Classical Nature of the Inflaton Field with Self-Interaction

Takahiro TANAKA*) and Masa-aki SAKAGAMI*,**)

Department of Earth and Space Science, Graduate School of Science
Osaka University, Toyonaka 560-0043, Japan

*Cosmology Group, Faculty of Integrated Human Studies
Kyoto University, Kyoto 606-8501, Japan

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Taking into account the effect of self-interaction, the dynamics of the quantum fluctuations of the inflaton field with a $\lambda\phi^4$ potential is studied in detail. We find that the self-interaction efficiently drives the initial pure state into a mixed state, which can be understood as a statistical ensemble. Further, the expectation value of the squared field operator is found to be converted into the variance of this statistical ensemble without giving any significant change in the amplitude of fluctuations. These results verify the ansatz of the quantum-to-classical transition that has been assumed in the standard evaluation of the amplitude of the primordial fluctuations of the universe.

§1. Introduction

The inflationary universe scenarios satisfactorily explain various aspects of the universe, such as homogeneity, isotropy and the amplitude of the primordial fluctuations observed in the microwave background radiation and in the large-scale structure.\textsuperscript{1) However, an incompleteness in the analysis of the evaluation of the primordial fluctuations has been pointed out in several recent papers.\textsuperscript{1)}

In most of inflationary universe scenarios, the seed of the inhomogeneity of the universe is traced back to the quantum fluctuations of the inflaton field generated by the accelerated expansion of the cosmic length scale. In this context, the amplification of quantum fluctuations is characterized by a large amount of squeezing of the state vector. A difficulty in interpreting this state vector exists in the fact that the expectation value of the linear field operator does not exhibit inhomogeneities, but rather vanishes, while that of the squared field operator becomes very large. Usually, one interprets this quantum state as if it were equivalent to a statistical ensemble which has the same amount of variance as the corresponding quantum operator in the sense of an expectation value, once the scale of the fluctuations of interest exceeds the Hubble horizon scale. Here we refer to the calculation based on this ad hoc classicalization ansatz as “the standard calculation”.

In order to justify the standard calculation, a stochastic approach to the inflationary universe scenario was proposed by Starobinsky\textsuperscript{2) and later investigated by many authors.\textsuperscript{3) The evolution of the order parameter that is defined as the expectation value of the spatially averaged field operator was studied. If the physical
size of the spatial averaging is kept constant, modes with larger and larger comoving wavenumbers are continuously added to the order parameter as the universe expands. The contribution from the newly added small scale modes affects the dynamics of the order parameter as a random noise. As a consequence, the evolution of the order parameter mimics Brownian motion, and thus the distribution of the order parameter can be closely approximated by a statistical ensemble of a classical system influenced by a stochastic noise.

In these early studies, a free scalar field was investigated, and little attention was paid to the effect of interaction. However, the fluctuations of a free scalar field in a Friedmann-Robertson-Walker spacetime can be decomposed into a set of harmonic oscillators which have time dependent spring constants. Then, all of these oscillators are decoupled from each other. No classicalization is expected in such a decomposed degree of freedom. Thus every mode that contributes to the order parameter during inflation retains its quantum nature. This fact implies that the analysis of the classicalization in the context of stochastic inflation is incomplete. In the true theory of the quantum-to-classical transition, the classical nature should be observed even if each decomposed mode is considered.

Along this line, recently Lesgourgues, Polarski and Starobinsky proposed a new idea to explain the quantum-to-classical transition. They claim that the equivalence between the large squeezing and the classicality of the state can be explained if one discards the tiny contribution due to the decaying mode of the perturbation. However, a free scalar field is considered in their approach, and the effect of interaction was not taken into account manifestly. Thus the elimination of the decaying mode is still artificial, and is not fully justified, although their result contains an important suggestion.

Only recently, the importance of interaction has become emphasized. As a useful tool to deal with the effect of interaction with environmental degrees of freedom, the closed-time path formalism has been developed. Here, the total system is divided into "the system" and "the environment", where the system contains quantities of interest. By using the closed-time path formalism, one can integrate out the environmental degrees of freedom to obtain the effective action for the system. Many cosmological issues, such as the back reaction to the expansion rate of the universe due to particle creation, have been investigated.

In particular, in Refs. 7), 8) and 10) the evolution of the fluctuations of the inflaton field is examined with the aid of the closed-time path formalism. It is repeatedly stressed in those works that the correct treatment of the effect of the environmental degrees of freedom may relax the constrains on the model parameters in the inflaton potential. Let us consider the $\lambda \phi^4$ model of chaotic inflation. In the standard calculation, $\lambda \sim 10^{-12}$ is required in order to explain the observed value of the primordial density fluctuations. Contrastingly, the authors of the above cited works suggest the possibility that the unnatural smallness of the coupling constant can be relaxed to $\lambda \sim 10^{-6}$. One of the main purposes of the present paper is to carefully examine this issue.

