The generalized gluing and resmoothing theorem originally proved by LeClair, Peskin and Preitschopf gives a powerful formula for the fused vertex obtained by contracting any two vertices in string field theories. Although the theorem is naturally expected to hold for the vertices at any loop level, the original proof was restricted to the vertices at tree level. Here we present a simplified proof for the tree level theorem and then prove explicitly the extended version at the one-loop level.

§1. Introduction

The basic ingredients of string field theories (SFT) are the vertices. We need to combine (or fuse) the various vertices in many situations, e.g., in showing the gauge invariance of the SFT action and in computing string scattering amplitudes in perturbation theory. LeClair, Peskin and Preitschopf (LPP) have developed a powerful method for defining general multistring vertices by conformally mapping the unit disks of participating strings into a complex plane and by using the correlation functions of conformal field theory (CFT) in the plane. They then proved a theorem, which they called "Generalized Gluing and Resmoothing Theorem" (GGRT), giving a general formula for the fused vertex obtained by the contraction of two vertices. They showed that the fused vertex equals their multistring vertex corresponding to the conformal mappings induced by gluing the two world sheets into one.

The point here is that this equality holds with weight one if the conformal anomaly (i.e., central charge $c$) is zero. Their proof is very thorough and even pedagogical. It is, however, a bit complicated and seems not easy to trace the sign of the equality.

In this paper we first present a much simplified proof for the LPP GGRT and determine the sign of the equality carefully. Our proof is inspired by the sewing method of two conformal field theories defined on two Riemann surfaces, which has been given, in particular, by Sonoda. Actually, the GGRT by LPP is a string field theory version of this general way of sewing two CFT's. The gluing and sewing are essentially the same and are just the insertion of the complete set of states. So, although the original GGRT by LPP is restricted to the vertices at tree level, it is naturally expected to hold at any loop level. Nevertheless, the SFT version is not so trivial. This is because the gluing in SFT must be performed by contracting two
strings, one each from the two vertices. On the other hand, the sewing of two CFT’s is performed by excising two holes freely, one each on the two Riemann surfaces. To do the same thing in the SFT case and to make contact with the definition of the vertices, one needs to map the string world sheets back and forth. These mappings give non-trivial conformal transformations on the operators, which must be traced neatly. We perform this procedure and prove explicitly an extended version of the GGRT at the one-loop level. It may be interesting to note that, as a byproduct, the formula for the CFT correlation function on the torus, \(10\)

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle_{\text{torus}} = \text{Tr} [(-1)^{N_{\text{FP}}} q^{2L_0} \mathcal{O}_1 \mathcal{O}_2 \cdots],
\]

is automatically derived by this procedure (up to an overall factor convention), where \(q = e^{i \pi T}\), and \(N_{\text{FP}}\) is the ghost number operator. The factor \((-1)^{N_{\text{FP}}}\) in this expression comes from the sign factor contained in the ket reflector.

This paper is organized as follows. First, in §2, we briefly review the definition of the vertices and the GGRT given by LPP. In §3, after making some remarks on the ambiguity present when \(e^{i \phi}\) in defining conformal transformation operators \(U_f\) corresponding to the mappings \(f(z)\), we present two propositions to clarify when \(U_f\) leaves the \(SL(2; \mathbb{C})\) bra and ket vacua inert, and we then give a simplified proof for the GGRT of LPP. The extension of GGRT to the one-loop level is given and proved in §4.

For simplicity of presentation, we assume henceforth that the strings are all bosonic open strings, so that the relevant conformal fields \(\phi(z)\) are string coordinates \(\partial X^\mu(z)\), the reparameterization ghost \(c(z)\), and the anti-ghost \(b(z)\), possessing dimensions \(d = 1, -1\) and 2, respectively. A closed string can be treated similarly, since it is more or less equivalent to a pair of open strings.

§2. GGRT at tree level

First, let us recall LPP’s definition of the tree level vertex which refers to the conformal field theory in the complex plane (a two-dimensional manifold \(M\) which is topologically equivalent to \(S^2\)); \(^{11},^{12}\)

\[
\langle v(n, \ldots, 2, 1) | A_1 \rangle_1 | A_2 \rangle_2 \cdots | A_n \rangle_n \equiv \left\langle h_1[O_{A_1}] h_2[O_{A_2}] \cdots h_n[O_{A_n}] \right\rangle_M.
\]

Here \(\langle v(n, \ldots, 2, 1)\rangle\) is the \(n\)-point LPP vertex, which is defined as a bra state in the product space \(\otimes_{i=1}^n \mathcal{H}_i\) of \(n\)-string Fock spaces \(\mathcal{H}_i\), and each string state \(|A_i\rangle_i \in \mathcal{H}_i\) is given in the form

\[
|A_i\rangle_i = O_A |0\rangle_i,
\]

where \(O_A\) is an operator creating the state \(A\) of string \(i\) from the \(SL(2; \mathbb{C})\) invariant vacuum \(|0\rangle_i\) in \(\mathcal{H}_i\); for instance, the tachyon state of momentum \(p\) is given by the vertex operator \(O(w) = c(w) \exp(ip \cdot X(w))\) at \(w = 0\), and the ladder operators \(\phi_n = \{a_n, c_n, b_n\}\) are given by the contour integration \(\oint dw/2\pi i w^{n+d-1}\phi(w)\) encircling the origin.

The meaning of the right-hand side of Eq. (2·1) is as follows: any LPP vertex \(\langle v(n, \ldots, 2, 1)\rangle\) is defined by specifying how the participating strings \(i\) are glued to
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We can regard each string world sheet from infinite past ($\tau_i = -\infty$) to the interaction time ($\tau_i = 0$) as a unit disk $|w_i| \leq 1$ with $w_i = \exp(\tau_i + i\sigma_i)$, and the world sheet formed by gluing those string sheets as a complex $z$-plane, which we call $M (\sim S^2)$, for the tree level vertex case. So this gluing can be simply specified by giving conformal mappings $h_i(w_i)$ of each string $w_i$-plane into the complex $z$-plane $M$, which is analytic and invertible inside each unit circle $|w_i| = 1$. Generally, any conformal mapping $f: w \rightarrow z = f(w)$ of the $w$-plane to the $z$-plane also defines a mapping of operators $\mathcal{O}$ in the $w$-plane to operators $f[\mathcal{O}]$ in the $z$-plane:

$$f[\mathcal{O}] = U_f \mathcal{O} U_f^{-1}.$$  \hfill (2·3)

If the operator $\mathcal{O}$ is a primary conformal field $\phi(w)$ of dimension $d_\phi$, this mapping is defined to be

$$f[\phi(w)] = \left( df(w) \frac{dw}{dw} \right)^{d_\phi} \phi(f(w)).$$  \hfill (2·4)

The operator representation $U_f$ in Eq. (2·3) of the conformal mapping $f$ is uniquely determined by this transformation law (2·4) of the primary fields up to a multiplicative constant. Since the Fourier components of the energy momentum tensor $T(z)$, $L_n \equiv \oint (dz/2\pi i)z^{n+1}T(z)$, generate infinitesimal conformal transformations, the operator $U_f$ for the finite transformation $f$ can be given in the form

$$U_f = \exp \left( \sum_n v_n L_n \right)$$  \hfill (2·5)

with certain parameters $v_n$. ($v(z) \equiv \sum_n v_n z^{n+1}$ and $f(z)$ are related by $f(z) = e^{v(z)\partial_z z}$.*) We should keep in mind, however, that this parameterization (2·5) for $U_f$ is not unique and that the very definition of $U_f$ by Eq. (2·3) has an ambiguity of a multiplicative constant. We shall come back to this problem later in the next section.

