Toroidal fluid motion at the top of the Earth’s core

D. Lloyd and D. Gubbins
Department of Earth Sciences, University of Cambridge, Bullard Laboratories, Madingley Rise, Madingley Road, Cambridge CB3 0EZ, UK

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SUMMARY

Geomagnetic secular variation is caused by flow of liquid iron in the core. Geomagnetic observations can be used to determine properties of the flow but such calculations in general have non-unique solutions. We prove a uniqueness theorem: the flow is determined uniquely if it is toroidal (zero horizontal divergence), the mantle is an insulator, the core a perfect conductor (the frozen-flux hypothesis), and there is no surface current in the boundary layer at the top of the core, and provided the magnetic field satisfies a simple point condition. The condition of no surface current allows use of the horizontal components of secular variation; previous studies have used only the radial component. Horizontal components allow simultaneous determination of the shear (radial derivatives of horizontal components of velocity).

We have devised a new numerical method for determining core flows based on a discrete vector spherical transform (DVST) which exactly transforms between a grid of points on the surface of a sphere and a truncated spherical harmonic series. The transform method is faster than methods involving Gaunt and Elsasser integrals and eliminates most of the lengthy algebra. We determine a flow from spherical harmonic models of main field and secular variation for epoch 1970 which satisfies more than 95 per cent of the weighted radial component of secular variation and more than 94 per cent of the entire secular variation vector. This fit is considerably better than can be achieved by geostrophic flows and only slightly worse than by general steady motions; the improvement in adding poloidal motions is comparable with that in relaxing the frozen flux hypothesis.

The main features of the flow are two gyres in the Atlantic hemisphere placed almost symmetrically about the equator with strong equatorial westerly flow and return easterly flows in high latitudes; it is similar to other published steady flows based on only the radial component of secular variation and severe damping. There is very little flow in the Pacific region. The shear indicates a radial length scale of about 600 km. It is very similar in form to the velocity and its sign is such that the velocity weakens with depth, as would be expected if the flow were driven from the core–mantle boundary rather than by convection from below. The overall pattern is consistent with flow in a density-stratified layer driven by lateral temperature gradients in the lower mantle, although the result allows other scenarios.

Key words: geomagnetism, toroidal core flows.

1 INTRODUCTION

The Earth’s magnetic field changes slowly due to the inductive effect of motion of liquid iron in the core. This secular variation of the magnetic field can be measured at a permanent observatory in one or two years, and can be detected in less accurate survey measurements after about a decade. The main features of secular variation are a slow fall in overall field strength, which has probably persisted since Roman times (Merrill & McElhinny 1983), and a westward drift in the Atlantic region. Attention has focused recently on determining estimates of the fluid motions in the core responsible for the secular variation, usually through inversions of models of the magnetic field and its time changes.

The best approach is to invert the magnetic observations...
themselves. However, it is much more convenient to use mathematical models of the magnetic field. We now have good models of the time-dependent field for the period 1820–1980 expressed as spherical harmonic series to degree 14 and Chebycheff polynomials to degree 10 in each of two time intervals, 1820–1900 and 1900–1980 (Bloxham 1987; Bloxham & Jackson 1989). These models fit virtually all of the available independent geomagnetic observations made since 1820 and incorporate covariance matrices for the coefficients. Thus any good statistical fit to the coefficients should produce a realistic fit to the original data; calculations made with the field models will be much quicker than inversion of the original measurements, which number over half a million. In fact the results of such a study may indicate a study involving original data would be unnecessary. All studies to date have used field models rather than original data except that of Whaler (1986) who used a very small data set.

We separate the mathematical problem of determining fluid velocity from magnetic field and its rate of change from the statistical problem of finding a satisfactory velocity from noisy magnetic field estimates and discuss the mathematical problem first. Fluid motion, \( v \), of a liquid metal with conductivity, \( \sigma \), induces magnetic field, \( B \), according to the induction equation (for example Jackson 1975)

\[
\dot{B} = \nabla \times (v \times B) + \left( \mu_0 \sigma \right)^{-1} \nabla^2 B
\]  
(1)

where \( \mu_0 \) is the permeability of free space and dot denotes differentiation with respect to time. Equation (1) applies at the core surface but the magnetic field is measured at the Earth's surface: the mantle is assumed an electrical insulator. They analysed the boundary layer and took the velocity inferred from frozen flux theory to be the flow at the top of the free stream, just beneath the boundary layer. The radial component of magnetic field will be continuous across the boundary layer, so that our estimates of \( B_r \) and its horizontal derivatives based on surface observations will apply at the top of the free stream, but if a surface current flows in the boundary layer there will be discontinuities in the horizontal components \( B_\theta \) and \( B_\phi \). Roberts & Scott (1965) argue that, in the limit \( \rho_m = \nu/\eta < 1 \) which is pertinent to the core, there are no appreciable surface currents in the boundary layer and all three components of \( B \) will be continuous across the boundary layer. Backus (1968) also investigated the boundary layer and came to no firm conclusion about the significance of the surface current, but in a subsequent analysis Hide & Stewartson (1972) concluded the jump in horizontal component of field would be negligible. Only Braginsky (1984) has produced a formulation for the secular variation which involves induction in the boundary layer; his formulation reduces to that of Roberts & Scott (1965) in the limit of perfect core conductivity.

The determination of \( v \) is not unique even with the frozen-flux hypothesis. Backus (1968) gives a careful study of the ambiguity and provides consistency tests for validity of the hypothesis and for continuity of horizontal field across the boundary layer. The ambiguity can be removed by imposing additional conditions on \( v \); several constraints have been tried including ad hoc regularization of the solution (Kahle, Vestine & Ball 1967) and dynamical constraints such as time independence (Voorhies & Backus 1985; Voorhies 1986a,b; Bloxham 1987; Whaler & Clarke 1988); geostrophy (LeMouel 1984; LeMouel, Gire & Madden 1985; Backus & LeMouel 1986), and stratification (Whaler 1980; Gubbins 1982; Bloxham 1988b). All these determinations use only the radial component of the induction equation.

Each condition can be considered for its physical intuitive appeal, or how well it suits our favoured view of core dynamics. Steady motions might be expected if the driving forces change on a longer time scale than the magnetic field (this presumably requires changes in magnetic forces associated with secular variation to be negligible); geostrophic flows are expected if the contributions of buoyancy and magnetic forces to the radial component of the vorticity equation are negligible, which are quite plausible assumptions; and density stratification, allowing only toroidal motions, would arise if the temperature at the top of the core were sub-adiabatic, as happens in some models of the Earth's thermal history (Gubbins, Thomson & Whaler 1982), or light material is concentrated near the core surface (Fearn & Loper 1981).