Here we adopt a conservative picture of the quantum-to-classical transition based on the paper by Joos and Zeh. There are two conditions for the system to be
of a classical nature. Based on this picture, we investigate the $\lambda \phi^4$ model, which is one of the simplest models of chaotic inflation. \(^1\) With the choice of parameter $\lambda \sim 10^{-12}$, the two conditions for the quantum-to-classical transition are proved to be satisfied, and the fluctuations of the scalar field, $\delta \phi$, become $O(H)$, as predicted in the standard calculation. Hence, we reach the conclusion that “the standard calculation” is justified in this simple model without any significant modification of the amplitude of the fluctuations, in contrast to the prediction given in the literature. \(^7\),\(^8\),\(^10\)

This paper is organized as follows. In §2 we explain the picture of the quantum-to-classical transition and clarify in what situation the system behaves classically. In §3 we explain our simple model and the basic assumptions. Integrating out the environmental degrees of freedom, we derive the effective action for the system. In §4, by using the closed-time path formalism, the dynamics of the system are analyzed in a less rigorous but very intuitive manner. A more rigorous treatment based on the quantum master equation for the reduced density matrix is presented in §5. Section 6 is devoted to summary and discussion.

§2. Decoherence measure between different worlds

In this section, we explain our picture of the quantum-to-classical transition. First, let us briefly summarize the standard discussion of decoherence based on the analysis of the reduced density matrix. Suppose that the whole system can be divided into two parts: one is the “system”, which contains the observables of interest, and the other is the “environment”, which is to be integrated out to obtain the reduced density matrix. We restrict our attention to the case in which the interaction term between the system and the environment does not contain the momentum variable of the system. We set the initial conditions for the whole system so that the density matrix is given by that of a pure state, and the correlation between the system and the environment is absent. Hence the state can be represented as

$$\rho(t_i) = |t_i\rangle_{\text{sys}} \otimes |t_i\rangle_{\text{env}} \otimes |t_i\rangle_{\text{sys}}.$$ \hfill (2·1)

The reduced density matrix is obtained by taking the partial trace over the environment,

$$\tilde{\rho}(x, x'; t) := \text{Tr}_{\text{env}} \langle x|\rho(t)|x'\rangle_{\text{sys}},$$ \hfill (2·2)

where $|x\rangle$ is the eigenstate of the coordinate of the system, $x$. If there is no interaction between the system and the environment, a loss of coherence does not take place, i.e., $\tilde{\rho}(x, x'; t)$ retains the form of a pure state, i.e., $\tilde{\rho}(x, x'; t) = |x\rangle_{\text{sys}} \langle x'|_{\text{sys}}$. Hence, $|\tilde{\rho}(x, x'; t)|^2 = 1$. However, in the presence of interaction, $\tilde{\rho}(x, x'; t)$ no longer retains the form of a pure state. If $|\tilde{\rho}(x, x'; t)|^2 = 1$, we may say that the coherence between different states labeled by $x$ and $x'$ disappears. We can determine the typical scale $\Delta x$ such that the coherence between states labeled by $x$ and $x'$ is lost when $|x - x'| > \Delta x$. As long as we observe the system with a resolution coarser than $\Delta x$, we may assume that the different states have no interference, and hence they can be recognized as independent worlds. However, in the above discussion, the stability of the state
through the time evolution was not taken into account. The dynamics of the system itself and the effects of the environment cause the broadening of the wave function of the system in general. If there is a large amount of broadening, it would be difficult to interpret the situation as one in which the state evolves into a statistical ensemble of many different "classical worlds". In the standard discussion of decoherence, this aspect is overlooked.

To take the broadening effect of wave functions into account, we propose to use a decomposition of the density matrix by using a set of wave packets. As mentioned in the Introduction, the basic idea is taken from the paper by Joos and Zeh. Our proposal is that the conditions necessary for the system to possess a classical nature are the following two.

1. The first condition is that the total system has a set of wave packets which have a sufficiently sharply peaked probability distribution in comparison with the accuracy of our measurement through the entire duration we consider. Suppose we label the wave packets by their initial peak position, \( s \), as \( |s; t\rangle \). Since the interaction term does not contain the momentum variable, in an approximate sense each wave packet can be written as a direct product:

\[
|s; t\rangle = |s; t\rangle_{\text{sys}} \otimes |s; t\rangle_{\text{env}}.
\]  

(2·3)

Here we should note that the state of the environment \( |s; t\rangle_{\text{env}} \) is also labeled by \( s \), since the interaction causes a correlation between the system and the environment during the course of their time evolution. Strictly speaking, what we require here is that \( |s; t\rangle_{\text{sys}} \) has a sharp peak (not \( |s; t\rangle \)) and that the peak is stable with respect to the evolution. Furthermore, we require that this set of wave packets is sufficiently complete that the initial state, \( |\Psi\rangle \), can be decomposed into a quantum mechanical superposition of these wave packets, i.e.,

\[
|\Psi\rangle = \sum_s c_s |s; t\rangle.
\]  

(2·4)

This means that each wave packet is sufficiently sharply peaked and sufficiently stable to be recognized as a distinguishable "world". We refer to this condition as "the classicality of the dynamics of the system".