Now, the meaning of the right-hand side of Eq. (2·1) is clear: it gives a correlation function of the mapped operators $h_i[\mathcal{O}_A]$ of the conformal field theory in the $z$-plane. Note that the crucial conceptual difference between the two sides of the defining equation (2·1) of the LPP vertex: the left-hand side is an inner product in the product space $\otimes_{i=1}^n \mathcal{H}_i$ of $n$-string Fock spaces $\mathcal{H}_i$, while the right-hand side is a correlation function of a single string conformal field theory in the $z$-plane.

We here note the fact that the tree level vertex $\langle v(n, \cdots, 1) \rangle$ is always Grassmann odd (fermionic), provided that we adopt a natural convention to take the ket $SL(2; \mathbb{C})$ vacuum $|0\rangle$ Grassmann even. This convention is natural since, then, the state $|A\rangle = \mathcal{O}_A |0\rangle$ and the corresponding operator $\mathcal{O}_A$ carry the same statistics.* Since the ghost number anomaly is 3 in the case of sphere, the net ghost number carried by the operators $\mathcal{O}_{A_1}, \cdots, \mathcal{O}_{A_n}$ must be 3 in order for the CFT correlation function in Eq. (2·1) not to vanish. This means that the product of states, $|A_1\rangle_1 |A_2\rangle_2 \cdots |A_n\rangle_n$, must be Grassmann odd, and so is the vertex $\langle v(n, \cdots, 1) \rangle$.

*) Indeed, then the sign changes by the ordering changes of the states $|A\rangle_i$ and the operators $\mathcal{O}_{A_i}$ on both sides of Eq. (2·1) become common, which makes the LPP vertex $\langle v(n, \cdots, 1) \rangle$ independent of the ordering of the arguments.
We need a little more preparation to state the GGRT. Let us introduce bra reflector $\langle R(1,2) \rangle$, which converts ket string states $|A\rangle$ to bra states $\langle A|$,

$$\langle R(1,2) \rangle |A\rangle_2 = |A\rangle_1,$$  \hspace{1cm} (2.6)

and ket reflector $|R(1,2)\rangle$ as its inverse:

$$\langle R(1,2) \rangle |R(2,3)\rangle = |1\rangle_{21}, \quad |1\rangle_{21} |A\rangle_1 = |A\rangle_2 .$$  \hspace{1cm} (2.7)

The reflectors $\langle R(1,2) \rangle$ and $|R(1,2)\rangle$ are just the metrics $g_{IJ}$ and $g^{IJ}$, respectively, if we use the notation $|A\rangle \equiv A^I$ and $\langle A| \equiv A_J$. So they can be defined by giving an inner product in the string Fock space $\mathcal{H}$. A natural inner product\(^3\) is defined by using the inversion $I(z) = -1/z$ as follows:

$$\langle A|B \rangle = \langle R(1,2) \rangle |A\rangle_2 |B\rangle_1 = \langle I[\mathcal{O}_A] \mathcal{O}_B \rangle .$$  \hspace{1cm} (2.8)

It is easy to find an explicit oscillator expression for the reflectors, as given in e.g., Refs. 2) and 13). Here we do not need this explicit expression but, rather, the following formal one. Let $\{ |\alpha\rangle \}$ be a complete set of the ket string states and $\{ \langle \tilde{\alpha}| \}$ be its orthonormal dual under this inner product; i.e., $\langle \tilde{\beta}|\alpha\rangle = \langle I[\mathcal{O}_\beta] \mathcal{O}_\alpha \rangle = \delta^\beta_\alpha$. Then we have the completeness relation

$$\sum_\alpha |\alpha\rangle \langle \tilde{\alpha}| = \sum_\alpha \mathcal{O}_\alpha |0\rangle \langle 0| I[\mathcal{O}_\tilde{\alpha}] = 1 \quad \text{in} \; \mathcal{H},$$  \hspace{1cm} (2.9)

where $\mathcal{O}_\alpha$ and $\mathcal{O}_{\tilde{\alpha}}$ are operators creating the states $|\alpha\rangle$ and $|\tilde{\alpha}\rangle$, respectively. Now we find the following formal expressions for the reflectors:

$$\langle R(1,2) \rangle = \sum_\alpha 1(\alpha|_2 \langle \tilde{\alpha}| = \sum_\alpha 1(0|_1 I[\mathcal{O}_\alpha^{(1)}]_2 0|_1 I[\mathcal{O}_{\tilde{\alpha}}^{(2)}],$$  \hspace{1cm} (2.10)

$$|R(1,2)\rangle = \sum_\alpha (-1)^{|\alpha|} |\alpha\rangle_1 |\tilde{\alpha}\rangle_2 = \sum_\alpha (-1)^{|\alpha|} \mathcal{O}_\alpha^{(1)} |0\rangle_1 \mathcal{O}_{\tilde{\alpha}}^{(2)} |0\rangle_2 .$$  \hspace{1cm} (2.11)

where $|\alpha|$ denotes the statistics index of the operator $\mathcal{O}_\alpha$ (or the ket state $|\alpha\rangle$) which is defined to be 0 (1) when $\mathcal{O}_\alpha$ is bosonic (fermionic). The validity of these expressions (2.10) and (2.11) can easily be confirmed by showing that the defining equations (2.6) and (2.7) are actually satisfied by them:

$$\langle R(1,2) \rangle |\beta\rangle_2 = \sum_\alpha 1(\alpha|_2 \langle \tilde{\alpha}|_2 = \sum_\alpha 1(\alpha| \delta^\beta_\alpha = 1|\beta|$$

$$\langle R(1,3) \rangle |R(2,3)\rangle = \langle R(1,3) \rangle \sum_\alpha (-1)^{|\alpha|} |\alpha\rangle_2 |\tilde{\alpha}\rangle_3$$

$$= \sum_\alpha |\alpha\rangle_2 \langle R(1,3) \rangle |\tilde{\alpha}\rangle_3 = \sum_\alpha |\alpha\rangle_2 |1\rangle_2 |\tilde{\alpha}\rangle = 1_{21}. \hspace{1cm} (2.12)$$

In the second equation, we have used the fact that the bra reflector $\langle R(1,2) \rangle$ is Grassmann odd, which we explain shortly. The sign factor $(-1)^{|\alpha|}$ contained in $|R(1,2)\rangle$ will become important in the following discussions.
Here we add some comments on the Grassmann even-oddness and symmetry properties of the reflectors. First, both the bra and ket reflectors are Grassmann odd.\(^*\) The reason is the same as for the tree level vertex \(\langle v(n, \cdots, 1) \rangle\). By the ghost number anomaly, \(O_\alpha\) and \(O_\bar{\alpha}\) must have opposite statistics in order to have non-vanishing inner product \(\langle \alpha | \bar{\alpha} \rangle = \langle I[O_\bar{\alpha}]O_\alpha \rangle = 1\). Thus, from the expressions (2·10) and (2·11), the reflectors are seen to be Grassmann odd. (The relation \(\{0\} c_{-1} c_{0} c_{1} \{0\} = 1\) also shows that the bra and ket \(SL(2; C)\) vacua, \(\{0\}\) and \(\{0\}\), have opposite statistics so that the product \(\{0\}\{0\}\) is Grassmann odd and the left-hand side of the completeness relation (2·9) is Grassmann even as it should be.) Second, the expressions (2·10) and (2·11) also clearly show the symmetry properties: under the interchange of the string labels 1 and 2, the bra reflector \(\langle R(1, 2) \rangle\) is symmetric and the ket reflector \(|R(1, 2)\rangle\) is anti-symmetric, since \(|\bar{x}| = |\alpha| + 1\).