Each condition may also be considered for its statistical appeal, or how well it performs in determining satisfactory unique core flows. The steady motion condition may be regarded as 'strong' because it provides uniqueness subject to the non-vanishing of a certain determinant, although this does not ensure availability of the required data (good estimates of the field are needed over times that are long enough to observe significant magnetic changes (Gubbins 1982; Voorhies & Backus 1985). The geostrophic condition is weaker because it only provides uniqueness over part of the CMB (Backus & LeMouël 1986), as is the toroidal...
condition, which allows determination of only one component of flow everywhere on the CMB except at critical points in $B$. (Roberts & Scott 1965). The relative 'strength' of geostrophic and toroidal conditions is unclear; both reduce the problem to that of determining a single scalar function of position.

In theory we can evaluate each condition by its consistency with observation, but in practice it is exceedingly hard to make a convincing case. There are two ways to demonstrate acceptability of a condition: the simplest is to find a core velocity that fits the condition and exceedingly hard to make a convincing case. There are and determine if any velocity model satisfying the constraint, find error estimates on the velocity model, and determine if any velocity model satisfying the constraint lies acceptably close. The two procedures are equivalent if done consistently. Unfortunately all of the velocity determinations performed so far have incorporated some ad hoc regularization, either in the form of severely truncated spherical harmonic expansions or by minimizing some norm of the velocity model, which guarantees uniqueness even without the dynamical constraint. It is therefore not possible to decide for the calculations done so far, whether the dynamical constraint has any effect on the solution.

For example, Bloxham (1988a) concludes 'the steady motions approximation may be reasonable over quite long time spans' but he does not show how large the time variations would have to be before they could be detected by his data; his conclusion could be restated in the more negative form 'this study does not have the resolution to determine changes in velocity'. There is considerable confusion in the literature over steady motions: Gubbins (1984) claimed a change in velocity during the geomagnetic 'jerk' of 1970 and was supported by Gire, Le Mouël & Madden (1986) who found the toroidal part of the motion to increase by a factor of two after the jerk, but Voorhies (1986a) found a satisfactory steady flow for a similar period and was supported by Bloxham (1988b) for a longer interval; Whaler & Clarke (1988) perform a statistical test which finds against the steady motion hypothesis but they further argue that the test may be invalid because of uncertainties in the velocity error estimates.

Confusion also surrounds investigations of geostrophic and toroidal motions. Whaler (1986) found significant poloidal motion from recent secular variation data and Voorhies (1986a) found 'the hypothesis of no upwelling ... is substantially worse than that of steady motion', while Bloxham (1988b) found 'better fits to the fields' (at different epochs) 'with toroidal flows than geostrophic motions to be non-steady but Bloxham (1988b) found Gire et al.'s (1986) motions to be ageostrophic at high degree; Gire & Le Mouël (1989) have now produced fully geostrophic flows and confirmed their earlier conclusions.

Part, but by no means all, of this confusion arises through failure of the frozen-flux hypothesis. Bloxham & Gubbins (1986) showed Backus' (1968) consistency conditions were violated by field changes between three field models at epochs 1959.5, 1969.5 and 1980.0, implying flux diffusion in the South Atlantic region. This result provoked a number of critiques of the use of model covariances based on Bayesian inference and stochastic inversion, the most recent of which is Backus (1988). These comments apply to most geophysical inversions which use Bayesian error estimates; a recent defence of their use for the geomagnetic problem is given in Bloxham, Gubbins & Jackson (1989). Bloxham & Gubbins (1985) give older maps of core fields which show that recent flux diffusion is part of a much larger long-term trend, starting at the beginning of the 20th century, associated with the region of very rapid secular change located off the coast of southern Africa today. The effects of flux diffusion on the century time scale certainly cannot be ignored; evidence for it is as strong as for any other feature of secular variation.

In this paper we view the frozen-flux hypothesis as a first approximation which is bound to fail if the observations are improved because diffusion affects magnetic fields of all time scales at sufficiently short wavelength and at all wavelengths at sufficiently long times. We also regard any dynamical constraint as a first approximation. We choose to investigate the toroidal flows for the following two reasons.

(1) The core fluid may be stably stratified, in which case toroidal flows are the only ones allowed. In the atmosphere the geostrophic constraint is applied locally and in the $\beta$-plane approximation it reduces to the toroidal motion condition for a stratified atmosphere (Pedlosky 1979, p. 323).

(2) Diffusion in the south Atlantic region, the only place where diffusion is obvious, is believed to be caused by flux expulsion (Bloxham & Gubbins 1985; Bloxham 1986). This would imply that the determination of poloidal flow should also include the effects of diffusion since they have the same order of effect on the secular variation.

Toroidal motions are thus a simple zeroth-order approximation. In future work we intend to simultaneously incorporate diffusion and upwelling. The present objective is simply to explain as much of the secular variation as possible with toroidal flow of a perfect conductor. Unlike previous authors, we use all three components of the induction equation. The influence of a surface current and concomitant jumps in horizontal components of field across the boundary layer have been investigated by Barralough, Gubbins & Kerridge (1989) and found negligible; it can also be assessed post hoc by the success of the theory in fitting the observations. We first prove, in Section 2, that toroidal motions are determined uniquely by all three components of $B$. In Sections 3 and 4 we introduce a new method of determining core flow which is computationally faster and more flexible than the usual method involving Gaunt and Elsasser integrals. A candidate velocity for epoch 1970 is determined and discussed in Section 6.

2 UNIQUENESS OF TOROIDAL MOTIONS

In the frozen flux approximation we write the induction equation (1) as

$$\dot{B} = (B \cdot V_\beta)v - (v \cdot V_\beta)B + B_v v'.$$

We assume that the core fluid is incompressible ($\nabla \cdot v = 0$), that there is no radial flow at the CMB ($v_r = 0$), and in the toroidal hypothesis, that there is no upwelling just under the
CMB (\(V_{1t} \cdot \mathbf{v} = 0\)), and write
\[ v'_r = 0. \]

From these assumptions, the radial component of (3) becomes
\[ \dot{B}_r = -\mathbf{v} \cdot \nabla_t B_r. \]

Equation (5) may be used to determine the component of \(v\) normal to contours of \(B_r\) on the CMB
\[ v_\perp = \frac{-\dot{B}_r}{|\nabla_t B_r|}. \]

Following Backus (1968) we now define an orthogonal curvilinear coordinate system \((r, \xi_2, \xi_3)\) where \(r\) is the usual radial coordinate, \(\xi_2 = B_r\), and \(\xi_3\) increases monotonically along contours of \(\dot{B}\). There are poles in \(\xi_2\) at maxima and minima of \(B_r\), and in \(\xi_3\) at saddle points of \(B_r\). Unit vectors \(\mathbf{e}_2\) and \(\mathbf{e}_3\) are defined in the usual way (for example \(\mathbf{e}_2 = \nabla_t B_r / |\nabla_t B_r|\)). The scale factors for these coordinates are
\[ h_r = 1; \ h_2 = \frac{1}{|\nabla_t B_r|}. \]
We leave \(h_3\) unspecified as it will not be needed.