2. The second condition is that the coherence between the different wave packets is quickly lost. We refer to this condition as "the decoherence between different worlds". For a total system which satisfies the first condition above, the density matrix at time \( t \) will be given by

\[
\rho(t) = \sum_{s, s'} c_s c_{s'}^* |s; t\rangle \langle s'; t|.
\]  

(2·5)

The partial trace over the environment gives the reduced density matrix,

\[
\tilde{\rho}(t) = \sum_{s, s'} \tilde{\rho}_{s, s'}(t), \quad \tilde{\rho}_{s, s'}(t) := c_s c_{s'}^* C(s, s'; t)|s; t\rangle_{\text{sys}} \langle s'; t|,
\]  

(2·6)

where \( C(s, s'; t) \) is given by

\[
C(s, s'; t) := \text{env}(s; t|s'; t)_{\text{env}}.
\]  

(2·7)
We refer to $\tilde{\rho}_{s,s'}(t)$ as the partial reduced density matrix. If we assume that the initial state is given by a direct product of the state of the system and that of the environment, we find $C(s, s'; t_i) = 1$. If there is no interaction between the system and the environment, a loss of coherence does not take place; i.e., $C(s, s'; t)$ remains a time independent constant for all $s$ and $s'$. However, in the presence of interaction, $|s; t\rangle_{\text{env}}$ and $|s'; t\rangle_{\text{env}}$ evolve differently, and hence $C(s, s'; t)$ no longer remains constant for $s \neq s'$, while $C(s, s; t) \equiv 1$.

If $|C(s, s'; t)|$ becomes quite small for $s \neq s'$, we can say that the diagonalization of $\tilde{\rho}$ has occurred. Thus the quantity $|C(s, s'; t)|$ characterizes the degree of decoherence between two different worlds. Here we note that

$$|c_s c_{s'} C(s, s'; t)|^2 = \text{Tr} \left( \tilde{\rho}_{s,s'}(t) \tilde{\rho}_{s,s'}^\dagger(t) \right).$$

(2·8)

In a strict sense, $|s; t\rangle$ cannot be written by a direct product as $|s; t\rangle_{\text{sys}} \otimes |s; t\rangle_{\text{env}}$. Thus the function $C(s, s'; t)$ given in (2·7) is not well defined. However, if we define $\tilde{\rho}_{s,s'}(t)$ by

$$\tilde{\rho}_{s,s'}(t) = \text{Tr}_{\text{env}} \langle s; t | \rho(t) | s'; t \rangle_{\text{sys}},$$

(2·9)

the expression on the right-hand side of Eq. (2·8) makes sense in any case and is expected to give the measure of decoherence between different worlds. As a more convenient alternative of the measure of decoherence, we propose to use

$$R = \frac{\text{max} |\tilde{\rho}_{s,s'}(t)|}{\text{max} |\tilde{\rho}_{s,s'}(t_i)|}.$$  

(2·10)

If $\tilde{\rho}_{s,s'}(t)$ takes a Gaussian form, $R$ equals $|C(s, s'; t)|$, ignoring the small correction due to the determinant factor that arises from the Gaussian integral. If $R$ becomes very small for $s \neq s'$, we claim that decoherence between different worlds is achieved.

In some sense, our picture is that of the third quantization of the universe\(^{13}\) or of the decoherence history.\(^{14}\) Our two requirements for the quantum-to-classical transition may be too strong.\(^{15}\) However, for the present purpose, we need not relax these conditions.

### §3. A simple model and the effective action

We consider the following simple model of inflation consisting of a single real scalar field, using an approximation similar to that introduced by Matacz,\(^8\) although the details are significantly modified. We assume that the spacetime can be approximated by a spatially flat de Sitter space,

$$ds^2 = dt^2 - a^2(t) d^2 x,$$

(3·1)

with

$$a(t) = \frac{1}{H} e^{Ht},$$

(3·2)
and that the Lagrangian of the inflaton field is given by
\[
S = \int_{t_i}^{t} ds \, a^3(s) \int_\Omega d^3x \left[ \frac{1}{2} \left( \frac{d\Phi(x,s)}{ds} \right)^2 - \frac{(\nabla \Phi(x,s))^2}{2a^2(s)} - V(\Phi(x,s)) \right],
\]
(3.3)
where \( \Omega \) is a finite comoving volume corresponding to the scale of the fluctuations of interest. Also, we assume that the effect from outside of this volume can be neglected. For simplicity, we choose \( \Omega \) as a cube with \( 0 \leq x, y, z \leq 1/2\pi k_c \). The lower boundary of the time integration, \( t_i \), is the time at which some appropriate initial conditions are set.

We focus on the dynamics of a spatially averaged field in this volume,
\[
\phi(s) := \frac{1}{\Omega} \int_\Omega d^3x \Phi(x,s).
\]
(3.4)
We stress that this averaging is performed only on a finite comoving volume, i.e., a part of the time-constant spatial surface. Hence \( \phi(s) \) does not represent the homogeneous part of the field \( \Phi(x,s) \). Rather, it represents a fluctuation mode on the comoving scale \( k_c^{-1} \). This point is discussed in detail in Appendix C. Although it is not essential, for definiteness, we set periodic boundary conditions on \( \Phi(x,s) \). Then \( \Phi(x,s) \) is decomposed as
\[
\Phi(x,s) = \phi(s) + \psi(x,s)
\]
\[
= \phi(s) + \sqrt{\frac{2}{\Omega}} \sum_k \left[ q^+_k \cos k \cdot x + q^-_k \sin k \cdot x \right],
\]
(3.5)
where \( k = k_c (i, j, k) \) is a non-vanishing vector with integers \( i, j \), and a non-negative integer \( k \). We denote the Heisenberg operator corresponding to \( q^\sigma_k \) by \( \hat{q}^\sigma_k \). Assuming that the potential can be approximated by
\[
V(\Phi) \sim V(\phi) + \psi V'(\phi) + \frac{\psi^2}{2} V''(\phi),
\]
(3.6)
the action reduces to
\[
S[\phi, q] = \Omega \int_{t_i}^{t} ds \, a^3(s) \left[ \frac{1}{2} \dot{\phi}^2(s) - V(\phi(s)) \right]
\]
\[
- \frac{1}{2} \sum_{\sigma} \sum_k \int_{t_i}^{t} ds \, a^3(s)V''(\phi(s)) (q^\sigma_k)^2
\]
\[
- \frac{1}{2} \sum_{\sigma} \sum_k \int_{t_i}^{t} ds \, a^3(s) \left[ \left( \hat{q}^\sigma_k \right)^2 - \frac{k^2}{a^2(s)} (q^\sigma_k)^2 \right]
\]
\[
=: S_{\text{sys}}[\phi] + S_{\text{int}}[\phi, q] + S_{\text{env}}[q],
\]
(3.7)
where the dot represents differentiation with respect to \( t \). As seen from the notation introduced in Eq. (3.7), the spatially averaged field is considered as the system, and the other short wavelength modes are the environment.
Similarly to Ref. 8, we calculate the reduced density matrix for \( \varphi \), integrating over the environmental degrees of freedom, \( q_k^q \). Here we assume that the density matrix of the total system is initially represented by the direct product as

