Renormalization Group Flow near the Superconformal Points in $N = 2$ Supersymmetric Gauge Theories

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The behavior of the beta-function of the low-energy effective coupling is determined near the superconformal points in moduli space for two cases, the $N = 2$ supersymmetric $SU(2)$ QCD with several massive matter hypermultiplets and the $SU(3)$ Yang-Mills theory. The renormalization group flow is unambiguously fixed by looking at limited types of deformation near the superconformal points. It is pointed out that the scaling dimension of the beta-function is controlled by the scaling behavior of moduli parameters, and the relation between them is explicitly worked out. Our scaling dimensions for the beta-functions are consistent in part with the results obtained recently by Bilal and Ferrari using a different method for the $SU(2)$ QCD.

§1. Introduction

Among many developments that have taken place after the work of Seiberg and Witten\(^1\) on the $N = 2$ supersymmetric Yang-Mills and QCD theories (see Ref. 2) for reviews), one of the most intriguing is concerned with the existence of superconformal symmetry. This symmetry is expected to be realized at some particular points in moduli space (superconformal points) where massless particles with mutually non-local charge exist. This has been observed in the $SU(3)$ gauge group without matter hypermultiplets,\(^3\) and in $SU(2)$ with massive hypermultiplets,\(^4\) and variations thereof. There is a great deal of strong evidence for the existence of such interacting non-trivial superconformal field theories, in particular on the basis of the representation theory of the superconformal algebra. The classification of the superconformal points in the general $SU(N_c)$ gauge group has also been carried out.\(^5\)

With conventional field theory techniques, the most straightforward way of examining the conformal invariance is to study the behavior of the beta-function or the renormalization group flow. Although there have been several attempts\(^6\)-\(^8\) to derive the beta-function in Seiberg-Witten theories, the idea of renormalization group flow is yet to be investigated in more detail. In the present paper, we inspect the behavior of the beta-function near the superconformal points by fitting the renormalization group idea à la Wilson to the Seiberg-Witten theories.

In the Wilson-type renormalization group, we start with an ultra-violet cutoff scale, say $M_0$, and with another intermediate energy scale $M$. We then integrate out all dynamical degrees of freedom in the momentum space between $M_0$ and $M$.

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The effective Lagrangian contains only a few relevant interaction terms. In the case of Seiberg-Witten theory with the $SU(2)$ gauge symmetry broken to $U(1)$, the low-energy effective theory is described by the $N = 2$, $U(1)$ gauge field multiplet $(W_{\alpha}, A)$:

$$\mathcal{L} = \text{Im} \frac{1}{4\pi} \left[ \int d^4 \theta \frac{\partial \mathcal{F}(A)}{\partial A} A^{\dagger} + \frac{1}{2} \int d^2 \theta \tau(A) W_{\alpha} W^{\alpha} \right]. \quad (1.1)$$

The function $\tau(A)$ is expressed as the second derivative of the prepotential, $\tau(A) = \partial^2 \mathcal{F}(A)/\partial A^2$. The bosonic part of $\tau(A)$ is related to the gauge coupling $g^2$ and the vacuum angle $\theta$ via

$$\tau(a) = \frac{8\pi}{g^2} + \frac{\theta}{\pi}, \quad (1.2)$$

where $a$ is the vacuum expectation value of $\phi$, $\phi$ being the scalar component of the gauge multiplet. If we include $N_f$ matter hypermultiplets with bare masses $m_i$, $(i = 1, \cdots, N_f)$, $\tau$ becomes a function of $u/A^2$ and $m_i/A$. Here $\Lambda$ is the QCD dynamical mass scale and $u$ is the expectation value $\text{Tr}(\phi^2)$. Seiberg and Witten have determined $\tau(u/A^2, m_i/A)$ completely on the basis of elliptic curves.

We are now interested in the linear response under the change of the energy scale $M$. The moduli parameters $u$ and $m_i$ are in principle dependent on this scale $M$ on the non-perturbative level. The low-energy Seiberg-Witten action does not seem to be telling us anything about the procedure of integrating out the dynamical degrees of freedom between $M_0$ and $M$. Near the superconformal points, however, we can still think of consistent renormalization group flows and discuss various critical exponents. Argyres et al.\textsuperscript{4)} in fact have considered consistent deformations near the superconformal points in the $SU(2)$ QCD case and have determined the scaling dimensions of $u$ and $m_i$. In recent interesting papers,\textsuperscript{9)} Bilal and Ferrari pointed out that the behavior of the beta-function near the superconformal points is constrained by the eigenvalues of monodromy matrices. They found a relation between the scaling dimensions of the beta-function and $u$. In this paper we study the behavior of the beta-function near the superconformal points using a more direct method. Our approach is along the line of Argyres et al.\textsuperscript{4)} and is more straightforward than that of Bilal and Ferrari.\textsuperscript{9)}