Equation (6) may now be written as
\[ v_2 = -h_2 \dot{B}_r. \]

If we know only the radial component of \(B\) then we can only find \(v_2\): any toroidal velocity \((0, 0, v_3)\) will be an annihilator for the problem and in Section 6 we demonstrate how seriously this can affect an inversion for a velocity. However, if all three components of \(B\) are known everywhere on the core surface, as we are going to assume, we can find all horizontal derivatives of \(B\) and \(B'_r\) which allows us to find the radial derivative of (8)
\[ v'_2 = -(h_2 \dot{B}_r)' = -h_2 \dot{B}_r' + \frac{\dot{B}_r}{|\nabla_t B_r|} |\nabla_t B_r|'. \]

We now look at the 2-component of the induction equation (3)
\[ \dot{B}_2 = \left( \mathbf{B} \cdot \nabla_t v_2 + \frac{B_2 v_3 h_3}{h_2 h_3} \frac{\partial h_2}{\partial \xi_3} \right) \]
\[ - \left( \mathbf{v} \cdot \nabla_t B_2 + \frac{B_2 v_3 h_3}{h_2 h_3} \frac{\partial h_2}{\partial \xi_3} \right) + B_2 v'_2 \]
\[ = \mathbf{B} \cdot \nabla_t v_2 - \frac{v_2}{h_2} \left( \frac{\partial B_2}{\partial \xi_3} + \frac{B_2 h_3}{h_2} \frac{\partial h_3}{\partial \xi_3} \right) \]
\[ + B_2 v'_2 - v_3 \frac{h_2}{h_3} \frac{\partial B_2}{\partial \xi_3}. \]

Since we already know \(v_2\) and \(v'_2\) and all necessary derivatives of \(B\), we can obtain \(v_3\) from equation (10) provided the condition
\[ \frac{h_2}{h_3} \frac{\partial}{\partial \xi_3} \left( \frac{B_2}{h_2} \right) \neq 0 \]
\[ \text{(11)} \]
is satisfied. By noting that
\[ B_2 = \frac{\mathbf{B} \cdot \nabla_t B_r}{|\nabla_t B_r|} \]
\[ \text{(12)} \]
and that
\[ \frac{1}{h_3} \frac{\partial}{\partial \xi_3} = \mathbf{e}_3 \cdot \nabla_t = (\mathbf{e}_x \times \mathbf{e}_2) \cdot \nabla_t \]
we can write the condition (11) in vector notation as
\[ (\mathbf{e}_x \times \nabla_t B_r) \cdot \nabla_t (\mathbf{B} \cdot \nabla_t B_r) \neq 0. \]

The 3-component of the induction equation (3) is
\[ \dot{B}_3 = \left( \mathbf{B} \cdot \nabla_t v_3 + \frac{B_3 v_2 h_2}{h_2} \frac{\partial h_2}{\partial \xi_2} \right) \]
\[ - \left( \mathbf{v} \cdot \nabla_t B_3 + \frac{B_3 v_2 h_2}{h_2} \frac{\partial h_2}{\partial \xi_2} \right) + B_3 v'_3 \]
\[ = \mathbf{B} \cdot \nabla_t v_3 - \frac{v_3}{h_3} \left( \frac{\partial B_3}{\partial \xi_2} + \frac{B_3 h_2}{h_2} \frac{\partial h_2}{\partial \xi_2} \right) \]
\[ + B_3 v'_3 - v_2 \frac{h_2}{h_3} \frac{\partial B_3}{\partial \xi_2}. \]

This equation can be used to determine \(v'_3\) once \(v_3\) is determined from equation (10), provided \(B_r \neq 0\). Note that we cannot find \(v'_3\) by differentiating \(v_3\) directly because the corresponding radial derivatives of \(B\) and \(B_r\) are not known at the core surface.

It is likely that the expression on the left-hand side of (14) will vanish along lines on the core surface but we can invoke continuity and horizontal differentiability of all components of \(v\) and \(B\) and derive a solution on such lines. There will, however, be a consistency condition on the observations at places where (14) fails obtained from (10) by setting the last term on the right-hand side to zero.

Condition (14) is sufficient to find a unique \(v_3\), but is it necessary? If the condition fails, we are left with only equation (15) to determine both \(v_2\) and \(v'_2\); clearly a unique solution for \(v_3\) cannot be found unless \(B_r = 0\). The case \(B_r = 0\) was considered by Backus (1968), who showed that \(v_3\) could be determined along such null-flux curves without the toroidal hypothesis, and also by Barraclough et al. (1989), who used recent field models to show the observations lacked the accuracy required for a meaningful determination of \(v_3\). Therefore we can say that the condition (14) is necessary except on null-flux curves.

We illustrate the uniqueness proof with two examples. We show first that any axisymmetric \(B\) fails condition (14) and does not allow determination of azimuthal flow. In this case contours of \(B\), are circles of latitude and so we can choose \(\xi_3 = \phi\). Symmetry implies that \(h_3 = 1/|\nabla_t B_r|\) and \(B_2\) are independent of \(\phi\) and hence \(B_2/h_2\) is independent of \(\phi\). Thus
\[ \frac{\partial}{\partial \xi_3} \left( \frac{B_2}{h_2} \right) = 0 \]
in violation of condition (11). \(v_3 (=v_\phi, \text{the azimuthal flow})\) cannot be found. This arises physically because rotation of an axisymmetric field around its axis induces no secular variation.

We next present a simple example in which the horizontal components of magnetic field resolve the ambiguity in the flow left by the radial component of (3). Consider a quadrupole poloidal magnetic field written \(S_{i}'(\mathbf{e}_x + \mathbf{e}_2)\) and \(T_{i}'(\mathbf{e}_x + \mathbf{e}_2)\) are poloidal and toroidal fields derived from a scalar...
field \( P^T_r(\cos \theta) Y_{nm}^r(m \phi) \) in the notation of Bullard and Gellman (1954). A rotating fluid (i.e. toroidal velocity \( T_r^T \)) induces a quadrupole poloidal field \( S^T_r \) and toroidal fields \( T_r^T, \ T_r^T \). Fluid flow along the contours of \( B_r \), (i.e. toroidal velocity \( T_r^T \)) will induce no poloidal field at all and since \( B_r \) is dependent only on poloidal field, this flow is an annihilator for the radial component of the induction equation. However the velocity \( T_r^T \) also interacts with the magnetic field to give toroidal fields \( T_r^T, \ T_r^T \) which will contribute to \( B_r \) and \( B_r \). Hence the horizontal components of the induction equation can be used to resolve the flow between \( T_r^T \) and \( T_r^T \).

In conclusion condition (14) will be violated only in exceptional special cases, and since the real geomagnetic field is highly asymmetric, we believe the condition for uniqueness will be satisfied almost everywhere and allow derivation of a unique toroidal motion from the geomagnetic field and its secular variation at any epoch, provided sufficiently accurate data consistent with the constraints are available.