\[
\rho_t = \tilde{\rho}(\varphi_t, \varphi'_t; t_i) \otimes \prod_k \prod_\sigma \rho_k^q(q_{ki}, q_{ki}', t_i). \tag{3·8}
\]

In order to define the initial quantum state of \( q_k^q \), we suppose that the interaction is switched off before \( t = t_i \), and we fix the quantum state of \( q_k^q \) by

\[
a_k^q|0\rangle = 0, \tag{3·9}
\]

where the annihilation operator \( a_k^q \) is defined by the decomposition of the Heisenberg operator,

\[
q_k^q = a_k^q u_k^q + a_k^{q*} u_k^{q*}, \tag{3·10}
\]

and the positive frequency function \( u_k^q \) is taken as

\[
u_k^q = u_0 := \frac{1}{\sqrt{2k}} \frac{e^{-ik\eta}}{a(t)} \left( 1 - \frac{i}{k\eta} \right), \tag{3·11}
\]

before \( t = t_i \). Here we introduced the conformal time coordinate by \( \eta := -e^{-Ht} \). We now consider the perturbative expansion with respect to the interaction term. Then the reduced density matrix is calculated to second order in \( S_{\text{int}}[\varphi, q] \) as

\[
\tilde{\rho}(\varphi, \varphi'; t) := \int dq \ \rho(\varphi, \varphi', q, q'; t) \\
= \int d\varphi_i \int d\varphi'_i \int_\phi \int_{\phi'} D\varphi D\varphi' \\
x \exp \left\{ i \left\{ S_{\text{sys}}[\varphi] - S_{\text{sys}}[\varphi'] \right\} + iS_{\text{IF}}[\varphi, \varphi'] \right\} \tilde{\rho}(\varphi_i, \varphi'_i; t), \tag{3·12}
\]

and

\[
iS_{\text{IF}}[\varphi, \varphi'] = -i \int_{t_i}^t ds \ \Delta(s) f(s) + i \int_{t_i}^t ds \int_{t_i}^s ds' \ \Delta(s) \Sigma(s') \mu(s, s') \\
- \int_{t_i}^t ds \int_{t_i}^s ds' \ \Delta(s) \Delta(s') \nu(s, s'), \tag{3·13}
\]

where

\[
\Delta(s) = V''(\varphi) - V''(\varphi'), \quad \Sigma(s) = \frac{1}{2} (V''(\varphi) + V''(\varphi')), \tag{3·14}
\]

and

\[
f(s) = \frac{a(s)^3}{2} \sum_\sigma \sum_k u_{0*}(s) u_{0}(s),
\]

\[
\mu(s, s') = -i \frac{a(s)^3 a(s')^3}{2} \sum_\sigma \sum_k \left( [u_{0*}(s)]^2 [u_{0}(s')]^2 - [u_{0}(s)]^2 [u_{0*}(s')]^2 \right),
\]