In some of the literature we sometimes find a conjecture that the exact relation found in Ref. 10) between the moduli parameter $u$ and the derivative of the prepotential $\mathcal{F}$ with respect to the QCD dynamical scale $\Lambda$, i.e. $\Lambda(\partial \mathcal{F}/\partial \Lambda)$, might have some relevance to the renormalization group idea. In fact in the weak coupling region, the $\Lambda$-derivative of the coupling $\tau(u/A^2, m_i/A)$ coincides with the coefficient of the beta-function. As we will see shortly, however, in the strong coupling region, there is no guarantee that the flow under the change of the energy scale $M$ and the $\Lambda$-derivative of $\tau(u/A^2, m_i/A)$ or $\mathcal{F}$ are directly connected.
§2. Renormalization group flow

Let us begin with the elliptic curves in the $SU(2)$ gauge group with several matter hypermultiplets ($N_f = 1, 2, 3$). They are given the structures, \(^1\)

\[ y^2 = x^2(x - u) + P_{N_f}(x, u, m_i, \Lambda), \] (2.1)

where

\[ P_1 = \frac{1}{4} \Lambda^2 m_1 x - \frac{1}{64} \Lambda^6, \] (2.2)

\[ P_2 = -\frac{1}{64} \Lambda^4 (x - u) + \frac{1}{4} \Lambda^2 m_1 m_2 x - \frac{1}{64} \Lambda^4 (m_1^2 + m_2^2), \] (2.3)

\[ P_3 = -\frac{1}{64} \Lambda^2 (x - u)^2 - \frac{1}{64} \Lambda^2 (x - u)(m_1^2 + m_2^2 + m_3^2) \]
\[ + \frac{1}{4} \Lambda m_1 m_2 m_3 x - \frac{1}{64} \Lambda^2 (m_1^2 m_2^2 + m_1^2 m_3^2 + m_2^2 m_3^2). \] (2.4)

We will use the same notation $\Lambda$ for the dynamical QCD scale for each case of different flavor numbers $N_f$, assuming that there will be no confusion.

Now we can put these curves into the Weierstrass form by changing the variables as $y = Y/2$ and $x = X + \text{const}$, i.e.,

\[ Y^2 = 4(X - e_1)(X - e_2)(X - e_3), \] (2.5)

where $e_1 + e_2 + e_3 = 0$. According to Seiberg-Witten theory, the coupling in (1.2) is given a very simple expression in terms of an integral over the two homology cycles:

\[ \frac{\partial \alpha(u, m_i, \Lambda)}{\partial u} = \sqrt{\frac{2}{4\pi}} \oint_{\gamma_1} \frac{dX}{Y} = \frac{\sqrt{2}}{2\pi} \frac{1}{\sqrt{e_1 - e_3}} K(k), \] (2.6)

\[ \frac{\partial \alpha_D(u, m_i, \Lambda)}{\partial u} = \sqrt{\frac{2}{4\pi}} \oint_{\gamma_2} \frac{dX}{Y} = i \frac{\sqrt{2}}{2\pi} \frac{1}{\sqrt{e_1 - e_3}} K(k'). \] (2.7)

Here the homology cycle $\gamma_1$ ($\gamma_2$) is defined as that encircling the two points $e_2$ and $e_3$ ($e_1$). The function $K(k)$ is the complete elliptic integral of the first kind, \(^{11}\)

\[ K(k) = \int_0^1 d\xi \frac{1}{\sqrt{(1 - \xi^2)(1 - k^2 \xi^2)}}, \] (2.8)

where we have introduced

\[ k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \] (2.9)

and $k'^2 = 1 - k^2$. In fact the coupling in Eq. (1.2) is a simple function of a single variable $k^2$:

\[ \tau \left( \frac{u}{\Lambda^2}, \frac{m_i}{\Lambda} \right) = \frac{\partial \alpha_D(u, m_i, \Lambda)}{\partial u} / \left( \frac{\partial \alpha(u, m_i, \Lambda)}{\partial u} \right) = i \frac{K(k')}{K(k)}. \] (2.10)
The superconformal points are expected to exist where the cycles shrink simultaneously. They are given by setting

\[ e_1 = e_2 = e_3 = -(e_1 + e_2) = 0. \tag{2.11} \]

We will denote these points generically by \((u^*/A^2, m_i^*/A)\). In order to argue that the points \((u^*/A^2, m_i^*/A)\) are actually the superconformal points, an intuitive argument usually goes in the following way. Suppose that we change the renormalization scale \(M\) continuously and that \((u/A^2, m_i/A)\) moves towards one of these points. If \(u/A^2\) and \(m_i/A\) move in such a way that \(k^2\) also goes towards some definite number, then Eq. (2.10) shows that the coupling \(\tau(u/A^2, m_i/A)\) approaches some definite number independent of any mass scale. This means that the coupling ceases to move there, and the theory becomes scale-invariant.

This argument may be phrased in the following manner. Let us define our "beta-function"

\[
\beta \left( \frac{u}{A^2}, \frac{m_i}{A} \right) = M \frac{d}{dM} \tau \left( \frac{u}{A^2}, \frac{m_i}{A} \right) = \left( \gamma_u \frac{\partial}{\partial u} + \sum_i \gamma_{m_i} \frac{\partial}{\partial m_i} \right) \tau \left( \frac{u}{A^2}, \frac{m_i}{A} \right). \tag{2.12} \]

Here we have introduced two basic quantities,

\[
\gamma_u \frac{1}{A^2} = M \frac{\partial}{\partial M} \left( \frac{u}{A^2} \right), \tag{2.13} \]

\[
\gamma_{m_i} \frac{1}{A} = M \frac{\partial}{\partial M} \left( \frac{m_i}{A} \right). \tag{2.14} \]

These are analogous to anomalous dimensions of \(u/A^2\) and \(m_i/A\), respectively.