3 STRATEGY FOR CALCULATING THE VELOCITY

A algorithm for determining \( v \) derived from (8) and (10) is not suitable for numerical implementation: it relies on a coordinate system dependent upon \( B_r \), which is likely to be noisy, and instabilities will develop when the left-hand side of (11) is small. A least-squares fit of velocity parameters to secular variation will be much more satisfactory.

We need finite representations for the secular variation \( \dot{B}(\theta, \phi) \), the velocity \( v(\theta, \phi) \) and its radial shear \( v' \) to solve the induction equation (3) numerically. The induction equation would then have the general form

\[
\ddot{g} = Ax
\]

where \( \dot{g} \) is the finite representation of the secular variation, \( x \) is the finite representation of velocity (including shear), and \( A \) is a matrix dependent on \( B \) and its derivatives. The error in secular variation is much larger than that in magnetic field, and we follow common practice in regarding \( A \) as perfectly known.

One way to discretize the secular variation and velocity is to represent the functions on a grid of points \( ((\theta, \phi), i = 1 \cdot N_i; j = 1 \cdot M) \) and reduce (17) to \( NM \) algebraic equations which may be solved by least squares. Previous authors have used a truncated spherical harmonic expansion to represent the velocity function and most authors prefer to represent the secular variation also as a truncated spherical harmonics series. The methods of Roberts & Scott (1965) and Kahle et al. (1967) required spherical harmonic coefficients for secular variation and used a formalism close to that of Bullard & Gellman (1954); so also did Whaler (1986), Voorhies (1986b), [although he had earlier used a less flexible method (Voorhies 1986a)], Gire et al. 1986, and Whaler & Clarke (1988). Alternatively Bloxham (1988a,b) used main field models, Whaler (1984) used observations of magnetic secular variation directly, and Voorhies (1986a) effectively represented the secular variation over a \( 2^\circ \times 2^\circ \) grid.

We must also decide whether to fit the secular variation at the Earth’s surface or at the core surface. The latter will emphasize the short-wavelength components and was chosen by Voorhies (1986a), but the measurements are made at the Earth’s surface making it the logical choice. However Whaler & Clarke (1988) show the issue becomes irrelevant if the \( \dot{g} \) are assigned appropriate weights, as we do here.

The elements of \( A \) are usually evaluated analytically as sums of integrals of triple products of spherical harmonics and their derivatives, the Gaunt and Elsasser integrals (for example, Roberts & Scott 1965; Whaler 1986; Whaler & Clarke 1988; Bloxham 1988a,b). Previous authors solved the radial component of (3) which involves only \( B_r \), \( B_r \), and its derivatives. The horizontal components of (3) used here introduce further complexities: \( B_r \) and \( B_r \) are derived from \( \partial Y^n_r / \partial \theta \) and \( \partial Y^n_r / \partial \phi \) respectively; the former of which is not an orthogonal set while the latter is not a complete set. Therefore it is not possible to resolve the \( \theta \)- and \( \phi \)-components of (3) into Gaunt & Elsasser integrals. Alternatively we could apply \( \mathbf{e} \cdot \nabla \mathbf{x} \) to the induction equation (3), but this is unsatisfactory as taking the curl increases considerably the algebraic complexity of the problem.

In Sections 4 and 5 we develop a method which involves evaluating the matrix elements in space on a finite grid and applying a vector harmonic transform. We show in Section 4 that, by suitable choice of the grid, the integration can be made exact and needs far fewer points that Voorhies’ (1986a) method. A scalar transform method was first devised by Orszag (1970) for fluid dynamic computations and applied in a form similar to ours by Young (1972). A more recent account of a transform method is given by Glatzmaier (1984) who uses it in a dynamo calculation.

Truncation of the spherical harmonic series requires some care. The triangle rule for Gaunt and Elsasser integrals shows that velocity harmonics up to degree \( L_v = L_B + L_e \) are capable of inducing secular variation up to degree \( L_v \) from a magnetic field truncated at degree \( L_B \). However, such a velocity would also induce secular variation up to harmonic degree \( L_{\text{ind}} = L_v + L_B = 2L_B + L_e \). The inversion for a velocity requires a very lengthy calculation, when this prescription is followed: for the model discussed in Section 6 \( L_B = L_v = 14 \Rightarrow L_{\text{ind}} = 28 \); \( L_{\text{ind}} = 42 \). Furthermore the secular variation is very poorly determined at the CMB beyond even a modest degree and it may be equally unrealistic to overconstrain small-scale flows. We therefore choose to truncate the velocity series somewhat earlier and constrain the velocity harmonics to have converged by this limit but it is still essential to retain the secular variation harmonics from degree \( L_v \) to \( L_{\text{ind}} \) in the solution and constrain them to be statistically small or a velocity may be produced which seems to fit the included secular variation well but which induces a large amount of small-scale secular variation.

4 THE VECTOR SPHERICAL TRANSFORM

We decompose the solenoidal vector fields \( v \) and \( \dot{B} \) on the core surface into toroidal and poloidal terms, for which we define a set of orthogonal vector harmonics (see for example Morse & Feshbach 1953, p. 1898)

\[
q^m_r = Y_r Y_{m, r}
\]
\[
\begin{align*}
\mathbf{s}^m_l &= \frac{1}{\sqrt{l(l+1)}} \mathbf{V}_l(r \mathbf{Y}^m_l), \\
\mathbf{t}^m_l &= \frac{1}{\sqrt{l(l+1)}} \mathbf{r} \times \mathbf{V}_l(r \mathbf{Y}^m_l)
\end{align*}
\]

where the \(\mathbf{Y}^m_l\) are fully normalized complex spherical harmonics. Any solenoidal vector field \(\mathbf{X}(\theta, \phi)\) may be written as a sum

\[
\mathbf{X}(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \mathbf{X}_{lQ}^m \mathbf{Q}^m_l + \mathbf{X}_{lS}^m \mathbf{s}^m_l + \mathbf{X}_{lT}^m \mathbf{t}^m_l \right)
\]

(21)

where we have dropped the \(l=0\) term in the above summation since \(\mathbf{X} \cdot \mathbf{r} = 0\), but the monopole component should not be forgotten in other applications.

The harmonic coefficients are obtained by

\[
\begin{align*}
\mathbf{X}_{lQ}^m &= \int \mathbf{q}^m_l \cdot \mathbf{X} \, dS, \\
\mathbf{X}_{lS}^m &= \int \mathbf{s}^m_l \cdot \mathbf{X} \, dS, \\
\mathbf{X}_{lT}^m &= \int \mathbf{t}^m_l \cdot \mathbf{X} \, dS
\end{align*}
\]

(22-24)

where we have used the orthogonality properties of the vector harmonics as given in the appendix. Relations (22)-(24) define the Vector Spherical Transform (VST) and relation (21) defines its inverse.

