possible significant effects of the scalar field are expected. However, as discussed briefly, some measurable effects in astronomical observations may exist.

Now suppose that a space rocket approaches a massive star for which $2M/R \gg 1$. If the rocket accidentally goes into a Schwarzschild radius of the star defined by $2M$, a spaceman in the rocket would be resigned to his fate to die. Now we know that, unfortunately for him, even if scalar-tensor theories describe classical gravity, he would hardly have a chance to return alive, because he could hardly meet a real false black hole. He has two possible futures, and they are equally tragic:

- If it is a black hole in terms of general relativity, he can never escape.
- If it is a naked singularity in terms of scalar-tensor theories, nobody knows what will happen when he touches it.

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Appendix A

--- Observational Constraints ---

In general, the coupling strength, $\alpha(\varphi)$, can be an arbitrary function of $\varphi$, and in the limit, $\alpha(\varphi) \to 0$, scalar-tensor theories approach general relativity. One defines

$$\alpha_0 \equiv \alpha(\varphi_0), \quad (A\cdot1)$$

$$\beta_0 \equiv \frac{d\alpha(\varphi)}{d\varphi} \bigg|_{\varphi=\varphi_0}, \quad (A\cdot2)$$

where $\varphi_0$ is the asymptotic value of $\varphi$ at spatial infinity. In the post-Newtonian approximation, the PPN parameters and the effective gravitational constant are expressed as follows: \cite{11}

$$1 - \gamma_E = \frac{2\alpha_0^2}{1 + \alpha_0^2}, \quad (A\cdot3)$$

$$\beta_E - 1 = \frac{\beta_0\alpha_0^2}{2(1 + \alpha_0^2)^2}, \quad (A\cdot4)$$

$$G = G_0 A^2(\varphi_0) \left(1 + \alpha^2(\varphi_0)\right). \quad (A\cdot5)$$
General relativity corresponds to the case that $\beta = \gamma = 1$.\textsuperscript{4,15} Experiments on the time delay and deflection of light in the solar system constrain $|1 - \gamma|$ as\textsuperscript{14}

$$|1 - \gamma| < 2 \times 10^{-3},$$

which constrains $\omega(\phi)$ and $\alpha_0$ as

$$\omega > 500, \quad \alpha_0^2 < 10^{-3}.$$ \hspace{1cm} (A·7)

The lunar-laser-ranging experiments constrain $|\beta - 1|$ as\textsuperscript{14}

$$|\beta - 1| \lesssim 6 \times 10^{-4},$$

which only constrains some combination of $\alpha_0$ and $\beta_0$. Consequently, if $\alpha_0$ tends to zero, the constraint on $\beta_0$ is effectively lost. However, by adopting a specific coupling function, $A = \exp\left(\frac{1}{2} \beta \phi^2\right)$, another constraint on $\beta_0$ is obtained from observations of the binary-pulsars, PSR1913+16, as\textsuperscript{7,14}

$$\beta_0 > -5.$$ \hspace{1cm} (A·9)

When we take (3·40) as a coupling function, the coupling strength is $\alpha(\varphi) = \beta \varphi$. Accordingly, $\alpha_0 = \beta \varphi_0$, and we obtain the constraint on $\varphi_0$. In this paper we take $\varphi_0 = 0$ for simplicity.

Appendix B

--- An Exterior Solution ---

A line element of the Einstein frame in the Just coordinate is\textsuperscript{16}

$$ds_* = -e^{\gamma(x)} dt^2 + e^{-\gamma(x)} d\chi^2 + e^{\lambda(x) - \gamma(x)} d\Omega^2.$$ \hspace{1cm} (B·1)

The field equations in the exterior space-time are

$$\gamma'' + \gamma' \lambda' = 0,$$ \hspace{1cm} (B·2)

$$-\gamma'' + \gamma' \lambda' - \chi'' + \gamma'' - 2 \lambda'' = 4 \varphi'^2,$$ \hspace{1cm} (B·3)

$$2 + e^\lambda (\gamma' \lambda' - \chi'' + \gamma'' - \lambda'') = 0,$$ \hspace{1cm} (B·4)

$$\varphi'' + \lambda' \varphi' = 0,$$ \hspace{1cm} (B·5)

where a prime denotes differentiation with respect to $\chi$. With (B·2), (B·4) and (B·5), the exterior solution can be obtained as

$$e^\lambda(x) = \chi^2 - a \chi,$$ \hspace{1cm} (B·6)

$$e^{\gamma(x)} = \left(1 - \frac{a}{\chi}\right)^{\frac{b}{a}},$$ \hspace{1cm} (B·7)

$$\varphi(\chi) = \varphi_0 + \frac{c}{a} \ln \left(1 - \frac{a}{\chi}\right),$$ \hspace{1cm} (B·8)
where $a$, $b$ and $c$ are constants of integration, and $\varphi_0$ denotes the asymptotic value of $\varphi$ at infinity. With (B·3), one finds

$$a^2 - b^2 = 4c^2.$$  \hfill (B·9)