We are interested in the behavior of the beta-function near the superconformal points, and we introduce a critical exponent of the beta-function or \(\tau(u/A^2, m_i/A)\) in the infrared limit

\[
\beta \left( \frac{u}{A^2}, \frac{m_i}{A} \right) \sim \text{const} \ (M_0/M)^{-\rho}, \tag{2.15} \]

or

\[
\rho = \lim_{M \to 0} \frac{\partial \log(\tau - \tau^*)}{\partial \log M}, \tag{2.16} \]

where we have denoted \(\tau(u^*/A^2, m_i^*/A)\) by \(\tau^*\). Since \(\tau(u/A^2, m_i/A)\) is a function of \(k^2\) alone, Eq. (2.16) may be written

\[
\rho = \lim_{M \to 0} \frac{\partial \log(k^2 - k^{*2})}{\partial \log M}, \tag{2.17} \]

where \(k^{*2}\) is the value of \(k^2\) at the superconformal point. We now study the exponent \(\rho\) in the \(SU(2)\) QCD with \(N_f = 1, 2, \) and \(3\) flavors and also in the \(SU(3)\) pure Yang-Mills theory.
§3. The SU(2), \(N_f = 1\) case

In the simplest \(N_f = 1\) case, the solutions to \(Y^2 = 0\) for the elliptic curve given by (2·1), (2·2) and (2·5) turn out to be

\[
e_1 = \frac{1}{3} \left( f_+^{1/3} + f_-^{1/3} \right),
\]
\[
e_2 = \frac{1}{3} \left( \omega f_+^{1/3} + \omega^2 f_-^{1/3} \right),
\]
\[
e_3 = \frac{1}{3} \left( \omega^2 f_+^{1/3} + \omega f_-^{1/3} \right).
\] (3·1) (3·2) (3·3)

Here \(\omega = \exp(2\pi i/3)\), and we have introduced \(f_\pm\) defined by

\[
f_\pm = u^3 - \frac{9}{8} m A^3 u + \frac{27}{128} A^6 \pm \frac{\sqrt{27}}{16} A^3 \sqrt{4 u^3 - 4 m^2 u^2 - \frac{9}{2} m A^3 u + \frac{27}{64} A^6 + 4 m^3 A^3}.
\] (3·4)

Superconformal invariance is expected to exist at the points where mutually non-local charges are present. They are obtained in this case by imposing (2·11), or equivalently by taking the limit \(f_\pm \to 0\). More explicitly, they turn out to be

\[
\left(\frac{u^*}{\lambda^2}, \frac{m^*}{\lambda}\right) = \left(\frac{3}{4}, \frac{3}{4}\right), \quad \left(\frac{3}{4} \omega, \frac{3}{4} \omega^2\right), \quad \left(\frac{3}{4} \omega^2, \frac{3}{4} \omega\right).
\] (3·5)

As a matter of fact, the function \(\tau(u/\lambda^2, m/\lambda)\) is ambiguous in general when \((u/\lambda^2, m/\lambda)\) is going to one of these points. Equation (2·9) shows that \(k^2\) is in fact \(0/0\). The renormalization group flow should be such that the motion of \((u/\lambda^2, m/\lambda)\) under the change of \(M\) conspires to give some definite numbers to \(k^2\) and \(\tau(u/\lambda^2, m/\lambda)\). While the motion of \((u/\lambda^2, m/\lambda)\) is dictated by (2·13) and (2·14), their explicit forms are not known to us. The information of the two basic quantities, (2·13) and (2·14), all comes from the high energy content of the theory. This is also the case with the limits of \(k^2\) and \(\tau(u/\lambda^2, m/\lambda)\). Their derivation starting from first principles is of course an important problem.

In the following, however, we get around this problem by making use of the method of Argyres et al. 4) What they did amounts to the following. Let us parameterize the deformation from the superconformal points by setting

\[
\frac{m - m^*}{\lambda} = D_1 t^{-\alpha},
\]
\[
\frac{u - u^*}{\lambda^2} = D'_1 t^{-\alpha} + D_2 t^{-\beta}.
\] (3·6) (3·7)

Here the two numbers \(\alpha\) and \(\beta\) will be called scaling dimensions of \(m\) and \(u\), respectively. The flow \((u/\lambda^2, m/\lambda)\) is parameterized by “time” \(t\), and the limit \(t \to +\infty\).
drives the point \((u/A^2, m/A)\) towards one of the superconformal points. Argyres et al. have shown that one can adjust \(k^2\) to any value, say \(k^*\), by fine-tuning if and only if we choose \(\beta = 3\alpha/2\) and \(D_1 = D'_1\).