By comparison, Bullard & Gellman (1954) defined the toroidal and poloidal vectors as

\[
\mathbf{T} = \left( 0, -\frac{1}{r \sin \theta} \frac{\partial \mathbf{T}}{\partial \phi}, \frac{1}{r} \frac{\partial \mathbf{T}}{\partial \theta} \right),
\]

(25)

\[
\mathbf{S} = \left( - \frac{\partial \mathbf{S}}{\partial \phi}, -\frac{1}{r \sin \theta} \frac{\partial \mathbf{S}}{\partial \theta}, \frac{1}{r \sin \theta} \frac{\partial \mathbf{S}}{\partial \phi} \right)
\]

(26)

and expanded \(\mathbf{S}, \mathbf{T}\) as spherical harmonic sums so that

\[
\begin{align*}
\mathbf{X}_{lQ}^m &= \frac{r^2}{l(l+1)} S_l^m, \\
\mathbf{X}_{lS}^m &= \frac{\sqrt{l(l+1)}}{r} S_l^m, \\
\mathbf{X}_{lT}^m &= \frac{\sqrt{l(l+1)}}{r} T_l^m
\end{align*}
\]

(27-29)

Our convention for \(\mathbf{X}_{lS}\) is the same as in most core velocity papers. Note that if we were interested in \(\mathbf{X}_{lQST}\) as a function of radius, then

\[
\mathbf{X}_{lQST} = \frac{2}{r} \mathbf{X}_{lQ} + \frac{\sqrt{l(l+1)}}{r} \mathbf{X}_{lS} + \frac{\sqrt{l(l+1)}}{r} \mathbf{X}_{lT}
\]

(30)

We form an inverse Discrete Vector Spherical Transform (DVST) by truncating the sum (21) at degree \(L\) and evaluating the vector \(\mathbf{X}\) on a grid of points \(\{\theta_i, \phi_j\}; i = 1 \cdots N_\theta; j = 1 \cdots M\) that are evenly spaced in \(\phi\) and at the zeroes of a Legendre polynomial of degree \(N\) for \(\theta\). The forward DVST is formed by replacing the integrals (22)-(24) with summations derived from the Gauss–Legendre Quadrature method (GLQ) of numerical integration for \(N\) points. We shall show that this integration method gives exact representations of the integrals provided \(N\) and \(M\) are sufficiently large.

We require orthogonality of the vector spherical harmonics; provided orthogonality passes from the continuous integral representation to the discrete transform then discrete forms of (22)-(24) will follow. Clearly the \(\{\mathbf{q}^m_l\}\) will still be orthogonal to the \(\{\mathbf{s}^m_l\}\) and \(\{\mathbf{t}^m_l\}\) orthogonal to the \(\{\mathbf{q}^m_l\}\). Consider next the orthogonality of the \(\{\mathbf{q}^m_l\}\), where

\[
\int q^m_{l1} \cdot q^m_{l2} \, dS = \int y^m_{l1} y^m_{l2} \, dS = \delta_{l1} \delta_{m1} \delta_{m2}.
\]

(31)

Replacing the integral with summations gives

\[
2\pi \sum_{l=1}^{M} P_{l1}^m(x_l) P_{l2}^m(x_l) w_l \sum_{j=1}^{M} e^{-im \phi_j} e^{im \phi_l} \phi_l \phi_j \delta_{m1} \delta_{m2} w_l.
\]

(32)

where the \(\{w_l\}\) are the weights for GLQ of order \(N\). The \(j\)-sum represents the \(\phi\)-integral and we know from the Nyquist criterion (for example Press et al. 1986, pp. 386-7) that if we choose the \(\phi\) points to be evenly spaced then the \(j\)-sum will be exact if \(m_1 + m_2 < M\). The Rodrigues’ formula for \(P_{l1}^m(x)\) is

\[
P_{l1}^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_{l1}(x)
\]

(33)

and the contribution to the \(\phi\)-sum in (32) is

\[
\sum_{l=1}^{N} (1-x_1^2)^{m_1+m_2} \frac{d^m}{dx^m} P_{l1}(x_1) \frac{d^m}{dx^m} P_{l2}(x_2) \delta_{m1} \delta_{m2} w_l.
\]

(34)

Setting \(m_1 = m_2\), since other contributions to (32) vanish by the \(\phi\)-sum, we note that (34) is a polynomial in \(x\) of degree \(l_1 + l_2\). GLQ is exact for polynomials in \(x\) of degree \(< 2N + 1\) (see for example Press et al. 1986, pp. 121-6), which proves the desired result of orthogonality in the discrete case.

From the Appendix it is clear that the \(\{\mathbf{s}^m_l\}\) and \(\{\mathbf{t}^m_l\}\) are orthogonal in the discrete sense provided the functions

\[
\begin{align*}
\frac{dP_{l1}^m}{dx} \frac{dP_{l1}^m}{dx} + \frac{m^2}{\sin^2 \theta} \frac{dP_{l1}^m}{dx} \frac{dP_{l1}^m}{dx}
\end{align*}
\]

(35)

are polynomials. Changing variables to \(x = \cos \theta\) gives

\[
(1-x^2) \frac{dP_{l1}^m}{dx} \frac{dP_{l1}^m}{dx} + \frac{m^2}{1-x^2} \frac{dP_{l1}^m}{dx} \frac{dP_{l1}^m}{dx}
\]

(36)

which is clearly polynomial when \(m = 0\). When \(m \neq 0\) Rodrigues’ formula (33) can be used to show that all terms are polynomials of degree \((l_1 + l_2)\) in \(x\). Thus the discrete orthogonality relations are exact if \(l_1 + l_2 \leq 2N + 1\) and \(m_1 + m_2 < M\). It follows immediately that the integrals (22)-(24) of the DVST are exact and will recover the transform coefficients \(\{X_{lQ}^m, X_{lS}^m, X_{lT}^m\}\) to within any accumulated rounding error. The DVST may be summarized as follows:

\[
\begin{align*}
\mathbf{X}(\phi_1, \phi_2) &= \sum_{l=0}^{L} \sum_{m=-l}^{l} \left[ X_{lQ}^m(\theta_1, \phi_1) + X_{lS}^m(\theta_1, \phi_1) + X_{lT}^m(\theta_1, \phi_1) \right], \\
X_{lQ}^m &= \sum_{l=1}^{N} \sum_{j=1}^{M} q^m_{l1}(\theta_j, \phi) \cdot \mathbf{X}(\theta_j, \phi) w_l,
\end{align*}
\]

(37-38)
These formulae transform a solenoidal vector function specified on a grid of points over a spherical surface to a truncated set of spherical harmonic poloidal and toroidal coefficients and back again. In Section 5 we drop the \( l = 0 \) monopole term from (37).

We can use the Fast Fourier transform (FFT) algorithm to do the \( \phi \)-sum, which yields \( M \) axial harmonics in \( O(M \log_2 M) \) operations as opposed to \( O(M^2) \). The DVST algorithm is as follows.