\[
\nu(s, s') = \frac{a(s)^3 a(s')^3}{4} \sum_\sigma \sum_k \left( [u_{0*}(s)]^2 [u_{0}(s')]^2 + [u_{0}(s)]^2 [u_{0*}(s')]^2 \right). \tag{3·15}
\]
In order to evaluate the summation over the modes, we replace it with an integral as
\[ \sum_{\sigma} \sum_{k} \rightarrow \frac{\Omega}{2\pi^2} \int_{k_{\text{min}}}^{\infty} k^2 \, dk, \] (3.16)
where we take \( k_{\text{min}} \) as a constant of \( O(k_c) \). Since the comoving length scale, \( k_c^{-1} \), of interest is that which corresponds to the large scale structure of the universe, we have \( a(t)k_c^{-1} \gg H^{-1} \) at the end of the inflation era. Furthermore, we neglect the effect of interaction before the scale \( k_c^{-1} \) crosses the Hubble horizon scale. Thus we concentrate on the case in which
\[ a(t)H = -\eta^{-1} \gg k_{\text{min}} \] (3.17)
is satisfied, and we set the initial conditions for the reduced density matrix at a time after the length scale of the fluctuations of interest exceeds the Hubble horizon scale. Under this condition, we evaluate the functions \( f(s) \), \( \mu(s, s') \) and \( \nu(s, s') \) approximately. The evaluation of \( f(s) \), \( \mu(s, s') \) and \( \nu(s, s') \) is rather complicated. The details of the computation are given in Appendix A. As shown there, the \( f \) and \( \mu \)-terms contain ultraviolet divergences that are absorbed by the mass and the coupling constant renormalizations. After subtraction of these divergences, keeping only the leading terms, we obtain
\[ f(s) \sim \frac{a^3(s)\Omega H^2}{8\pi^2} \log \left( \frac{H}{p_{\text{min}}(s)} \right), \]
\[ \nu(s, s') \sim \frac{a^3(s)\Omega H^4}{48\pi^2 p_{\text{min}}^3(s)} e^{-3H(s-s')}, \] (3.18)
where we have introduced \( p_{\text{min}}(s) := a^{-1}(s)k_{\text{min}} \). As shown in Appendix A, in order to subtract the divergent portion in the \( \mu \)-term of \( S_{\text{IF}} \), integration by parts with respect to \( s' \) is necessary. Then \( \mu \)-term is evaluated as
\[ S_{\text{IF}}^{(\mu)} = \int_{t_i}^{t} ds \Delta(s) \int_{t_i}^{s} ds' \left[ 2 \left( \mu_i(s, s') + \mu_a(s, s') \right) \Sigma(s') + \mu_b(s, s') \bar{\Sigma}(s') \right], \] (3.19)
where
\[ \mu_i(s, s') \sim \frac{a^3(s)\Omega}{16\pi^2} \log \left( \frac{p_{\text{min}}(s)}{H} \right) \delta(s - s'), \]
\[ \mu_a(s, s') \sim \frac{a^3(s)\Omega H}{16\pi^2} \log \left( k_{\text{min}}(\eta - \eta') \right), \]
\[ \mu_b(s, s') \sim \frac{a(s)a^2(s')\Omega}{16\pi^2} \log \left( k_{\text{min}}(\eta - \eta') \right). \] (3.20)

Precisely speaking, the approximated values of \( \mu \)s are different from the true value by a factor of order unity. Thus \( 16\pi^2 \) can be replaced by, say, \( 24\pi^2 \). However, these errors do not change the discussion given below because, as we show, the effect of these terms is small and can be neglected.
§4. An intuitive interpretation of the effective action

Here we give a less rigorous but quite intuitive analysis of the evolution of the averaged field $\phi$ in the model with $V = \lambda \phi^4/4!$. We defer a more rigorous treatment to the next section, but the essential points that we wish to claim in this paper are contained in this section.

The effective action is rewritten in terms of

$$S[\phi, \phi'] = \Omega \int_{t_i}^{t} ds \ a^3(s) \left\{ \phi_+(s) \phi_-(s) - V'(\phi(s)) \varphi_\Delta \right\} - \lambda \int_{t_i}^{t} ds \ \phi_+(s) \varphi_\Delta(s) f(s)$$

$$+ i \lambda^2 \int_{t_i}^{t} ds \phi_+(s) \varphi_\Delta(s) \int_{s_i}^{s} ds' \ \nu(s, s') \phi_+(s') \varphi_\Delta(s') + S^{(\mu)}[\phi, \phi'],$$

and

$$S^{(\mu)}[\phi, \phi'] = \lambda^2 \int_{t_i}^{t} ds \phi_+(s) \varphi_\Delta(s) \int_{s_i}^{s} ds'$$

$$\times \left[ \left( \mu_i(s, s') + \mu_a(s, s') \right) \phi_+^2(s') + \mu_b(s, s') \phi_+(s') \varphi_\Delta(s') \right],$$

where cubic and higher order terms with respect to $\varphi_\Delta$ are neglected.

Since $\nu(s, s')$ decays fast as $s - s'$ becomes large, here we approximate it as

$$\nu(s, s') \sim \delta(s - s') a^6(s) \Omega^2 \Xi^2(s)/\phi_+^2(s),$$

where

$$\Xi(s) := \frac{\phi_+(s)}{a^3(s) \Omega} \left[ \int_{s_i}^{s} ds' 2\nu(s, s') \right]^{1/2} \sim \sqrt{\frac{H^3}{72\alpha' \pi^2}} \phi_+(s),$$

with $\alpha' := \Omega k_{\min}^3$. Then the $\nu$-term in the action reduces to

$$S_{\nu}[\phi, \phi'] = i \lambda^2 \Omega^2 \int_{t_i}^{t} ds \ a^6(s) \Xi^2(s) \frac{\varphi_\Delta^2(s)}{2}.$$
of the wave packets, $\Psi_\psi(\phi, t)$, where $\psi$ is a label used to distinguish different wave packets. Here we use the initial peak position as the label of wave packets; i.e., $\psi = \phi_{\psi}(t)$. The wave packet labeled by $\psi$ is assumed to have a sharp peak at $\phi_{\psi}(t)$ with a width $(\delta \phi)_{WP}$ that is much smaller than $(\delta \phi)_{QF} \sim H$. We write the reduced density matrix at the initial time as

$$\rho[\phi, \phi'; t_i] = \int d\psi \int d\psi' C(\psi)C^*(\psi') \tilde{\rho}_{\psi, \psi'}[\phi, \phi'; t_i], \quad (4.8)$$

and the initial condition for the partial reduced density matrix is given by

$$\tilde{\rho}_{\psi, \psi'}[\phi, \phi'; t_i] = \Psi_\psi(\phi, t_i)\Psi_{\psi'}^*(\phi', t_i). \quad (4.9)$$

Then we consider the evolution of the partial reduced density matrix, which obeys the same equation as the reduced density matrix. But in this section, we use an alternative method and do not directly solve the evolution of the partial reduced density matrix.