Let us now go one step further to identify the "time" variable with \(M_0/M\). The large time limit corresponds to the infrared limit \(t = M_0/M \to +\infty\). We mean by this that the flow of the moduli point \((u/A^2, m/A)\) under the change of \(t\) is the only possibility for the renormalization group flow. This identification immediately enables us to derive the formulae for (2.13) and (2.14),

\[
\frac{\gamma_m}{A} = \alpha \left( \frac{m - m^*}{A} \right),
\]

\[
\frac{\gamma_u}{A^2} = \frac{3}{2} \alpha \left( \frac{u - u^*}{A^2} \right) - \frac{1}{2} \alpha \left( \frac{m - m^*}{A} \right).
\]

The beta-function can thus be evaluated by looking for various possible deformations around superconformal points.

With the ratio \(\beta = 3\alpha/2\), Eq. (2·9) turns out to be the following form near the superconformal points:

\[
k^2 - k^* = \text{const } t^{-\alpha/2}.
\]

Thus we can immediately conclude that the critical exponent \(\rho\) becomes

\[
\rho = \frac{1}{2} \alpha.
\]

If we normalize the critical exponent of \(a(u, m, A)\) to be unity, the exponent \(\alpha\) should be set equal to 4/5, and thus we have \(\rho = 2/5\). This result agrees with the corresponding quantity obtained by Bilal and Ferrari. 9)

\[\text{§4. The SU(2), } N_f = 2 \text{ case}\]

The calculational steps discussed in the \(N_f = 1\) case apply in the \(N_f = 2\) case as well with little modification. If we set the masses of hypermultiplets equal \((m_1 = m_2)\), we can expect higher criticality. In the following, we keep our argument as general as possible, and we introduce

\[
m = \frac{1}{2} (m_1 + m_2), \quad C_2 = \frac{1}{2} (m_1 - m_2)^2
\]

instead of \(m_1\) and \(m_2\).

We now have three parameters, \(u, m\) and \(C_2\), while the condition for the superconformal points, Eq. (2·11), gives only two constraints. This indicates that superconformal invariance is expected to exist along a line of complex-dimension one in the moduli space. In order to scrutinize the one-dimensional line, let us rewrite the elliptic curve in the form \(Y^2 = 4(X^3 + fX + g)\), where

\[
f = -\frac{1}{3} \left\{ u^2 - \left( \frac{3A^2}{8} \right)^2 \right\} + \frac{A^2}{4} \left\{ m^2 - \left( \frac{A}{2} \right)^2 \right\} - \frac{A^2}{8} C_2,
\]
\[ g = -\frac{2}{27} \left\{ u^2 - \left( \frac{3A^2}{8} \right)^2 \right\} u + \frac{1}{12} \left( u - \frac{3A^2}{8} \right) m^2 A^2 \]
\[ - \left( \frac{1}{24} A^2 u + \frac{1}{64} A^4 \right) C_2. \]

(4.3)

The point \((u^*/A^2, m^*/A, C^*_2/A^2)\) on the line corresponds to the solution of the equations \(f = 0\) and \(g = 0\). The analysis of the scaling dimensions goes exactly in the same way as in the \(N_f = 1\) case for generic values of \((u^*/A^2, m^*/A, C^*_2/A^2)\) on the line. Something peculiar happens, however, if we impose the further condition \(C^*_2 = 0\), namely, at the points

\[ \left( \frac{u^*}{A^2}, \frac{m^*}{A}, \frac{C^*_2}{A^2} \right) = \left( \frac{3}{8}, \frac{1}{2}, 0 \right), \quad \left( \frac{3}{8}, \frac{1}{2}, 0 \right). \]

(4.4)

Here the two masses, \(m_1\) and \(m_2\), are set equal, and the first derivatives of the function \(g\) with respect to \(m\) and \(u\) both vanish. The scaling behavior at this particular point differs from other points, and we would like to concentrate our investigation on the point \((u^*/A^2, m^*/A, C^*_2/A^2) = (3/8, 1/2, 0)\) hereafter.

We again use the limiting procedure of Argyres et al. which is in this case put in the form

\[ \frac{m - m^*}{A} = D_3 t^{-\alpha}, \]
\[ \frac{u - u^*}{A^2} = D'_3 t^{-\alpha} + D_4 t^{-\beta}. \]

(4.5) \hspace{1cm} (4.6)

We should also consider the deformation \(C_2 - C^*_2\), whose contributions are, however, always sub-leading and need not be considered in the following discussion.

We require again that \(k^2\) can be sent to a generic value \(k^2\) in the limit \(t \to \infty\). This is possible if and only if \(\beta = 2\alpha\) and \(D_3 = D'_3\). As before, the identification \(t = M_0/M\) immediately gives us expressions for \(\gamma_u\) and \(\gamma_m:\)

\[ \gamma_m = \alpha \left( \frac{m - m^*}{A} \right), \]
\[ \frac{\gamma_u}{A^2} = 2\alpha \left( \frac{u - u^*}{A^2} \right) - \alpha \left( \frac{m - m^*}{A} \right). \]

(4.7) \hspace{1cm} (4.8)

Putting (4.5) and (4.6) into Eq. (2.9), the infrared behavior of \(k^2\) is approximated as

\[ k^2 = \left( \frac{1}{2} - \frac{D_3}{\sqrt{D_4}} \right) + \frac{D^2_3 - D_4}{2\sqrt{D_4}} t^{-\alpha} + \ldots. \]