- Loop over \( \theta_i \) (GLQ)
  1. Evaluate \( P_l^m(x_i, \phi_i) \)
  2. FFT \( (X_\theta(x_i, \phi_i) \rightarrow (X_\theta(x_i, m)) \)
  3. Calculate the partial terms \( \psi(l, m) \)

Evaluating the associated Legendre functions takes \( O(L^2) \) operations, the FFT requires \( O(M \log_2 M) \) operations and the final sum takes \( O(L(L + 1)) \) operations. Thus the DVST requires \( O(L^3) \) operations yielding \( O(L^2) \) coefficients.

A discrete version of Parseval's theorem also applies for the DVST; it follows directly from the discrete form of the orthogonality relations for the vector spherical harmonics.

\[
\frac{2\pi}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} \cos(x_i, \phi_i) w_i = \sum_{l=0}^{L} \sum_{m=-l}^{l} [X_l^m]^2 + X_{l,5}^m + X_{l,T}^m].
\]  (41)

This result is needed in demonstrating that a least-squares fit on the space grid is equivalent to a fit to suitably weighted harmonic coefficients.

### 5 Method for Calculating Velocity

We now specialize to toroidal velocities and reduce the induction equation (3) to a finite-dimensional matrix equation of the form (17)

\[
\mathbf{g} = \mathbf{A} \mathbf{x}
\]  (42)

by writing \( \mathbf{B} \) and \( \mathbf{v} \) as a vector harmonic sums

\[
\mathbf{B} = \sum_{l,m} \left( g_l^m \mathbf{X}_l^m + g_l^m \mathbf{S}_l^m + g_l^m \mathbf{T}_l^m \right),
\]  (43)

\[
\mathbf{v} = \sum_{l,m} \chi_l^m \mathbf{t}_l^m.
\]  (44)

Substituting (44) into the induction equation (3) gives

\[
\mathbf{B} = \sum_{l,m} \left( \chi_l^m \left( \mathbf{B} \cdot \mathbf{V}_{l+1} \right) \mathbf{t}_l^m - (\mathbf{t}_l^m \cdot \mathbf{V}_{l+1}) \mathbf{B} \right) + \chi_l^m \frac{B_r}{r} \mathbf{t}_l^m
\]  (45)

where the \( \chi_l^m \) are the harmonic coefficients for the shear \( \mathbf{v}' \) at the CMB. We evaluate \( \mathbf{B} \) and its horizontal derivatives at our mesh points \( (\theta_i, \phi_i) \) for use in calculating the elements of the interaction matrix \( \mathbf{A} \) from (42). Similarly we evaluate \( \mathbf{t}_l^m (\theta_i, \phi_i) \) and its horizontal derivatives up to the required truncation limit for the velocity, \( L_x \). The terms

\[
\mathbf{B} \cdot \mathbf{V}_{l+1} \mathbf{t}_l^m - (\mathbf{t}_l^m \cdot \mathbf{V}_{l+1}) \mathbf{B}
\]  (46)

\[
\frac{B_r}{r} \mathbf{t}_l^m
\]  (47)

from the right-hand side of (45) are calculated at the mesh points \( (\theta_i, \phi_i) \), for all \( (l, m) \) and the DVST applied to give the coefficients of \( \mathbf{A} \)

\[
A_{l,m}^{l,m} = \iint \left( \begin{array}{c} q_{l2}^m \\ s_{l2}^m \\ s_{l2}^m \\ \vdots \end{array} \right) \left( \begin{array}{c} \mathbf{B} \cdot \mathbf{V}_{l+1} \mathbf{t}_l^m - (\mathbf{t}_l^m \cdot \mathbf{V}_{l+1}) \mathbf{B} \\ \frac{B_r}{r} \mathbf{t}_l^m \end{array} \right) dS
\]  (48)

in the matrix equation

\[
\begin{pmatrix} \mathbf{g}_l^m \\ \mathbf{g}_{l+1}^m \end{pmatrix} = \mathbf{A} \begin{pmatrix} \mathbf{x}_l^m \\ \mathbf{x}_{l+1}^m \end{pmatrix}
\]  (49)

This is solved by using weighted least squares with \( \mathbf{g} \) as data: we form the normal equations

\[
\mathbf{A}^T \mathbf{C}_c^{-1} \mathbf{g} = (\mathbf{A}^T \mathbf{C}_c^{-1} \mathbf{A})^{-1} \mathbf{x}
\]  (50)

where \( \mathbf{C}_c \) is the covariance matrix for the data. In Section 3 we discussed the truncation limits of the harmonic series and concluded that the velocity spectrum has to converge by \( L_x \). LeMoli et al. (1985) have discussed the techniques available to impose convergence on the velocity model. We choose an ad hoc damping varying as \( \lambda^{-4} \) (a form used previously by Whaler & Clarke 1988) with separate damping constants \( \lambda \) and \( \mu \) for the velocity and the radial shear respectively. Note we are only damping to produce a smooth solution rather than a unique solution.

### 6 Velocity Model for 1970

The time-dependent model of Bloxham & Jackson (1989) was used for the main field and secular variation for 1970. Unfortunately we did not have the covariance matrix for the time-dependent model and had to construct our own weights for the secular variation coefficients. We assumed that relative variances of main field coefficients were representative of relative variances of secular variation coefficients and took the average variances of the main field coefficients for each degree from the 1980 MAGSAT model of Gubbins & Bloxham (1985) to form a diagonal covariance matrix. Only the relative values of these variances \( (\sigma_{ij}/\sigma_{ii}) \) have meaning, their absolute values being arbitrary. These errors peak at degree 12, beyond which the a priori information takes over and the errors diminish. Induction terms generate secular variation out to spherical harmonic degree \( L_{\text{ind}} = L_{x} + L_{B} = 28 \) in this case. Harmonics of degree \( L_{B} + 1 \) to \( L_{\text{ind}} \) are set to zero and harmonics of degree 13 to \( L_{\text{ind}} \) are assigned weights corresponding to
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Figure 1. Standard errors for secular variation coefficients used in the velocity calculation. They are based on variances calculated for a main field model continued to high degree and order and are the same for all spherical harmonic order for the same degree.

the peak error, that for degree 12. Fig. 1 graphs the secular variation error against spherical harmonic degree. Errors $\sigma_{L,5}$ and $\sigma_{L,T}$ in the horizontal component terms were both taken as $4\sigma_{Q,Q}$ as suggested by equations (19) and (20). These values are little more than educated guesses. The uncertainties in the horizontal component terms will depend on the electrical conductivity of the lower mantle, which controls leakage of toroidal flux and which we have not yet considered. However, our aim is to derive a self-consistent velocity model and to assess its validity afterwards, and this weighting scheme provides just such a model.