Now we define the contraction (or fusion) of two vertices appearing in the GGRT. Let \(\langle v(C, \{A_i\}) \rangle\) be an \((n+1)\)-point LPP vertex for the strings \(A_i\) \((i = 1, 2, \cdots, n)\) and \(C\) defined by conformal mappings \(h_{A_i}\) and \(h_C\):

\[
\langle v(C, \{A_i\}) \rangle \prod_{i=1}^{n} |A_i\rangle_{A_i} |C\rangle_C = \left\langle \prod_{i=1}^{n} h_{A_i}[O_{A_i}] h_C[O_C] \right\rangle_M.
\] (2·13)

Then, let \(\langle v(D, \{B_j\}) \rangle\) be another \((m+1)\)-point LPP vertex defined similarly:

\[
\langle v(D, \{B_j\}) \rangle \prod_{j=1}^{m} |B_j\rangle_{B_j} |D\rangle_D = \left\langle \prod_{j=1}^{m} h_{B_j}[O_{B_j}] h_D[O_D] \right\rangle_N.
\] (2·14)

Note that we have called the \(z\)-planes for the two cases \(M\) and \(N\), for distinction, although they are both topologically equivalent to \(S^2\). Then we can define a fused vertex \(\langle v(\{B_j\}, \{A_i\})\rangle_{\text{fused}}\) of these two vertices by gluing the strings \(C\) and \(D\) in each with the help of the ket reflector \(|R(D, C)\rangle\):

\[
\langle v(\{B_j\}, \{A_i\})\rangle_{\text{fused}} \equiv \langle v(D, \{B_j\})\rangle \langle v(C, \{A_i\})\rangle |R(D, C)\rangle.
\] (2·15)

Intuitively, the fusion gives a new Riemann surface which is formed by cutting out the images of the unit disks of strings \(C\) and \(D\) in \(M\) and \(N\), respectively, and then gluing smoothly the remaining pieces of \(M\) and \(N\) together. This Riemann surface again becomes a complex plane, a manifold \(M_{\infty}N\) topologically equivalent to \(S^2\). This gluing also induces conformal mappings \(\hat{h}_{A_i}(w_i)\) and \(\hat{h}_{B_j}(w_j)\) of the unit disks \(|w_i| \leq 1\) and \(|w_j| \leq 1\) of \(n + m\) strings \(\{A_i\}, \{B_j\}\) into the plane \(M_{\infty}N\), which are again analytic and invertible inside each disk.

Now we can state the GGRT, which was first proved by LPP:\(^2\)

**Theorem [LPP]** Let \(\langle v(\{B_j\}, \{A_i\})\rangle\) be the LPP vertex defined by this set of mappings,

\[
\langle v(\{B_j\}, \{A_i\})\rangle \prod_{i=1}^{n} |A_i\rangle_{A_i} \prod_{j=1}^{m} |B_j\rangle_{B_j} = \left\langle \prod_{i=1}^{n} \hat{h}_{A_i}[O_{A_i}] \prod_{j=1}^{m} \hat{h}_{B_j}[O_{B_j}] \right\rangle_{M_{\infty}N}.
\] (2·16)

\(^*\) For the case of a closed string, however, the reflector \(\langle R(1, 2)\rangle\), as well as \(|R(1, 2)\rangle\), is Grassmann even since it is a product of two ‘open string’ reflectors corresponding to the holomorphic and anti-holomorphic modes.
Then, if the conformal anomaly is zero, the fused vertex (2.15) is equal to this LPP vertex:

\[
\langle v(D, \{B_j\})|v(C, \{A_i\})|R(D, C)\rangle = \langle v(\{B_j\}, \{A_i\}) \rangle .
\] (2.17)

If the conformal anomaly is present, the equality (2.17) is violated by a multiplicative c-number factor which depends non-trivially on the mappings \(g, h_D\) and \(h_C\).

LPP analyzed the above gluing procedure in Eq. (2.15) more carefully, as shown in Fig. 1. First, the complex planes \(M\) and \(N\) defining the vertices \(v(C, \{A_i\})\) and \(v(D, \{B_j\})\) are mapped by \(h_C^{-1}\) and \(I \circ h_D^{-1}\) so that the exterior region of string \(C\) in \(M\) is mapped to the region outside a unit circle and the exterior region of string \(D\) in \(N\) to the inside of a unit circle, respectively. Then the region outside the unit circle in the plane \(h_C^{-1}(M)\) and the region inside the unit circle in the plane \(I \circ h_D^{-1}(N)\) are glued smoothly as they stand. However, unless the mappings \(h_C\) and

![Figure 1](https://academic.oup.com/ptp/article-abstract/100/2/437/1853619/10024371853619)

Fig. 1. Gluing and subsequent smoothing in the contraction of two vertices.
$h_D$ are $SL(2; C)$ transformations, neither they nor their inverses will be one-to-one mappings outside the unit circles. So the glued surface $h^{-1}_C(M) \circ I \circ h^{-1}_D(N)$ will generally possess branch-cut singularities. Since the covering surface nevertheless has the topology of $S^2$, there exists a mapping $g$ which carries the surface into the plane $M \circ N$, smoothing out the branch cuts. Therefore the conformal mappings $h_{A_i}$ and $h_{B_j}$ of the strings $\{A_i\}$ and $\{B_j\}$ into the plane $M \circ N$, mentioned in the Theorem, can thus be identified with

$$\hat{h}_{A_i} = g \circ h^{-1}_C \circ h_{A_i}, \quad \hat{h}_{B_j} = g \circ I \circ h^{-1}_D \circ h_{B_j}.$$  \hspace{1cm} (2·18)

The last step mapping $g$ corresponds to a resmoothing procedure. This accounts for the name 'Gluing and Resmoothing Theorem'.

§3. A simple proof for the tree level GGRT

We give in this section a proof for the GGRT at tree level. This proof is much simpler than the original one by LPP and, therefore, makes it easy to trace correctly the sign factors that appear.

As promised, we first discuss the parameterization forms for the conformal transformation operator $U_f$ introduced in Eq. (2·3), and the ambiguity involving the multiplicative constant which exists if the conformal anomaly (central charge $c$) is nonzero. As in the usual Lie group, there are a variety of ways of representing the group elements $U_f$ in terms of the Virasoro generators $L_n$. We refer to the parameterization form

$$U_f = \exp\left(\sum_n v_n L_n\right)$$  \hspace{1cm} (3·1)

already cited in Eq. (2·5), as the 'canonical form'. This is most commonly used in Lie group theory. Another useful parameterization form, which we refer to as the 'normal ordered' form, is given by

$$U_f = \exp\left(\sum_{n \geq 2} v_n L_{-n}\right) \exp\left(\sum_{k=0, \pm 1} v_k L_k\right) \exp\left(\sum_{m \geq 2} v_{-m} L_m\right).$$  \hspace{1cm} (3·2)

Note that the middle factor $\exp\left(\sum_{k=0, \pm 1} v_k L_k\right)$ is the element belonging to the $SL(2; C)$ subgroup. Of course we can convert various parameterization forms from one to another by using the commutation relations of $L_n$. But the point is that, if the conformal anomaly is nonzero, there appears a non-trivial multiplicative $c$-number factor in front in this rewriting: for instance, we have a relation like

$$\exp\left(\sum_{n \geq 2} v_n L_{-n}\right) \exp\left(\sum_{k=0, \pm 1} v_k L_k\right) \exp\left(\sum_{m \geq 2} v_{-m} L_m\right) = e^a \exp\left(\sum_n v_n L_n\right).$$  \hspace{1cm} (3·3)

The $c$-number factor $e^a$ depends on the central charge $c$ (the exponent $a$ is linear in $c$), but the other group element part is uniquely determined independently of $c$. This means that, in the presence of nonzero central charge, the conformal transformation
operator $U_f$ has an ambiguity of an overall factor depending on which parameterization form is adopted in defining $U_f$. This is due to the fact that the unit ‘operator’ 1 is also one of the generators of the extended Virasoro algebra with central charge.