(4.9)

This gives us the relation for scaling dimensions,

\[ \rho = \alpha. \]

(4.10)

The normalization of the scaling dimension of \(\alpha(u, m_i, A)\) being equal to unity leads us to the relation \(\rho = 2/3\), which agrees again with the results obtained by Bilal and Ferrari. 9)
§5. The SU(2), $N_f = 3$ case

Let us turn to the $N_f = 3$ case. We consider the combination of the bare masses of matter hypermultiplets,

$$m = \frac{1}{3} \sum_{i=1}^{3} m_i, \quad C_2 = \sum_{i=1}^{3} (m_i - m)^2, \quad C_3 = \sum_{i=1}^{3} (m_i - m)^3, \quad (5.1)$$

instead of $m_1$, $m_2$ and $m_3$. In terms of these, the elliptic curve given by (2·4) and (2·5) is put into the form $Y^2 = 4(X^3 + fX + g)$, where

$$\tilde{f} = -\frac{1}{3} \left\{ u - 2 \left( \frac{A}{8} \right)^2 \right\}^2 + \left( \frac{A}{8} \right) \left( m - \frac{A}{8} \right)^2 \left( 2m + \frac{A}{8} \right),$$

$$-C_2 \left\{ m \left( \frac{A}{8} \right) + \left( \frac{A}{8} \right)^2 \right\} + \frac{2}{3} \left( \frac{A}{8} \right) C_3, \quad (5.2)$$

$$\tilde{g} = -\frac{2}{27} \left\{ u - 2 \left( \frac{A}{8} \right)^2 \right\}^2 \left\{ u + \frac{11}{8} \left( \frac{A}{8} \right)^2 \right\} - 3 \left( \frac{A}{8} \right)^2 \left( m^2 - \frac{u}{2} \right)^2$$

$$+ \frac{1}{3} \left\{ u + \left( \frac{A}{8} \right)^2 \right\} \left\{ 2 \left( \frac{A}{8} \right) \left( m - \frac{A}{8} \right)^2 \left( m + \frac{1}{2} \left( \frac{A}{8} \right) \right) \right\}$$

$$+ \frac{2}{3} \left( \frac{A}{8} \right) C_3 - C_2 \left\{ m \left( \frac{A}{8} \right) + \left( \frac{A}{8} \right)^2 \right\}$$

$$+ \left( \frac{A}{8} \right)^2 u C_2 + \left( \frac{A}{8} \right)^2 \left( 2m C_3 - \frac{C_2^2}{4} \right). \quad (5.3)$$

The condition (2·11) for superconformal invariance to exist is given by the solutions to the equations $\tilde{f} = 0$ and $\tilde{g} = 0$. These equations define a surface of complex-dimension two in the moduli space. The analysis of the scaling dimensions proceeds in a similar way at a generic point on the surface, as was done in the $N_f = 1$ and $N_f = 2$ cases. As we realize rather easily from the expressions (5·2) and (5·3), however, a different type of scaling is expected at

$$\left( \frac{u^*}{A^2}, \frac{m^*}{A}, \frac{C_2^*}{A^2}, \frac{C_3^*}{A^3} \right) = \left( \frac{1}{32}, \frac{1}{8}, 0, 0 \right). \quad (5.4)$$

Actually, the first derivatives of $\tilde{f}$ and $\tilde{g}$ with respect to $u$ and $m$ all vanish, and the critical behavior becomes different.

Let us restrict ourselves to this particular point (5·4), and we again parameterize the deformation around (5·4):

$$\frac{m - m^*}{A} = D_5 t^{-\alpha}, \quad (5.5)$$

$$\frac{u - u^*}{A^2} = D'_5 t^{-\alpha} + D''_5 t^{-\beta} + D_6 t^{-\gamma}. \quad (5.6)$$
The deviations $C_2 - C_2^*$ and $C_3 - C_3^*$ should also be given due consideration, but they turn out to be irrelevant to our following consideration.

Suppose that we would like to be able to tune the value $k^2$ to a generic number $k^{*2}$. This is possible if and only if $D_5' = 3D_5/8$, $D_6' = D_6^*$, $\beta = 2\alpha$, and $\gamma = 3\alpha$. For later reference, we write down the functional form of $\gamma_u$ and $\gamma_m$ in this case,

$$\frac{\gamma_m}{\Lambda} = \alpha \left( \frac{m - m^*}{\Lambda} \right),$$

$$\frac{\gamma_u}{\Lambda^2} = 3\alpha \left( \frac{u - u^*}{\Lambda^2} \right) - \frac{3}{4} \alpha \left( \frac{m - m^*}{\Lambda} \right) - \alpha \left( \frac{m - m^*}{\Lambda} \right)^2.$$

As before, we put (5·5) and (5·6) into Eqs. (3·1) ~ (3·4) and we find the leading behavior of $k^2$ near its value $k^{*2}$ at the superconformal points,

$$k^2 - k^{*2} = \text{const.} t^{-\alpha}.$$  

This gives us the relation

$$\rho = \alpha.$$  

With the scaling dimension of $a(u, m_i, \Lambda)$ being set to unity ($2\alpha = 1$), we obtain $\rho = 1/2$. This, however, differs from the corresponding result of Bilal and Ferrari by a factor of 2. The origin of the discrepancy could be explained by the difference in the manner of dealing with the scaling behavior of the bare masses; in the case of Bilal and Ferrari the bare masses are set to the fixed value $m^*$, while we are considering the deviation $(m - m^*)/\Lambda$ as parameterized in (5·5). The scaling dimension of the beta-function for the case of $N_f = 3$ is affected by the difference in this procedure, although the cases of $N_f = 1$ and 2 are not.