Our preferred velocity, with a rms velocity of 13.6 km yr$^{-1}$ and a rms shear of 80.45 km yr$^{-1}$ is mapped in Fig. 2. Flow is along streamlines at a rate given by the density of streamlines. An undamped solution is presented in Fig. 3. Rms velocities for these models are given in Table 1. The similarity between the damped and undamped solutions suggests that uniqueness is provided by the horizontal components and damping serves mainly to remove small-scale velocities. Contrast this with the differences between our preferred velocity model based on the radial component of secular variation alone, shown in Fig. 4, and the corresponding undamped solution (Fig. 5). Flow along the contours of $B_2$ is not determined by the radial component, and is constrained only by the truncation point of the spherical harmonic series used to represent the model. The undamped flow is 10$^4$ times faster than the damped flow; flow along contours of $B_2$ dominate the solution and Fig. 5 is virtually identical with a map of radial magnetic field with suitable contour interval. Obviously we can attach no physical significance to the solution given in Fig. 5, we include it merely to illustrate the strong influence exerted by the horizontal components of secular variation in providing a unique solution.

Our preferred solution in Fig. 2 compares quite well with that based on radial component alone in Fig. 4. There are some differences, particularly at high latitudes, which we attribute to the ad hoc regularization procedure used to force uniqueness from the radial equation. Our preferred solution also compares quite well with the toroidal motions obtained by Bloxham (1989) from the radial equation.

Statistical results for these models are shown in Table 1. The misfit is measured by the root sum of the weighted residuals for each of the terms in the representation (43) of rate of change of magnetic field induced at the CMB.

$$r_o = \left[ \sum_{l,m} \left( \frac{\delta m \cdot \delta m}{\sigma_{Q,Q}} \right)^2 \right]^{1/2}$$

(51)

Our preferred solution in Fig. 2 compared quite well with that based on radial component alone in Fig. 4. There are some differences, particularly at high latitudes, which we attribute to the ad hoc regularization procedure used to force uniqueness from the radial equation. Our preferred solution also compares quite well with the toroidal motions obtained by Bloxham (1989) from the radial equation.

Figure 2. Streamfunction for the preferred velocity (model CV70), based on all three components of secular variation for epoch 1970, plotted on Mercator projection. Rms velocity is 13.6 km yr$^{-1}$. 
Figure 3. Streamfunction for the velocity model CVUD70, based on all three components of secular variation for epoch 1970 as for CV70 but with no damping, plotted on Mercator projection. It is similar to CV70 apart from small-scale noise, indicating uniqueness of the solution is provided by the horizontal components. Rms velocity is 19.5 km yr\(^{-1}\).

with similar definitions for \(r_5\), \(r_7\), and where \(\vec{g}\) represents the induced secular variation. The absolute values of the weights are arbitrary and it is useful to discuss the residuals as percentages of the root sum of univariate data:

\[
\left[ \sum_{m=0}^{14} \left( g_{r,m}^m / \sigma_r^m \right)^2 \right]^{1/2}.\quad (52)
\]

Note this is equal to the corresponding \(S\) sum but the \(T\) sum is zero because there is no toroidal part to the observed potential field.

The residuals in Table 1, presented as percentages of (52) show that we are fitting over 95 per cent of the weighted data in \(Q\), \(S\), and \(T\) separately and 96 per cent in all. Table 1 also gives the misfit to the radial component of the secular variation at the Earth's surface. This quantity has no statistical significance because we used weighted data, but it is a useful indicator of how well the velocity model explains geomagnetic observations. Table 1 also gives the norms for the velocity and, for the first two models, the shear. These are minimum norm solutions and the procedure minimizes these quantities. Table 1 also gives the rms velocity and shear in km yr\(^{-1}\); these quantities are similar to the norm in that they measure smoothness of the models; they have no

Figure 4. Streamfunction for velocity model CVR70, based on only the radial component of secular variation at epoch 1970, plotted on Mercator projection. It is very similar to our preferred model, indicating we have not paid a high price in fitting additional components of secular variation. Rms velocity is 15.4 km yr\(^{-1}\).
Figure 5. Streamfunction for the velocity model CVRUD70, based on only the radial component of secular variation for epoch 1970, as CVR70 but with no damping, plotted on Mercator projection. This map is virtually identical to a plot of $B$, at the CMB because the flow along contours of $B$, is not determined by the radial component. Rms velocity is $1.5 \times 10^9$ km yr$^{-1}$; note the large difference in magnitude from the other models.

Table 1. Statistical results for the velocity models. The residuals are expressed as percentages of the univariate data. The SV error is the misfit to $B$, at the Earth’s surface.

<table>
<thead>
<tr>
<th>Model</th>
<th>Residuals</th>
<th>Model Norms</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$r_q$</td>
<td>$r_s$</td>
<td>$r_T$</td>
<td>$RMS$</td>
<td>$SV$</td>
<td>$v'$</td>
<td>$RMS$</td>
<td>$v'$</td>
</tr>
<tr>
<td>CV70</td>
<td>4.5</td>
<td>3.1</td>
<td>2.2</td>
<td>20135</td>
<td>36966</td>
<td>13.6</td>
<td>km yr$^{-1}$</td>
<td>80.45 km yr$^{-1}$</td>
</tr>
<tr>
<td>CVUD70</td>
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<td>1.6</td>
<td>1.9</td>
<td>46857</td>
<td>66028</td>
<td>15.5</td>
<td>km yr$^{-1}$</td>
<td>149.7 km yr$^{-1}$</td>
</tr>
<tr>
<td>CVR70</td>
<td>3.9</td>
<td>--</td>
<td>--</td>
<td>21492</td>
<td>--</td>
<td>15.4</td>
<td>km yr$^{-1}$</td>
<td>--</td>
</tr>
<tr>
<td>CVRUD70</td>
<td>0.9</td>
<td>--</td>
<td>--</td>
<td>$1.4 \times 10^9$</td>
<td>--</td>
<td>$1.5 \times 10^6$ km yr$^{-1}$</td>
<td>--</td>
<td>0.51 nT yr$^{-1}$</td>
</tr>
</tbody>
</table>

Figure 6. Streamfunction $\gamma'$ for the shear $\partial \partial / \partial r$. Mercator projection. It is very similar in form to the preferred solution in Fig. 2; its magnitude suggests a vertical length scale for the velocity of 600 km and its sign indicates the velocity will decrease with distance away from the CMB. Rms shear is $80.4$ km yr$^{-1}$.
statistical significance but, like the misfit to surface data, they give an indication of how much energy is in the flow.

The results in Table 1 show a number of interesting features. The undamped model CVRUD70 has a very large normal and rms velocity because of the problem of non-uniqueness; the corresponding model CVUD70 does not because the horizontal equations constrain the flow. The fit to the radial equation \( Q \) is similar for both models CV70 and CVR70 indicating that we have not paid a high price by incorporating horizontal equations. This suggests the shear terms, which are not included in the model CVR70, are instrumental in fitting the horizontal components. The rms shear gives an estimate of the vertical length scale of the motions needed to match the horizontal components. A typical length scale is obtained by dividing the rms shear into the rms velocity (giving a result in units of core radii): for model CV70 the length scale is 590 km, which is several times larger than the skin depth, a necessary requirement for consistency with the assumptions of frozen flux. We examined the weighted residual mapped over the CMB to see if there were any obvious features in the secular variation that our flow could not generate (as might perhaps be expected in a region of strong upwelling) but no such features were found.