The same problem of a multiplicative c-number factor arises also in the composition law of two group elements. Whatever parameterization convention is adopted for $U_f$, the multiplication of two elements $U_f$ and $U_g$ yields $U_{fog}$ of the composite mapping $f \circ g$ only up to a constant $e^a$:

$$U_f \cdot U_g = e^a U_{fog}. \quad (3.4)$$

Again the constant $a$ is linear in the central charge $c$ (and has a complicated dependence both on the mappings $f$ and $g$ and on the parameterization convention). So the naive composition law is violated unless the conformal anomaly is zero. This is the crucial property which is responsible for the fact that the GGRT holds only in the critical dimension.

The characteristic feature of the normal ordered form (3.2) for $U_f$ is that it manifestly satisfies

$$
\langle 3| U_f |0\rangle = \langle 0| U_f |3\rangle = 1 \quad \left( \langle 3| \equiv \langle 0| c_{-1}c_0 c_1, \quad |3\rangle \equiv c_{-1}c_0 c_1 |0\rangle \right) \quad (3.5)
$$
even when $c \neq 0$. (Recall that the $SL(2;\mathbb{C})$ invariant vacuum is normalized by the condition $\langle 0| c_{-1}c_0 c_1 |0\rangle = \langle 3|0\rangle = \langle 0|3\rangle = 1$.) This property follows because the $SL(2;\mathbb{C})$ invariant vacuum (either $|0\rangle$ or $|0\rangle$) is literally invariant under the $SL(2;\mathbb{C})$ transformation $\exp(\sum_{k=0,\pm 1} v_k L_k)$, the ghost-number 3 ket state $|3\rangle \equiv c_{-1}c_0 c_1 |0\rangle$ is invariant under $\exp(\sum_{m \geq 2} v_m L_m)$, since $L_m |3\rangle = L_m c_{-1}c_0 c_1 |0\rangle = 0$ for $m \geq 1$, and similarly, $\langle 3|$ is invariant under $\exp(\sum_{n \geq 2} v_n L_{-n})$, since $\langle 3| L_{-n} = \langle 0| c_{-1}c_0 c_1 L_{-n} = 0$ for $n \geq 1$.

Now we have the following simple proposition, which was essentially stated and used in LPP:2)

**Proposition 1** Let $f(z)$ be a conformal mapping satisfying $f(0) = 0$. If $f(z)$ is analytic and invertible in the neighborhood of $z = 0$, then the corresponding operator $U_f$ defined by Eqs. (2.3) and (2.4) leaves the $SL(2;\mathbb{C})$ ket vacuum inert up to a multiplicative constant:

$$U_f |0\rangle = e^a |0\rangle. \quad (3.6)$$

If $U_f$ is taken to be of the normal ordered form (3.2), or the conformal anomaly is absent, this constant $e^a$ equals 1.

**Proof** The $SL(2;\mathbb{C})$ ket vacuum $|0\rangle$ is characterized by the property that $\phi(z) |0\rangle$ remains regular as $z \to 0$ for any primary field $\phi(z)$. This property implies for a primary field of dimension $d$, $\phi(z) = \sum_n \phi_n z^{-n-d}$, that

$$
\lim_{z \to 0} \phi(z) |0\rangle = \text{regular} \iff \phi_n |0\rangle = 0 \quad \text{for} \quad n \geq 1 - d. \quad (3.7)
$$

So, if we can show that

$$
\lim_{z \to 0} f[\phi(z)] |0\rangle = \text{regular} \quad (3.8)
$$
for any primary field \( \phi \), then since \( f[\phi(z)] = \sum_n f[\phi_n]z^{-n-d} \), we can deduce

\[
f[\phi_n]|0\rangle = U_f\phi_n U_f^{-1}|0\rangle = 0 \quad \rightarrow \quad \phi_n U_f^{-1}|0\rangle = 0 \quad \text{for} \quad n \geq 1 - d. \tag{3.9}
\]

This implies that \( U_f^{-1}|0\rangle \) is proportional to the vacuum \( |0\rangle \), \( U_f^{-1}|0\rangle = e^{-\alpha}|0\rangle \), or equivalently, \( U_f|0\rangle = e^{\alpha}|0\rangle \), with some constant \( \alpha \). But, applying (3) to this relation from the left and using the normalization condition \( \langle 3|0\rangle = 1 \), we have

\[
\langle 3|U_f|0\rangle = e^{\alpha}. \tag{3.10}
\]

If \( U_f \) is of the normal ordered form (3.2), or the conformal anomaly is absent, the left-hand side is 1 by Eq. (3.5), and \( e^{\alpha} = 1 \) follows.

Thus we have now only to prove Eq. (3.8). The mapped field \( f[\phi(z)] \) is explicitly given by Eq. (2.4) for primary fields, and so it is expanded as

\[
f[\phi(z)] = (f'(z))^d \phi(f(z)) = \sum_n \phi_n \cdot (f(z))^{-n-d} (f'(z))^d. \tag{3.11}
\]

By the assumption of analyticity of \( f(z) \) around the origin and \( f(0) = 0 \), \( f(z) \) behaves as \( f(z) = f_1z + O(z^2) \). Moreover, \( f'(0) \neq 0 \) by the assumption of the invertibility of \( f(z) \) around \( z = 0 \), and hence \( f_1 \neq 0 \). So, clearly, singular terms in the expansion (3.11) as \( z \to 0 \) are only those with \( -n - d \leq -1 \), but they all vanish when acting on the vacuum \( |0\rangle \), by Eq. (3.7): \( \phi_n|0\rangle = 0 \) for \( n \geq 1 - d \). Thus the condition (3.8) actually holds. q.e.d.

We can always rewrite \( U_f \) into the normal ordered form up to a multiplicative constant. Then, let the general operator \( U_f \) of the normal ordered form (3.2) act on the ket vacuum:

\[
U_f|0\rangle = \exp\left(\sum_{n \geq 2} v_n L_{-n}\right) \exp\left(\sum_{k=0,\pm 1} v_{-k} L_k\right) \exp\left(\sum_{m \geq 2} v_{-m} L_m\right)|0\rangle
= \exp\left(\sum_{n \geq 2} v_n L_{-n}\right)|0\rangle. \tag{3.12}
\]

where use has been made of \( L_n|0\rangle = 0 \) for \( n \geq -1 \). So, if this equals \( |0\rangle \), we must have \( v_n = 0 \) for \( n \geq 2 \). This implies that \( U_f \) actually has the simple form

\[
U_f = \exp\left(\sum_{k=0,\pm 1} v_{-k} L_k\right) \exp\left(\sum_{m \geq 2} v_{-m} L_m\right) \tag{3.13}
\]

for such \( f(z) \) analytic and invertible around the origin. In the case of an infinitesimal transformation, \( f(z) = z + \delta z \), this result is expected from the beginning, since \( L_{-n} \) is a generator of \( \delta z = z^{-n+1} \), which is singular at \( z = 0 \) for \( n \geq 2 \).