To manage the effect of the $\nu$-term, we introduce an auxiliary field, $\xi$, which represents the Gaussian white noise:

$$\langle \xi(s)\xi(s') \rangle_{EA} = \delta(s - s'), \quad (4.10)$$

where the subscript EA stands for the ensemble average. Further, we introduce the action with noise by

$$S_{\xi}[\phi, \phi'] = \Omega \int_{t_i}^{t_f} ds \ a_3(s) \left\{ \dot{\phi}_+(s)\dot{\phi}_\Delta(s) - V'(\phi(s))\varphi_\Delta \right\} - \lambda \int_{t_i}^{t_f} ds \left( \phi_+(s) f(s) - a_3(s)\Omega \Xi(s)\xi(s) \right) \varphi_\Delta(s) + S(\mu)[\phi, \phi']. \quad (4.11)$$

Then, from the fact that

$$\exp \left( iS[\phi, \phi'] \right) = \langle \exp \left( iS_{\xi}[\phi, \phi'] \right) \rangle_{EA}, \quad (4.12)$$

one may expect that the action with noise, $S_{\xi}$, gives the evolution of the field that includes the effect of the fluctuations induced by the environment through the $\nu$-term.

To examine the first condition for classicality, we consider the equation for the expectation value of $\dot{\phi}_+$. One may think it strange to study the expectation value $\langle \dot{\phi}_+ \rangle$ to examine the broadening of wave packets. However, we claim that the broadening effect due to the environment is included in the dynamics of $\langle \dot{\phi}_+ \rangle$ determined by the action with noise. Since this action contains a stochastic variable $\xi$, the motion of $\langle \dot{\phi}_+ \rangle$ becomes a random walk. As a result, $\langle \dot{\phi}_+ \rangle$ has a variance when we consider the ensemble of $\xi$. We interpret this variance as the broadening of wave packets due to the existence of the environment. Unfortunately, we do not have a simple reasoning which justifies this naive expectation. We put off the justification of this interpretation to the next section.
Taking the variation of $S_\xi$ with respect to $\varphi_\Delta$, we obtain the Heisenberg equation for the operator $\hat{\varphi}_+(t)$. Sandwiching thus obtained Heisenberg equation in between the "bra" and "ket" vectors of the initial state, the equation for the expectation value becomes

$$
\frac{d^2}{dt^2}\langle \hat{\varphi}_+(t) \rangle + 3H \frac{d}{dt}\langle \hat{\varphi}_+(t) \rangle + \frac{\lambda}{6}\langle \hat{\varphi}_+^3(t) \rangle + \frac{\lambda f(t)}{a^3(t)\Omega}\langle \hat{\varphi}_+(t) \rangle
$$

$$
-\frac{\lambda^2}{a^3(t)\Omega}\int_{t_i}^t ds \left[(\mu_i(t,s) + \mu_a(t,s))\langle \hat{\varphi}_+(t)\hat{\varphi}_+(s) \rangle + \mu_b(t,s)\langle \hat{\varphi}_+(t)\hat{\varphi}_+(s)\hat{\varphi}_+(s) \rangle \right]
$$

$$
= \Xi(t)\xi(t).
$$

(4.13)

If we set

$$
\hat{\varphi}_+ = \langle \hat{\varphi}_+ \rangle + \hat{\varphi},
$$

(4.14)

$\langle \hat{\varphi} \rangle = 0$ follows by construction. Thus we find

$$
\langle \hat{\varphi}_+^3(t) \rangle = \langle \hat{\varphi}_+(t) \rangle^3 + 3\langle \hat{\varphi}_+(t) \rangle \langle \hat{\varphi}_+(t) \rangle^2 + \langle \hat{\varphi}_+(t) \rangle^3.
$$

(4.15)

Assuming that the quantum fluctuations are small in the sense that $\langle \hat{\varphi}_+^2(t) \rangle \ll \langle \hat{\varphi}_+(t) \rangle^2$, we obtain

$$
\frac{d^2}{dt^2}\langle \hat{\varphi}_+(t) \rangle + 3H \frac{d}{dt}\langle \hat{\varphi}_+(t) \rangle + \frac{\lambda}{6}\langle \hat{\varphi}_+(t) \rangle^3 + \frac{\lambda f(t)}{a^3(t)\Omega}\langle \hat{\varphi}_+(t) \rangle
$$

$$
-\frac{\lambda^2}{a^3(t)\Omega}\langle \hat{\varphi}_+(t) \rangle\int_{t_i}^t ds \left[(\mu_i(t,s) + \mu_a(t,s))\langle \hat{\varphi}_+(s) \rangle^2 + \mu_b(t,s)\langle \hat{\varphi}_+(s) \rangle^2 \frac{d\langle \hat{\varphi}_+(s) \rangle}{ds} \right]
$$

$$
= \Xi(t)\xi(t).
$$

(4.16)

In the remainder of this section, we simply use $\varphi(t)$ instead of $\langle \hat{\varphi}_+(t) \rangle$. The $f$- and $\mu$-terms give the correction to the evolution of $\varphi$ in a deterministic manner, while the effect of the $\nu$-term (the r.h.s. of Eq. (4.16)) can be interpreted as a stochastic force.