§6. The $SU(3)$ Yang-Mills theory

Finally let us come to the case of the $SU(3)$ gauge group broken to $U(1) \times U(1)$ in the $N = 2$ supersymmetric pure Yang-Mills theory. In this case we begin with the hyper-elliptic curve

$$y^2 = (x^3 - ux - v)^2 - \Lambda^6 = \prod_{i=1}^{6} (x - e_i),$$

with the two moduli parameters, $u$ and $v$. The branch points $e_i$ are given by

$$e_1 = f_{1+} + f_{1-}, \quad e_4 = f_{2+} + f_{2-},$$  
$$e_2 = \omega f_{1+} + \omega^2 f_{1-}, \quad e_5 = \omega f_{2+} + \omega^2 f_{2-},$$  
$$e_3 = \omega^2 f_{1+} + \omega f_{1-}, \quad e_6 = \omega^2 f_{2+} + \omega f_{2-}.$$
where \( \omega = \exp(2\pi i/3) \), and

\[
\begin{align*}
 f_{1\pm} &= 2^{-1/3} \left\{ (v - \Lambda^3) \pm \sqrt{(v - \Lambda^3)^2 - \frac{4}{27}u^3} \right\}^{1/3}, \\
 f_{2\pm} &= 2^{-1/3} \left\{ (v + \Lambda^3) \pm \sqrt{(v + \Lambda^3)^2 - \frac{4}{27}u^3} \right\}^{1/3}.
\end{align*}
\]

The low-energy coupling \( \tau^{ij} \) is given in terms of 2 \( \times \) 2 matrices as

\[
\tau^{ij} = B \cdot A^{-1},
\]

where

\[
A = \begin{pmatrix}
\partial_u a_1 & \partial_v a_1 \\
\partial_u a_2 & \partial_v a_2
\end{pmatrix}, \quad
B = \begin{pmatrix}
\partial_u a_D^1 & \partial_v a_D^1 \\
\partial_u a_D^2 & \partial_v a_D^2
\end{pmatrix}.
\]

We can obtain the explicit form of each matrix element of \( A \) and \( B \) from the exact solution. This yields

\[
\begin{align*}
\frac{\partial a_i}{\partial u} &= \frac{\sqrt{2}}{4\pi} \int_{\alpha_i} \frac{dx}{y}, \quad \frac{\partial a_i}{\partial v} = \frac{\sqrt{2}}{4\pi} \int_{\alpha_i} \frac{dx}{y}, \\
\frac{\partial a_D^i}{\partial u} &= \frac{\sqrt{2}}{4\pi} \int_{\beta_i} \frac{dx}{y}, \quad \frac{\partial a_D^i}{\partial v} = \frac{\sqrt{2}}{4\pi} \int_{\beta_i} \frac{dx}{y}.
\end{align*}
\]

We will use the same definition of the cycles \( \alpha_i \) and \( \beta_i \) as that used by Argyres and Douglas: 3) The \( \alpha_1 \) (or \( \beta_1 \)) cycle is that encircling \( e_3 \) and \( e_2 \) (or \( e_1 \)), and the \( \alpha_2 \) (or \( \beta_2 \)) cycle that encircling \( e_6 \) and \( e_5 \) (or \( e_4 \)).

Let us think of a special situation in which massless particles with mutually non-local charge appear. This situation corresponds to the existence of superconformal symmetry and is realized if \( a_1 = a_D^1 = 0 \), or \( a_2 = a_D^2 = 0 \). In other words, the two points \( (u^*/\Lambda^2, v^*/\Lambda^3) = (0, \pm 1) \) are the superconformal points. Without loss of generality, we confine ourselves to only one of the two, namely, \( (u^*/\Lambda^2, v^*/\Lambda^3) = (0, 1) \). The branch points, Eqs. (6·2) \( \sim \) (6·4), for this particular choice of \( u \) and \( v \) turn out to be

\[
e^*_1 = e^*_2 = e^*_3 = 0, \quad (e^*_4, e^*_5, e^*_6) = 2^{1/3} \Lambda(1, \omega, \omega^2).
\]