In a similar manner, one can prove that the bra vacuum \( \langle 0| \) remains intact under conformal transformations which have the same properties around the point at infinity.

**Proposition 2** Let \( f(z) \) be a conformal mapping satisfying the same conditions as in Proposition 1. Then the mapping \( g \equiv I \circ f \circ I \) satisfies (schematically
writing) \( g(\infty) = \infty \) and is analytic and invertible in the neighborhood of \( z = \infty \). The corresponding operator \( U_g = U_{1ofoI} \) leaves the \( SL(2; C) \) bra vacuum inert up to a multiplicative constant:

\[
\langle 0 | U_g = e^a \langle 0 | . \tag{3·14}
\]

If \( U_g \) is taken to be of the normal ordered form (3·2), or the conformal anomaly is absent, this constant \( e^a \) equals 1.

Alternatively, this could also be proved as follows if we use the form (3·13) for \( U_f \) and the transformation property of \( L_n \) under inversion, \( I[L_n] = (-1)^n L_{-n} \):

\[
U_g = U_{1ofoI} = I \left[ \exp \left( \sum_{k=0, \pm 1} v_{-k}L_k \right) \exp \left( \sum_{m \geq 2} v_{-m}L_m \right) \right] = \exp \left( \sum_{m \geq 2} (-1)^m v_{-m} L_{-m} \right) \exp \left( \sum_{k=0, \pm 1} (-1)^k v_{-k} L_{-k} \right). \tag{3·15}
\]

Then \( \langle 0 | U_g = \langle 0 | \) is clear, since \( \langle 0 | L_{-n} = 0 \) for \( n \geq -1 \).

Without loss of generality, we can assume both \( h_C \) and \( h_D \) to map the origin of the unit disk to the origins in \( M \) and \( N \), respectively:

\[
h_C(w=0) = 0, \quad h_D(w=0) = 0. \tag{3·16}
\]

This is achieved, if necessary, by applying the \( SL(2; C) \) transformation to \( M \) and \( N \), since the CFT correlation functions are \( SL(2; C) \) invariant. Then, the mappings \( h_C^{-1} \) and \( h_D^{-1} \), as well as \( h_C \) and \( h_D \), satisfy the required properties of the propositions, and so we have

\[
U_{h_C^{-1}} |0\rangle = |0\rangle, \quad \langle 0 | U_{I o h_D^{-1} o I} = \langle 0 | . \tag{3·17}
\]

We shall frequently use the simple formula below which follows from Eq. (2·3) immediately:

\[
\langle f[\mathcal{O}_1] f[\mathcal{O}_2] \cdots \rangle = \langle 0 | U_f \mathcal{O}_1 \mathcal{O}_2 \cdots U_f^{-1} |0\rangle . \tag{3·18}
\]

Now we begin the proof of GGRT, Eq. (2·17). Using the LPP mapping relations (2·18) and the formula (3·18), we first rewrite Eq. (2·16) as

\[
\langle v(\{B_j\}, \{A_i\}) \prod_{i=1}^n |A_i\rangle_{A_i} \prod_{j=1}^m |B_j\rangle_{B_j} \]

\[
= \langle \prod_{i=1}^n h_{A_i} [\mathcal{O}_{A_i}] \prod_{j=1}^m h_{B_j} [\mathcal{O}_{B_j}] \rangle_{M \infty N} \]

\[
= \langle \prod_{i=1}^n g \circ h_C^{-1} \circ h_{A_i} [\mathcal{O}_{A_i}] \prod_{j=1}^m g \circ I \circ h_D^{-1} \circ h_{B_j} [\mathcal{O}_{B_j}] \rangle_{M \infty N} \]

\[
= \langle 0 | U_g \prod_{i=1}^n h_{A_i}^{-1} \circ h_{A_i} [\mathcal{O}_{A_i}] \prod_{j=1}^m I \circ h_D^{-1} \circ h_{B_j} [\mathcal{O}_{B_j}] U_g^{-1} |0\rangle . \tag{3·19}
\]
Inserting the completeness relation (2.9) in the middle here, and applying the formula (3·18) for \( f = h_C^{-1} \) and \( I \circ h_D^{-1} \circ I \), we have

\[
\sum_{\alpha} \langle 0 | U_g \prod_{i=1}^{n} h_C^{-1} \circ h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} | 0 \rangle \langle 0 | I \circ h_D^{-1} \circ h_{B_j} [\mathcal{O}_{B_j}] U_g^{-1} | 0 \rangle
\]

\[
= \sum_{\alpha} \langle 0 | U_g U_{h_C}^{-1} \prod_{i=1}^{n} h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} U_{h_C}^{-1} | 0 \rangle \times \langle 0 | U_{I \circ h_D^{-1} \circ I} I \circ h_D [\mathcal{O}_{\alpha}] \prod_{j=1}^{m} I \circ h_{B_j} [\mathcal{O}_{B_j}] U_{I \circ h_D^{-1} \circ I}^{-1} | 0 \rangle.
\]

But, by Eq. (3·17), we can use \( U_{h_C}^{-1} | 0 \rangle = | 0 \rangle \) and \( \langle 0 | U_{I \circ h_D^{-1} \circ I} = \langle 0 | \) to obtain

\[
\sum_{\alpha} \langle 0 | U_g U_{h_C}^{-1} \prod_{i=1}^{n} h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} | 0 \rangle \times \langle 0 | I \circ h_D [\mathcal{O}_{\alpha}] \prod_{j=1}^{m} I \circ h_{B_j} [\mathcal{O}_{B_j}] U_{I \circ h_D^{-1} \circ I}^{-1} U_g^{-1} | 0 \rangle.
\]

Now suppose that the relations

\[
\langle 0 | U_g U_{h_C}^{-1} = \langle 0 | , \quad U_{I \circ h_D^{-1} \circ I} U_g^{-1} | 0 \rangle = | 0 \rangle,
\]

hold. Then we have

\[
= \sum_{\alpha} \langle 0 | \prod_{i=1}^{n} h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} | 0 \rangle \cdot \langle 0 | I \circ h_D [\mathcal{O}_{\alpha}] \prod_{j=1}^{m} I \circ h_{B_j} [\mathcal{O}_{B_j}] | 0 \rangle
\]

\[
= \sum_{\alpha} \left( \prod_{i=1}^{n} h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} \right) \cdot \left( I \circ h_D [\mathcal{O}_{\alpha}] \prod_{j=1}^{m} I \circ h_{B_j} [\mathcal{O}_{B_j}] \right)
\]

\[
= \sum_{\alpha} \left( \prod_{i=1}^{n} h_{A_i} [\mathcal{O}_{A_i}] h_{C[\mathcal{O}_{\alpha}]} \right) \cdot \left( h_D [\mathcal{O}_{\alpha}] \prod_{j=1}^{m} h_{B_j} [\mathcal{O}_{B_j}] \right),
\]

where use has been made of the invariance of the CFT correlation functions under inversion, i.e., \( \langle I[\mathcal{O}] \rangle = \langle \mathcal{O} \rangle \).\(^{a)}\) But the last expression is identical to the definition of the vertices \( \langle v(C, \{A_i\}) \rangle \) and \( \langle v(D, \{B_j\}) \rangle \) in Eqs. (2·13) and (2·14). So, we obtain

\[
= \sum_{\alpha} \langle v(C, \{A_i\}) | \prod_{i=1}^{n} \mathbb{I}_{A_i} | \alpha \rangle_C \cdot \langle v(D, \{B_j\}) | \tilde{\alpha} \rangle_D \prod_{j=1}^{m} \mathbb{I}_{B_j} \rangle
\]

\[
= \sum_{\alpha} (-1)^{1+|A|+|\alpha|} \langle v(D, \{B_j\}) | v(C, \{A_i\}) | \prod_{i=1}^{n} \mathbb{I}_{A_i} | \alpha \rangle_C \cdot \tilde{\alpha} \rangle_D \prod_{j=1}^{m} \mathbb{I}_{B_j} \rangle
\]