Now we show that the effect of the $f$- and $\mu$-terms is negligible under the present conditions. The effect of $f$-term is just to modify the potential. Thus we compare the force due to the $f$-term (the fourth term in Eq. (4.16)),

$$
\frac{\lambda f(t)\varphi(t)}{a^3(t)\Omega} \sim \frac{1}{8\pi^2}\frac{\lambda \varphi(t)H^2 \log \left( \frac{H}{p_{\text{min}}(t)} \right)}{p_{\text{min}}(t)},
$$

(4.17)

with that due to the bare potential (the third term in Eq. (4.16)),

$$
V' (\varphi(t)) = \frac{\lambda \varphi^3(t)}{6}.
$$

(4.18)

Hence, the contribution from the $f$-term can be neglected if $\varphi^2 \gg H^2 \log(H/p_{\text{min}}(t))$. In the inflationary universe scenario, a typical value of $\varphi$ at the time when the comoving scale of the fluctuations of interest crosses the Hubble horizon scale during inflation is known to become the order of the Planck scale, $m_p$. Therefore the above inequality holds. Thus we conclude that the $f$-term can be neglected.
The contribution from the $\mu$-term is not a simple change of the potential but has a hereditary effect. Again, the force coming from the $\mu_i$ and $\mu_a$-terms is roughly evaluated as

$$-rac{\lambda^2}{a^3(t)\Omega} \dot{\phi}(t) \int_{t_i}^{t} ds \left[ \mu(t, s) \phi^2(s) \right] 
\sim \left| \frac{\lambda^2}{16\pi^2} \phi^3(t) \int_{t_i}^{t} ds' \left[ \log \left( \frac{\mu_{\min}(s')}{\mu_{\max}(s')} \right) \delta(t - s') + H \log \left( k_{\min}(\eta_k - \eta') \right) \right] \right| 
\lesssim \frac{\lambda^2}{16\pi^2} \phi^3(t) (H \Delta t)^2, \quad (4.19)$$

where $\eta_k$ is the conformal time corresponding to the cosmological time $t$, and $\Delta t$ is the maximum value assumed by $t - t_i$. Here we neglected the time-dependence of $\dot{\phi}(s)$, because we expect the slow rolling condition to be satisfied. The details of evaluation of the integral are given in Appendix B. Equation (4.19) is to be compared with Eq. (4.18). Then we find that the $\mu_i$ and $\mu_a$-terms can be neglected as long as the condition $1 \gg (\lambda/16\pi^2)(H \Delta t)^2$ holds. This condition is satisfied for typical values of the model parameters such as $v \sim 10^{-12}$ and $H \Delta t \sim 60$.

We now turn to the contribution from $\mu_b$. In the same way, it is evaluated as

$$-rac{\lambda^2}{a^3(t)\Omega} \dot{\phi}(t) \int_{t_i}^{t} ds \mu_b(t, s) \phi(s) \dot{\phi}(s) 
\sim \left| \frac{\lambda^2}{16\pi^2} \phi^2(t) \dot{\phi}(t) \int_{t_i}^{t} ds' \frac{\eta_k^2}{\eta^2} \log \left( k_{\min}(\eta_k - \eta') \right) \right| 
\lesssim \frac{\lambda^2}{32\pi^2} \phi^2(t) \dot{\phi}(t) \Delta t. \quad (4.20)$$

The details of this calculation too are provided in Appendix B. This term should be compared with the friction term due to the cosmic expansion (the second term in Eq. (4.16)):

$$3H \dot{\phi}(t). \quad (4.21)$$

Then, the ratio of the $\mu_b$-term to the above is evaluated as $(\lambda/48\pi^2)(v/H^2) H \Delta t$, where $v$ is a typical value of the inflaton mass squared: $v \sim \lambda \phi(t)^2/2$. For the $\lambda \phi^4$ model, $v$ is given by $v/H^2 \sim 1/100$. Hence we conclude that the $\mu_b$-term can also be neglected. Thus we concentrate on the effect of the $v$-term, neglecting the $f$- and $\mu$-terms in the remainder of this section.

Under the condition that the fluctuations, $\delta\phi(t) := \phi(t) - \phi(t)|_{t \equiv 0}$, caused by $\xi(t)$ are small, Eq. (4.16) reduces to

$$\delta\dot{\phi}(t) + 3H \delta\dot{\phi}(t) + V''(\phi(t)) \delta\phi(t) = \lambda \Xi(t) \xi(t). \quad (4.22)$$

Then we can evaluate the variance of $\delta\phi$ caused by the stochastic force, following the standard analysis of Brownian motion. Approximating $V''(\phi(t))$ and $\Xi(t)$ by the constants $\nu$ and $\Xi$, respectively, the equation can be solved as

$$\delta\phi(t) = -\frac{\lambda \Xi}{D} \left[ \int_{t_i}^{t} ds e^{-\lambda_1(t-s)} \xi(s) - \int_{t_i}^{t} ds e^{-\lambda_2(t-s)} \xi(s) \right], \quad (4.23)$$
where

\[ \lambda_1 = \frac{3H + D}{2}, \quad \lambda_2 = \frac{3H - D}{2}, \]

\[ D = \sqrt{9H^2 - 4v}. \tag{4·24} \]