We would like to analyze the behavior of \( \tau^{ij} \) in the vicinity of this superconformal point. By changing the integration variables in (6·9) and (6·10), we obtain a more useful form amenable to further analysis,

\[
\begin{align*}
\frac{\partial a_1}{\partial v} &= \frac{\sqrt{2}}{4\pi} \frac{4}{\sqrt{(e_2 - e_1)(e_4 - e_2)(e_5 - e_2)(e_6 - e_2)}} \\
&\times \int_0^1 d\xi \frac{1}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)(1 - \delta_4 \xi^2)(1 - \delta_5 \xi^2)(1 - \delta_6 \xi^2)}}, \\
\frac{\partial a_1}{\partial u} &= \frac{\sqrt{2}}{4\pi} \frac{4}{\sqrt{(e_2 - e_1)(e_4 - e_2)(e_5 - e_2)(e_6 - e_2)}} \\
&\times \int_0^1 d\xi \frac{(e_3 - e_2)k^2e_2}{\sqrt{(1 - \xi^2)(1 - k^2\xi^2)(1 - \delta_4 \xi^2)(1 - \delta_5 \xi^2)(1 - \delta_6 \xi^2)}},
\end{align*}
\]

where \( \delta_i \) are the zeros of the branch cuts.
where $k^2 = (e_3 - e_2)/(e_1 - e_2)$ and $\delta_i = (e_3 - e_2)/(e_i - e_2), (i = 4, 5, 6)$. We obtain $\partial_v a^1_D$ and $\partial_u a^1_D$ in a similar way by exchanging $e_1$ and $e_2$. In other words, $k^2$ and $\delta_i$ in (6·12) and (6·13) should be replaced by $k'^2 = 1 - k^2$ and $\delta'_i = (e_3 - e_1)/(e_i - e_1)$, respectively.

The above concise expression enables us to evaluate the infrared behavior of (6·9) and (6·10) near the superconformal point by taking $k^2 \to k'^2$, $k^2 \to k'^2$, $\delta_i \to 0$, and $\delta'_i \to 0$, i.e.,

$$\frac{\partial a_1}{\partial v} \to \frac{1}{\pi A^{3/2}} \frac{1}{\sqrt{e_2^* - e_1^*}} K(k), \quad \frac{\partial a^1_D}{\partial v} \to \frac{1}{\pi A^{3/2}} \frac{i}{\sqrt{e_2^* - e_1^*}} K(k'),$$

$$\frac{\partial a_1}{\partial u} \to 0, \quad \frac{\partial a^1_D}{\partial u} \to 0.$$  

Equation (6·14) is reminiscent of the $SU(2)$ case, described by Eqs. (2·6) and (2·7).

Let us now have a closer look at the limiting behavior (6·14) and (6·15) near the superconformal point, following the argument of the work of Argyres et al. 4) We consider the deviation from the superconformal point by introducing a parameter $t$ as

$$\frac{u - u^*}{A^2} = E_1 t^{-\alpha}, \quad \frac{v - v^*}{A^3} = E_2 t^{-\beta}.$$  

The ratio of the exponents $\beta/\alpha$ should be determined in order for the coupling $\tau^{11}$ to go to a generic value. This condition provides us uniquely with $\beta = 3\alpha/2$. Plugging this result, we can evaluate the asymptotic behavior of each integral. We find

$$\frac{\partial a_1}{\partial v} \sim \frac{1}{\pi A^{3/2}} \frac{1}{\sqrt{e_2^* - e_1^*}} \left\{ K(k) + \mathcal{O}(t^{-3\alpha/2}) \right\},$$

$$\frac{\partial a^1_D}{\partial v} \sim \frac{1}{\pi A^{3/2}} \frac{i}{\sqrt{e_2^* - e_1^*}} \left\{ K(k') + \mathcal{O}(t^{-3\alpha/2}) \right\},$$

$$\frac{\partial a_1}{\partial u} \sim \mathcal{O}(t^{-\alpha/4}),$$

$$\frac{\partial a^1_D}{\partial u} \sim \mathcal{O}(t^{-\alpha/4}).$$

In the same way, the other integrals associated with the larger homology cycle turn out to be

$$\frac{\partial a_2}{\partial v} = \frac{\sqrt{2}}{4\pi} \frac{4}{\sqrt{(e_4^* - e_5^*)(e_6^* - e_5^*)^3}} \left\{ J_1 + \mathcal{O}(t^{-\alpha}) \right\},$$

$$\frac{\partial a^2_D}{\partial v} = \frac{\sqrt{2}}{4\pi} \frac{4i}{\sqrt{(e_4^* - e_5^*)(e_6^* - e_5^*)^3}} \left\{ J_2 + \mathcal{O}(t^{-\alpha}) \right\},$$

$$\frac{\partial a_2}{\partial u} = \frac{\sqrt{2}}{4\pi} \frac{4}{\sqrt{(e_4^* - e_5^*)(e_6^* - e_5^*)}} \left\{ J_1 + \mathcal{O}(t^{-\alpha}) \right\},$$

$$\frac{\partial a^2_D}{\partial u} = \frac{\sqrt{2}}{4\pi} \frac{4i}{\sqrt{(e_4^* - e_5^*)(e_6^* - e_5^*)}} \left\{ J_2 + \mathcal{O}(t^{-\alpha}) \right\}.$$
where

\[ I_1 = \int_0^1 d\xi \frac{1}{\sqrt{(1 - \xi^2)(1 - \ell^2 \xi^2)(\xi^2 + \kappa^2)}} \]  
\[ J_1 = \int_0^1 d\xi \frac{1}{\sqrt{(1 - \xi^2)(1 - \ell^2 \xi^2)(\xi^2 + \kappa)}} \]  