\(^{a)}\) Incidentally, this invariance leads to the identity \( \langle A|B \rangle = (-1)^{|A||B|} \langle B|A \rangle \), since \( \langle A|B \rangle = \langle I[\mathcal{O}_{A}] | \mathcal{O}_{B} \rangle = \langle \mathcal{O}_{A} | I[\mathcal{O}_{B}] \rangle = (-1)^{|A||B|} \langle I[\mathcal{O}_{B}] | \mathcal{O}_{A} \rangle = (-1)^{|A||B|} \langle B|A \rangle \). This implies the symmetry property of the reflector, \( \langle R(1, 2) \rangle = \langle R(2, 1) \rangle \).
\begin{align}
&= (-1)^{1+|A|} \langle v(D, \{B_j\}) \rangle \langle v(C, \{A_i\}) \rangle \prod_{i=1}^{n} |A_i\rangle_{A_i} |R(C, D)\rangle \prod_{j=1}^{m} |B_j\rangle_{B_j} \\
&= - \langle v(D, \{B_j\}) \rangle \langle v(C, \{A_i\}) \rangle |R(C, D)\rangle \prod_{i=1}^{n} |A_i\rangle_{A_i} \prod_{j=1}^{m} |B_j\rangle_{B_j},
\end{align}

where \(|A| \equiv \sum_i |A_i|\) and use has been made of the fact that the (tree level) vertices \(\langle v(C, \{A_i\}) \rangle\) and \(\langle v(D, \{B_j\}) \rangle\) are both fermionic in going to the second line. We have also used the expression (2.11) for the reflector \(|R(C, D)\rangle\) in going to the third line, and the Grassmann oddness of \(|R(C, D)\rangle\) to the last expression. We thus obtain the desired identity by using the anti-symmetry property of \(|R(C, D)\rangle\):

\[\langle v(\{B_j\}, \{A_i\}) \rangle = \langle v(D, \{B_j\}) \rangle \langle v(C, \{A_i\}) \rangle |R(C, D)\rangle = + \langle v(D, \{B_j\}) \rangle \langle v(C, \{A_i\}) \rangle |R(D, C)\rangle.\]

Now it is, therefore, sufficient to show that the relations (3.22) actually hold. Recall that \(g\) is a mapping which smooths out the branch cuts generated in the plane \(h_C^{-1}(M)\cap h_D^{-1}(N)\). So, although the mappings \(g\) as well as \(h_C^{-1}\) and \(h_D^{-1}\) are singular, the composite mapping \(g \circ h_C^{-1}\) from \(M\) to \(M \cap N\), is analytic and invertible outside the image of the unit disk, \(h_C(|w| < 1)\), in the \(M\) plane, and so is the mapping \(g \circ h_D^{-1}\) from \(N\) to \(M \cap N\), outside the image of the unit disk, \(h_D(|w| < 1)\), in the \(N\) plane. Again, using the freedom of the \(SL(2; \mathbb{C})\) transformation in the \(M \cap N\) plane, we can make the mapping \(g\) to satisfy

\[g \circ h_C^{-1}(\infty) = \infty, \quad g \circ \circ \circ h_D^{-1} \circ I(0) = 0.\]

Then, the mappings \(g \circ h_C^{-1}\) and \(g \circ h_D^{-1} \circ I\) are analytic and invertible in the neighborhoods of \(z = \infty\) and \(z = 0\), respectively, and so we can apply the propositions 2 and 1 to obtain

\[\langle 0 | U_{g \circ h_C^{-1}} = \langle 0 |, \quad U_{g \circ h_D^{-1} \circ I} |0\rangle = |0\rangle.\]

These give the desired relations (3.22) if

\[U_{g \circ h_C^{-1}} = U_g U_{h_C^{-1}}, \quad U_{g \circ h_D^{-1} \circ I} = U_g U_{h_D^{-1} \circ I} .\]

These actually hold with weight one when the conformal anomaly is zero. Other than in the critical dimension, there appears a very non-trivial multiplicative \(c\)-number factor by which Eq. (2.17) is violated. This finishes the proof of GGRT.

\section*{§4. GGRT at one loop}

We now prove a generalized version of GGRT at the one-loop level, that is, the theorem for the fused vertex with double contractions by two reflectors, which reads something like

\[\langle v(D, F, \{B_j\}) \rangle \langle v(C, \{A_k\}, E) \rangle |R(D, C)\rangle |R(E, F)\rangle = \epsilon_L \langle v_L(\{B_j\}, \{A_k\}) \rangle .\]
(Note the ordering of the arguments in the two reflectors.) Here the suffix \(L\) denotes quantities at the one-loop level whose more precise definitions will be given in the course of the proof.

Using the GGRT (2·17) at the tree level, we can first rewrite the left-hand side as

\[
\langle v(D, F, \{B_j\}) | R(D, C) | R(E, F) \rangle = \langle v(F, \{\Phi_i\}, E) | R(E, F) \rangle,
\]

with the abbreviation \(\{\Phi_i\}\) denoting the combined set of states \(\{B_j\}\) and \(\{A_k\}\), where \(\langle v(F, \{\Phi_i\}, E) \rangle\) is a tree level LPP vertex obtained by the fusion of the two vertices by a single contraction \(|R(D, C)|\). Since this vertex \(\langle v(F, \{\Phi_i\}, E) \rangle\) is a tree level vertex, it corresponds to a plane, which we call \(M\), and there are mappings \(h_F\), \(h_E\) and \(h_{\phi_i}\) which map the unit disks of the strings \(F\), \(E\) and \(\phi_i\) into the plane \(M\), analytic and invertible inside each unit disk, respectively. Using also the expression (2·11) for the ket reflector, we can write the right-hand side of Eq. (4·2) as

\[
\langle v(F, \{\Phi_i\}, E) | R(E, F) \rangle \prod_i |O_{\phi_i}\rangle_{\phi_i} = \sum_\alpha (-1)^{\alpha} \langle v(F, \{\Phi_i\}, E) | \alpha \rangle_E |\bar{\alpha}\rangle_F \prod_i |O_{\phi_i}\rangle_{\phi_i},
\]

\[
= \sum_\alpha (-1)^{\alpha} \langle h_E [O_{\alpha}] h_F [\bar{O}_{\bar{\alpha}}] \prod_{i=1}^n h_{\phi_i} [O_{\phi_i}] \rangle_M.
\]

Using the freedom of \(SL(2; C)\), we can assume without loss of generality that

\[
h_E(w=0) = 0, \quad h_F(w=0) = \infty.
\]

These mappings are schematically shown in Fig. 2. Further contraction by \(|R(E, F)|\) in Eq. (4·2), or summation over \(\alpha\) in Eq. (4·3), corresponds to the gluing of the two boundaries \(h_E(|w|=1)\) and \(h_F(|w|=1)\) in this plane \(M\). This makes the plane a torus, which we call \(M8\). The torus \(M8\) can be represented by a complex plane with the identification

\[
z \sim q^2 z, \quad \exists q \equiv e^{i\pi\tau}.
\]

This means that there is a smooth mapping of \(M\) into the torus plane \(M8\), and there are mappings \(h_{\phi_i}\) of the unit disks to \(M8\) which are analytic and invertible inside each unit circle.