The rms flow speed of the model, 13.6 km yr^{-1}, is close to that of Whaler & Clarke (1988) in their steady flow model centred on 1970. Our model has the fast flow clearly confined to the Atlantic hemisphere; it is roughly symmetric across the equator. The flow in the Pacific hemisphere is very slow, but here also there is a faint suggestion of symmetry about the equator. Our model has strong westward flow under equatorial Africa and the Indian Ocean, in common with many other models, but it differs from the flows of Whaler & Clarke (1988) and Bloxham (1988b) in returning eastward in high latitudes under the North and South Atlantic, rather than carrying on northwesterly towards Alaska and skirting the northern Pacific.

The stream function for the shear is plotted in Fig. 6. We intend to study the shear terms in more detail in future work, but we remark in passing that shear in confined to the Atlantic hemisphere, like the velocity, and is remarkably similar in form to the velocity (compare Fig. 2). The sign is such that the velocity will decrease with depth into the core.

7 CONCLUSIONS

We have shown that toroidal core flows can be found uniquely by using all three components of the induction equation, provided \( B \) obeys a simple point condition which the geomagnetic field appears to satisfy. We have found a simple toroidal fluid motion for epoch 1970 which fits over 95 per cent of the radial component of secular variation at the CMB, and rather more of the horizontal components. This result supports the findings of Barraclough et al. (1989), that geomagnetic observations are consistent with an insulating mantle and perfectly conducting core with a boundary layer across which the jump in horizontal field is negligible.

The fit to the observations is very good and suggests that any remaining improvements should involve relaxation of the frozen flux hypothesis as well as the addition of poloidal flow. This is in agreement with the findings of Bloxham (1988b), who decided on the basis of radial components alone that toroidal motions provided an adequate fit to the secular variation and a better fit than geostrophic motions, contrary views of other authors (Voorhies 1986a; Gire et al. 1986; Whaler 1986) notwithstanding. We do not need to rely on damping to produce a reasonable velocity, but have used some damping to avoid over-fitting the secular variation models.

The shear terms indicate that the vertical length scale of the velocity is about 600 km. The shear has the same form as the velocity and its sign is such that the velocity decreases in strength with depth. This would result from driving from the CMB by, for example, lateral variations in temperature in the lower mantle, as envisaged by Bloxham & Gubbins (1987), in a density-stratified core where poloidal flows were inhibited.

We have developed a new numerical procedure, the transform method, for determining core motions from geomagnetic field models. The transform has a number of advantages.

1. It is computationally faster than the analytical method, even discounting the time taken to evaluate the Gaunt and Elsasser integrals: calculating the elements of \( A \) using the DVST as in (48) takes \( O(T^3L^3|\text{ind}) \) operations whereas a Gaunt & Elsasser technique would take \( O(T^2L^2|\text{ind}) \) operations. The Fast Fourier transform algorithm can be used to sum over \( \phi \)-points, making the process faster still. The numerical integration scheme is exact.

2. It avoids the analysis needed to express the matrix elements in terms of Gaunt and Elsasser integrals. The analysis is particularly lengthy for some calculations involving the horizontal components of the induction equation.

3. The algorithm for performing the DVST is not easy to code, but once done it allows the remaining calculations to be done with minimal analysis and programming: only simple space products are needed once a routine is available to transform between space and transform domains.

ACKNOWLEDGMENTS

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APPENDIX: ORTHONORMALITY OF VECTOR SPHERICAL HARMONICS

We now demonstrate the orthonormality of the vector spherical harmonics (18), (19) and (20). The orthonormality of \( q_{lm}^m \) is trivial

\[
\int \int q_{lm}^m \cdot \bar{q}_{lm}^m \, dS = \int \int Y_{l+1}^m \cdot \bar{Y}_{l+1}^m \, dS = \delta_{ll} \delta_{m1} \tag{53}
\]

where the integral is taken over the surface of a unit sphere. \( \bar{q}_{lm}^m \) is trivially orthogonal to \( q_{lm}^m \) and \( \bar{q}_{lm}^m \cdot \bar{q}_{lm}^m \) is also trivially orthogonal to \( \bar{q}_{lm}^m \). The self-orthogonality and \( q_{lm}^m \) require a little more work.

\[
\int \int \bar{q}_{lm}^m \cdot \bar{q}_{lm}^m \, dS \tag{54}
\]

where

\[
N_{ll} = \frac{1}{(l+1)(2l+1)} \tag{55}
\]

which since \( \partial Y_{lm}^m / \partial \phi = imY_{lm}^m \) becomes

\[
N_{ll} \int_0^{2\pi} \frac{1}{2l+1} \left( \frac{\partial Y_{lm}^m}{\partial \theta} \frac{\sin^2 \theta}{\partial \phi} \partial \phi \right) \, d\theta \, d\phi \tag{56}
\]
and taking the $\phi$-integral
\[
2\pi N_{i_1l_1}[\int_0^\pi \left( \frac{dP_{i_1}^{m_1}}{d\theta} \frac{dP_{i_2}^{m_2}}{d\theta} + \frac{m_1 m_2}{\sin^2 \theta} p_{i_1}^{m_1} p_{i_2}^{m_2} \right) \delta_{m_1 m_2} \sin \theta \, d\theta].
\]

Integrating the first term (with $\sin \theta$) by parts gives
\[
2\pi N_{i_1l_1}[\int_0^\pi P_{i_1}^{m_1} \frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{dP_{i_2}^{m_2}}{d\theta}) + \frac{m_2^2}{\sin^2 \theta} P_{i_2}^{m_2} \delta_{m_1 m_2} \sin \theta \, d\theta]
\]
which from the associated Legendre equation which by the argument above gives
\[
2\pi N_{i_1l_1}(l_1 + 1) \int_0^\pi P_{i_1}^{m_1} P_{i_2}^{m_2} \delta_{m_1 m_2} \sin \theta \, d\theta
\]

\[\begin{align*}
\text{Toroidal fluid motion at the top of the Earth's core} & \quad 467 \\
\text{which from the orthogonality relation for } P_i^m & \text{ gives}
\end{align*}\]

\[
\int \int \xi_{i_1}^{m_1*} \xi_{i_2}^{m_2*} \, dS = \delta_{i_1i_2} \delta_{m_1m_2}
\]

and similarly for $\xi_i^m$:

\[
\int \int \xi_{i_1}^{m_1*} \xi_{i_2}^{m_2} \, dS
\]

which by the argument above gives

\[
\int \int \xi_{i_1}^{m_1*} \xi_{i_2}^{m_2} \, dS = \delta_{i_1i_2} \delta_{m_1m_2}
\]