Here we use \( \ell^2 = -\omega \) and \( \kappa = 1/(\omega - 1) \). We can also obtain expressions for \( I_2 \) and \( J_2 \) by replacing \( \ell^2 \) and \( \kappa \) in (6·25) and (6·26) by \( \ell'^2 = 1 - \ell^2 = -\omega^2 \) and \( \kappa' = 1/(\omega^2 - 1) \), respectively. The simple relations \( I_2 = I_1^* \) and \( J_2 = J_1^* \) are immediately recognized because \( \ell' = \ell^* \) and \( \kappa' = \kappa^* \). It is also amusing to see that the value of \( J_2/J_1 = J_1^*/J_1 \) can be calculated analytically; i.e., \( J_2/J_1 = (1 - \sqrt{3}\kappa)/2 \).

By using the above asymptotic form of the integrals (6·17) ~ (6·24), we have analyzed the asymptotic behavior of the effective coupling \( \tau^{ij} \). Our final result is as shown below:

\[ \tau^{11} = i \frac{K(k')}{K(k)} + O(t^{-\alpha/2}), \]  
\[ \tau^{22} = ie^{i\pi/6} \left( \frac{J_2}{J_1} \right) + O(t^{-\alpha/2}) \]  
\[ = e^{i\pi/3} + O(t^{-\alpha/2}), \]  
\[ \tau^{12} = \tau^{21} = O(t^{-\alpha/4}). \]

We are thus led to conclude that the slope parameters of the beta-function associated with the coupling \( \tau^{ij} \) are

\[ \rho = \alpha/2 \quad \text{for} \quad \tau^{11} \quad \text{and} \quad \tau^{22}, \]  
\[ \rho = \alpha/4 \quad \text{for} \quad \tau^{12} \quad \text{and} \quad \tau^{21}. \]

The exponent of \( \alpha_1 \) being unity as in the \( SU(2) \), \( N_f = 1 \) case, \( \alpha \) should be set equal to \( 4/5 \). This leads us to \( \rho = 2/5 \) for \( \tau^{11} \) and \( \tau^{22} \), and \( \rho = 1/5 \) for \( \tau^{12} \) and \( \tau^{21} \).

The difference between \( \tau^{11} \) and \( \tau^{22} \) results from the fact that, while the infrared limit of \( \tau^{22} \) has been determined by (6·28) uniquely, the value of \( \tau^{11} \) at the superconformal point has not: The value of \( \tau^{11} \) there depends on the coefficients \( E_1 \) and \( E_2 \) in (6·16), whose determination is made by the procedure of integrating out the high-energy content of the theory. These facts could be connected with the observation that the \( U(1) \) gauge theory with coupling \( \tau^{11} \) is an interacting non-trivial superconformal field theory, while the \( U(1) \) gauge theory with coupling \( \tau^{22} \) is a free trivial one.

Argyres et al. claim that the superconformal field theory considered in the present section, i.e., with the \( SU(3) \) pure Yang-Mills theory, is the same \( (1, 1) \) type as that of the superconformal field theory of \( SU(2) \) QCD with one flavor. Our result involving the scaling dimension of \( \tau^{11} \) is a non-trivial check of their claim.
§7. Summary and discussion

In the present paper we have investigated the renormalization group flow near the superconformal points in the $SU(2)$ QCD and $SU(3)$ pure Yang-Mills theories. We have determined the slope parameter (2.16) of the beta-function, which agrees partly with results obtained previously by Bilal and Ferrari using a different method.

As mentioned briefly in the Introduction, there exist a number of elaborate works on the renormalization group equation in Seiberg-Witten theory. It is often argued in the literature that taking the derivative of $\tau$ with respect to the QCD dynamical scale $\Lambda$ reproduces the correct beta-function in the weak coupling region. Since the coupling $\tau$ in the $SU(2)$ case is a function of $u/\Lambda^2$ and $m_i/\Lambda$, the beta-function in the weak coupling region turns out to be expressed by

$$-\Lambda \frac{\partial}{\partial \Lambda} \tau \left( \frac{u}{\Lambda^2}, \frac{m_i}{\Lambda} \right) = \left\{ \frac{2u}{\Lambda^2} \frac{\partial}{\partial u} + \sum_i m_i \frac{\partial}{\partial m_i} \right\} \tau \left( \frac{u}{\Lambda^2}, \frac{m_i}{\Lambda} \right).$$

This indicates that our two basic quantities $\gamma_u$ and $\gamma_{mi}$, used extensively in our work, are given in the semi-classical region by $2u$ and $m_i$, respectively. In the strong coupling region, on the other hand, we have successfully determined the functional form of $\gamma_u$ and $\gamma_{mi}$ from the consistency argument. We have, however, no convincing way to obtain $\gamma_u$ and $\gamma_{mi}$ in the entire region of moduli space. The procedure of integrating out the high energy content of the theory above $M$ has hidden information concerning the full $M$ dependence of $u$ and $m_i$. The point of our work is that, despite the lack of complete information on the $M$-dependence, we are still able to determine the renormalization group flow near the superconformal point by looking for possible deformation thereof.

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