In string field theory, this mapping can be decomposed into the following steps in a manner very similar to that in the case of LPP at tree level. As shown in Fig. 3,
the complex plane $M$ is mapped in two ways, one by $h_{E}^{-1}$ and the other by $I \circ h_{F}^{-1}$, so that the exterior region of string $E$ in $M$ is mapped to the region outside the unit circle and the exterior region of string $F$ in $M$ to the inside of the unit circle, respectively. Then the region outside the unit circle in the plane $h_{E}^{-1}(M)$ and the region inside the unit circle in the plane $I \circ h_{F}^{-1}(M)$ are glued smoothly as they stand. But, again, the glued surface generally possesses branch cuts unless the mappings $h_{C}$ and $h_{D}$ are $SL(2; C)$ transformations. Since we know that the glued surface is a covering space of a torus in any case, there exists a mapping $g$ which carries the surface into the torus plane $M8$ (the plane with identification $z \sim q^{2}z$), smoothing out the branch cuts. Therefore the conformal mappings $h_{\Phi_{i}}$ of the strings $\{\Phi_{i}\}$ into the torus plane $M8$, mentioned above, are identified with

$$
\hat{h}_{\Phi_{i}} = g \circ h_{E}^{-1} \circ h_{\Phi_{i}}.
$$

But, in this loop case, the mappings via the other route, $g \circ I \circ h_{F}^{-1} \circ h_{\Phi_{i}}$, should be
equally good mappings. Indeed, the whole region $R$ in $M$ outside the two images of the unit disks of strings $E$ and $F$ is mapped to the two adjacent regions in $M^{8}$ displaced with a period $q^{2}$ if we follow the two routes of mappings, $g \circ h^{-1}_{E}$ and $g \circ I \circ h^{-1}_{F}$. This can be easily seen by inspecting Fig. 3. That is, we have the following equation for $\forall z \in R$ in $M$:

$$g \circ I \circ h^{-1}_{F}(z) = q^{2} \times g \circ h^{-1}_{E}(z). \quad (4.7)$$

This is a key relation in this one-loop case. This is purely a c-number relation between the two conformal mappings. The corresponding operator relation of course reads

$$U_{g \circ I \circ h^{-1}_{F}} = U_{q^{2} \circ g \circ h^{-1}_{E}}, \quad (4.8)$$

where $q^{2}$ as a mapping denotes $q^{2}(z) = q^{2} \cdot z$. We know that the operator representation of this Weyl transformation $q^{2}$ is given by $U_{q^{2}} = q^{2}L_{0}$. Therefore, if the conformal anomaly is zero, we can use the composition law for the group elements freely to obtain

$$U_{g \circ I \circ h^{-1}_{F}} = q^{2}L_{0} \quad U_{g \circ h^{-1}_{E}}^{-1} \quad \rightarrow \quad U_{h^{-1}_{F} \circ I} = U_{I \circ h^{-1}_{F}}^{-1} = U_{g}^{-1} \quad q^{2}L_{0} \quad U_{g \circ h^{-1}_{E}}^{-1}. \quad (4.9)$$

If $c \neq 0$, there will appear non-trivial multiplicative c-number factors in these equations.

With these equations in hand, we can now prove the GGRT at one loop as follows. We rewrite Eq. (4.3) as

$$\sum_{\alpha} (-1)^{|\alpha|} \left< h_{E}[\alpha] \ h_{F}[\alpha_{1}] \ \prod_{i=1}^{n} \ h_{\Phi_{i}}[\alpha_{1}] \right>_{M}$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \epsilon' \left< h_{F}[\alpha] \ \prod_{i=1}^{n} \ h_{\Phi_{i}}[\alpha_{1}] \ h_{E}[\alpha] \right>_{M}$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \epsilon' \left< 0 \right| U_{h^{-1}_{F} \circ I} I[\alpha] U_{h^{-1}_{F} \circ I}^{-1} \ \prod_{i=1}^{n} \ h_{\Phi_{i}}[\alpha_{1}] \ h_{E}[\alpha] \left| 0 \right> \quad (4.10)$$

with the sign factor $\epsilon' = (-1)^{|\alpha|(|\alpha|+\sum_{i} |\Phi_{i}|)}$. Fortunately, however, this sign factor is 1 since $|\alpha| + |\alpha_{1}| + \sum_{i} |\Phi_{i}| = 1$ mod 2, in order for the correlation function in $M \sim S^{2}$ to be non-zero, implying $|\alpha| (|\alpha| + \sum_{i} |\Phi_{i}|) = |\alpha| (|\alpha| + 1) = 0$ mod 2. Here, since $h_{F} \circ I(\infty) = \infty$ and $h_{F} \circ I$ is analytic and invertible in the neighborhood of $z = \infty$, we can use $\left< 0 \right| U_{h^{-1}_{F} \circ I} = \left< 0 \right|$ by Proposition 2. Moreover, we can use the above key relation (4.9) when $c = 0$. Then we further proceed as follows:

$$= \sum_{\alpha} (-1)^{|\alpha|} \left< 0 \right| I[\alpha] U_{g}^{-1} q^{2}L_{0} U_{g \circ h^{-1}_{E}}^{-1} \ \prod_{i=1}^{n} \ h_{\Phi_{i}}[\alpha_{1}] \ h_{E}[\alpha] \left| 0 \right>$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \left< 0 \right| I[\alpha] U_{g}^{-1} q^{2}L_{0} U_{g \circ h^{-1}_{E}}^{-1} U_{h_{E}} \ \prod_{i=1}^{n} \ h_{\Phi_{i}}[\alpha_{1}] \ h_{E}[\alpha] \ h_{E}^{-1} \left| 0 \right> \quad (4.10)$$
\[
= \sum_{\alpha} (-1)^{\alpha} \langle 0 | I[\mathcal{O}_{\alpha}] U_g^{-1} q^{2L_0} U_g \prod_{i=1}^{n} h_E^{-1} \circ h_{\Phi_i}[\mathcal{O}_{\Phi_i}] \mathcal{O}_{\alpha} | 0 \rangle \\
= s\text{Tr} \left[ U_g^{-1} q^{2L_0} U_g \prod_{i=1}^{n} h_E^{-1} \circ h_{\Phi_i}[\mathcal{O}_{\Phi_i}] \right]. \quad (4·11)
\]

Here, in going to the third line, we have used \( U^{-1}_h | 0 \rangle = | 0 \rangle \) from Proposition 1 and \( U g \circ h^{-1}_E U h_E = U_g \), which holds again when \( c = 0 \). In obtaining the last expression we have used the definition

\[
s\text{Tr}[\cdots] = \sum_{\alpha} (-1)^{\alpha} \langle 0 | I[\mathcal{O}_{\alpha}] \cdots \mathcal{O}_{\alpha} | 0 \rangle = \sum_{\alpha} (-1)^{\alpha} \langle \hat{\alpha} | \cdots | \alpha \rangle. \quad (4·12)
\]

It may seem strange to call this a ‘definition’ since this \( s\text{Tr} \) is of course just the usual trace for the bosonic mode sector and the usual super trace for the ghost non-zero mode sector. However, the trace operation for the ghost zero-mode sector is not self-evident (as will be explained later), and Eq. (4·12) gives its definition. The usual cyclic identity for bosonic operators also holds for this trace.