The coordinate transformation between the Schwarzschild coordinate, $r$, and the Just coordinate, $\chi$, is given by

$$r^2 = \chi^2 \left( 1 - \frac{a}{\chi} \right)^{a-b}.$$  \hfill (B·10)

Note that $r \to \chi$ at spatial infinity. In the Schwarzschild coordinate, a line element becomes

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\mu(r)}dr^2 + r^2d\Omega^2.$$  \hfill (B·11)

The exterior solution in the Schwarzschild coordinate is given by

$$e^{2\nu(r)} = \left( 1 - \frac{a}{\chi(r)} \right)^{\frac{b}{a}},$$  \hfill (B·12)
$$e^{2\mu(r)} = \left( 1 - \frac{a}{\chi(r)} \right)^{-2} \left( 1 - \frac{a+b}{2\chi(r)} \right)^{-2}.$$  \hfill (B·13)

Asymptotic behavior of the exterior solution at spatial infinity is as follows:

$$e^{2\nu(r)} \to 1 - \frac{b}{r},$$  \hfill (B·14)
$$e^{2\mu(r)} \to 1 + \frac{b}{r},$$  \hfill (B·15)
$$\varphi(r) \to \varphi_0 - \frac{c}{r}.$$  \hfill (B·16)

**Appendix C**

--- An Interior Solution: Numerical Methods ---

Using the variables, $m_*(r)$ and $\nu(r)$, defined by

$$f_*(r) \equiv e^{2\nu(r)}, \quad h_*(r) \equiv \left[ 1 - \frac{2m_*(r)}{r} \right]^{-1},$$  \hfill (C·1)

the field equations (3·6) $\sim$ (3·9) become

$$\frac{dm_*}{dr} = 4\pi G_* A^4(\varphi) r^2 \rho + \frac{1}{2} r (r - 2m_*) \psi^2,$$  \hfill (C·2)
$$\frac{d\nu}{dr} = \frac{m_* + 4\pi G_* A^4(\varphi) r^3 \rho}{r(r - 2m_*)} + \frac{1}{2} r \psi^2 \equiv \Phi(r),$$  \hfill (C·3)
$$\frac{d\varphi}{dr} = \psi,$$  \hfill (C·4)
$$\frac{d\psi}{dr} = \frac{4\pi G_* A^4(\varphi) r}{r - 2m_*} [\alpha(\varphi)(\rho - 3p) + (\rho - p)r\psi] - \frac{2(r - m_*)}{r(r - 2m_*)} \psi,$$  \hfill (C·5)
$$\frac{dp}{dr} = -(\rho + p)(\Phi + \alpha(\varphi)\psi).$$  \hfill (C·6)
The total baryon mass measured in the physical frame is

\[
\bar{m} = m_b \int n\sqrt{-g}u^0 d^3x = m_b \int_0^{r_s} 4\pi n A^3(\varphi) r^2 \left(1 - \frac{2m_*}{r}\right)^{-\frac{1}{2}} dr. \quad (C.7)
\]

Given the equation of state, we can numerically integrate the above field equations outward from the center, \( r = 0 \), with the boundary conditions as follows:

\[
\begin{align*}
m_*(0) &= 0, \\
\varphi(0) &= \varphi_c, \\
\psi(0) &= 0, \\
p(0) &= p_c, \\
\rho(0) &= \rho_c,
\end{align*}
\]

where \( \rho_c \) and \( p_c \) are given by replacing \( n \) in (4.1) and (4.2) with \( n_c = n(0) \). The surface of a star, \( r = r_s \), is determined by the condition, \( p(r_s) = 0 \). A numerically obtained interior solution is to be matched to the exterior one by the conditions:

\[
\begin{align*}
\varphi_0 &= \varphi_s + \frac{\psi_s}{\sqrt{\nu_s^2 + \psi_s^2}} \tanh^{-1} \left( \frac{\nu_s' + \psi_s^2}{\nu_s' + 1/r_s} \right), \\
b &= 2r_s^2 \sqrt{1 - 2m_*} \exp \left(-\frac{\nu_s'}{\sqrt{\nu_s^2 + \psi_s^2}} \tanh^{-1} \left( \frac{\nu_s^2 + \psi_s^2}{\nu_s' + 1/r_s} \right) \right), \\
c &= \frac{\psi_s}{2\nu_s'} b, \\
a &= \sqrt{b^2 + 4c^2},
\end{align*}
\]

where a prime denotes differentiation with respect to \( r \), and the subscript, \( s \), refers to quantities evaluated at the surface, \( r_s \). The central value of \( \varphi, \varphi_c \), is chosen such that we have \( \varphi_0 = 0 \).

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