It is now possible to rewrite the last trace expression into the final form:

\[
= s\text{Tr} \left[ U_g^{-1} q^{2L_0} U_g \prod_{i=1}^{n} h_E^{-1} \circ h_{\Phi_i}[\mathcal{O}_{\Phi_i}] \right] = s\text{Tr} \left[ q^{2L_0} U_g \prod_{i=1}^{n} h_E^{-1} \circ h_{\Phi_i}[\mathcal{O}_{\Phi_i}] U_g^{-1} \right] \\
= s\text{Tr} \left[ q^{2L_0} \prod_{i=1}^{n} g \circ h^{-1}_E \circ h_{\Phi_i}[\mathcal{O}_{\Phi_i}] \right] = s\text{Tr} \left[ q^{2L_0} \prod_{i=1}^{n} \hat{h}_{\Phi_i}[\mathcal{O}_{\Phi_i}] \right], \quad (4·13)
\]

where the mapping relation (4·6) has been used.

Thus, if we define the CFT correlation function on the torus by

\[
\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle_{\text{torus \ \tau}} = s\text{Tr}[q^{2L_0} \mathcal{O}_1 \mathcal{O}_2 \cdots], \quad (4·14)
\]

and the LPP vertex at one-loop level by

\[
\langle v_L(\{\Phi_i\}; \tau) | \prod_i | \mathcal{O}_{\Phi_i} \rangle_{\text{torus \ \tau}} = \left\langle \prod_i \hat{h}_{\Phi_i}[\mathcal{O}_{\Phi_i}] \right\rangle_{\text{torus \ \tau}}, \quad (4·15)
\]

then, what we have proved is summarized in the following GGRT at one loop, by Eqs. (4·2), (4·3), (4·10) and (4·13).

**Theorem** When the conformal anomaly is zero, the fused vertex obtained by twice contracting with two reflectors, equals the LPP vertex at the one-loop level:

\[
\langle v(D, F, \{B_j\}) | v(C, \{A_k\}, E) | R(D, C) | R(E, F) \rangle = \langle v_L(\{B_j\}, \{A_k\}; \tau) \rangle. \quad (4·16)
\]

If the conformal anomaly is present, this equality (4·16) is violated by a multiplicative \( c \)-number factor.

A few remarks are in order here:

We noted before that the arguments of the LPP vertex can be written in an arbitrary order (as far as the cyclic order is kept among the open-string arguments),
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thanks to our convention taking the $SL(2; C)$ ket vacuum $|0\rangle$ Grassmann even. If we take this into account, we can rewrite the GGRT formulas, Eq. (2·17) at the tree level and Eq. (4·16) at the loop level, respectively, into the following forms which look more natural and are also helpful as mnemonics:

$$\langle v({\{B_j}\}, D) | v(C, \{A_i\}) | R(D, C) \rangle = \langle v({\{B_j\}, \{A_i\}}) \rangle ,$$

$$\langle v({\{B_j}\}, D, F) | v(E, C, \{A_i\}) | R(F, E) \rangle | R(D, C) \rangle = \langle v_L({\{B_j\}, \{A_k\}}; \tau) \rangle .$$

(4·17)

If we define correlation functions on a torus for a system possessing non-zero central charge $c$, it is known better to replace $q^{2L_0}$ in Eq. (4·14) with $q^{2(L_0-c/24)}$:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \rangle_{\text{torus}} \tau = sTr[q^{2(L_0-c/24)} \mathcal{O}_1 \mathcal{O}_2 \cdots] .$$

(4·18)

This operator $L_0 - c/24$ can be identified with $(L_0)_{\text{cylinder}}$ on the cylinder ($\rho$-plane), and $-c/24$ comes from the Schwarzian derivative term in the anomalous transformation law of the energy momentum tensor under the coordinate change $\rho \rightarrow z = e^\rho$. Indeed, with this factor $(q^2)^{-c/24}$, the vacuum functional (partition function) becomes invariant under the modular transformation $\tau \rightarrow -1/\tau$, and the other correlation functions also come to have more natural modular transformation properties. However, note that this ‘improvement’ does not cure at all the violation of the above GGRT when the net central charge is non-zero: the multiplicative $c$-number factor violating the theorem is a number which depends on all the details of the mappings and moduli parameters which cannot be cancelled by such a simple factor like $(q^2)^{-c/24}$. If the net central charge is zero, then one may of course calculate the correlation functions for each sector possessing central charges separately, by using the improved definition Eq. (4·18).

It is interesting that the super trace formula Eq. (4·12) appeared automatically in our derivation. It is by no means a priori clear how the trace should be defined for the ghost zero-mode sector, since it has an off-diagonal metric structure:

$$\langle 2|1\rangle = \langle 1|2\rangle = \langle \mathcal{O}_1 | c_0 | \mathcal{O}_1 \rangle = 1 ,$$

(4·19)

where $|1\rangle \equiv c_1 |0\rangle \equiv |\mathcal{O}_1\rangle$ and $|1\rangle \equiv |0\rangle c_{-1} \equiv |\mathcal{O}_1\rangle$ are ket and bra Fock vacua, and $|2\rangle \equiv c_0 |\mathcal{O}_1\rangle$ and $|2\rangle \equiv |\mathcal{O}_1 | c_0 \rangle$. If we follow the definition (4·12) of the supertrace $sTr$, then noting that $|\alpha\rangle = 1$ and $0$ for $|\alpha\rangle = |1\rangle$ and $|2\rangle$, respectively, the trace in the ghost zero-mode sector, denoted by $sTr_0$, is calculated as

$$sTr_0[1] = (-1)^1 \langle 2|1|1\rangle + (-1)^0 \langle 1|1|2\rangle = -1 + 1 = 0 ,$$

$$sTr_0[c_0] = (-1)^1 \langle 2|c_0|1\rangle + (-1)^0 \langle 1|c_0|2\rangle = 0 + 0 = 0 ,$$

$$sTr_0[b_0] = (-1)^1 \langle 2|b_0|1\rangle + (-1)^0 \langle 1|b_0|2\rangle = 0 + 0 = 0 ,$$

$$sTr_0[c_0b_0] = (-1)^1 \langle 2|c_0b_0|1\rangle + (-1)^0 \langle 1|c_0b_0|2\rangle = 0 + 1 = 1 ,$$

(4·20)

and $sTr_0[b_0c_0] = sTr_0[1 - c_0b_0] = -1$, of course. These equations precisely demonstrate that the ghost correlation functions on the torus vanish unless both $c_0$ and $b_0$...
modes appear at least once, in conformity with the fact that there is one zero mode each for \( c(z) \) and \( b(z) \) in the torus case (that is, a conformal Killing vector and a holomorphic quadratic differential, respectively).

It may be noted that the supertrace can also be rewritten into a form given by Freedman et al.,\(^{10}\)

\[
s\text{Tr}[\cdots] = -\text{Tr}[-1]^N_{\text{FP}} \cdots], \tag{4.21}
\]

in terms of the ‘usual’ trace \( \text{Tr} \) (with the understanding that \( \text{Tr}[\mathcal{O}] \equiv \langle 2|\mathcal{O}|1 \rangle + \langle 1|\mathcal{O}|2 \rangle \) in the zero mode sector) and the FP ghost number defined by

\[
N_{\text{FP}} = c_0 b_0 + \sum_{n \geq 1} (c_{-n} b_n - b_{-n} c_n), \tag{4.22}
\]

which counts the ghost number from the Fock vacuum.

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