Since \( v \) is much smaller than \( H^2 \) for the \( \lambda \phi^4 \) model, we approximately have \( \lambda_1 \sim 3H \) and \( \lambda_2 \sim v/3H \). Then the fluctuations caused by this stochastic field \( \xi(t) \) are evaluated as

\[
\langle (\delta \phi)^2 \rangle_{\text{EA}} \sim \frac{\lambda^2 \Sigma^2}{2D^2} \left( \frac{D^2}{3Hv} \frac{1}{\lambda_2} e^{-2\lambda_2(t-t_i)} \right)
\sim \frac{\lambda}{432\alpha^6\pi^2} \left( 1 - e^{-2\lambda_2(t-t_i)} \right) H^2. \tag{4·25} \]

As stated above, our interpretation is that the \( \nu \)-term broadens the peak width of each wave packet as much as \( \langle (\delta \phi)^2 \rangle_{\text{EA}} \). This effect will be negligibly small as long as the width of the packet, \( (\delta \phi)^2_{\text{WP}} \), is much larger than \( \langle (\delta \phi)^2 \rangle_{\text{EA}} \). Since \( \lambda \) is a small number, typically of \( O(10^{-12}) \), we can choose \( (\delta \phi)^2_{\text{WP}} \) to satisfy \( (\delta \phi)^2_{\text{QF}} \gg (\delta \phi)^2_{\text{EA}} \). In Refs. 7) and 8), this quantity \( \langle (\delta \phi)^2 \rangle_{\text{EA}} \) was interpreted as the real fluctuation, which is expected to become classical. Thus our interpretation is totally different from theirs. Here we do not discuss this issue further. Again we note that a justification of our interpretation is provided in the next section.

As for the first condition, we must also mention the effect related to the uncertainty relation, which has been neglected in the above consideration. We decomposed the initial quantum state of the system into a superposition of wave packets with a small variance with respect to the variables in configuration space, \( \phi \). According to the uncertainty principle, the small variance in \( \phi \) necessarily indicates the existence of a large variance in its conjugate variable, \( \Omega a^3(t)\dot{\phi}(t) \). This variance may induce a broadening of wave packets. This possibility was first pointed out by Matacz. \(^{16}\) The induced variance in \( (\delta \phi)^2 \) will become of \( O([\int_{t_i}^{t} ds \delta \phi(s)]^2) \). The amplitude of \( \delta \phi(s) \) due to the uncertainty relation can be evaluated by \( \Omega a^3(t)(\delta \phi(t))_{\text{UR}} \sim 1 \) as

\[
(\delta \phi)^2_{\text{UR}} := \left( \int_{t_i}^{t} ds (\delta \phi(s))_{\text{UR}} \right)^2 \sim \frac{H^2}{9} \left( \frac{H^2}{(\delta \phi)^2_{\text{WP}}} \right) \left( \frac{2\pi p_c(t_i)}{H} \right)^6, \tag{4·26} \]

where \( p_c(t) := k_c/a(t) \). Thus we find that this effect is also small compared with \( \langle (\delta \phi)^2 \rangle_{\text{QF}} \sim H^2 \) if \( t_i \) is set at a time well after the scale of interest exceeds the Hubble horizon scale and unless \( (\delta \phi)^2_{\text{WP}} \) is extremely small. Further, we note that

\[
\frac{(\delta \phi)^2_{\text{UR}}}{(\delta \phi)^2_{\text{WP}}} \sim \left\{ \frac{H^2}{3(\delta \phi)^2_{\text{WP}}} \left( \frac{2\pi p_c(t_i)}{H} \right)^3 \right\}^2. \tag{4·27} \]

Hence, if \( t_i \) is set to satisfy \( p_{\text{min}}(t_i) \ll H \), we can also choose \( (\delta \phi)^2_{\text{WP}} \) to satisfy \( (\delta \phi)^2_{\text{QF}} \gg (\delta \phi)^2_{\text{WP}} \gg (\delta \phi)^2_{\text{UR}} \). The restriction to the initial time obtained here is
consistent with the general belief that the quantum fluctuations of the inflaton field become classical only after the horizon crossing.

Now we turn to the second condition. The main effect of the $\nu$-term is to reduce the off-diagonal elements of the density matrix, and it brings the quantum state into a decohered state. Looking at Eq. (4·6), the evolution of the initial state (4·8) under the influence of the environment is approximately given by

$$\rho[\phi, \phi'; t] \sim \int d\psi \int d\psi' C(\psi) \Psi_\psi(\phi, t) C^*(\psi') \Psi_{\psi'}^*(\phi', t) \times \exp \left( -\lambda^2 \Omega^2 a^6(t) \Xi^2 (\psi - \psi')^2 \right)$$

(4·28)

where we have treated $\Xi(t)$ and $\phi_\psi(t) - \phi_{\psi'}(t)$ as constants and denoted them as $\Xi$ and $\psi - \psi'$, respectively. The latter replacement is justified because $\dot{\phi}_\psi$ is nearly constant when the slow rolling condition is satisfied. Equation (4·28) indicates that the off-diagonal elements are exponentially suppressed when

$$(\psi - \psi')^2 \gg (\delta \phi)^2_{dec} = \frac{6H}{\lambda^2 \Omega^2 a^6(t) \Xi^2} = 216\pi^2 \alpha' \left( \frac{2\pi p_c(t)}{H} \right)^6 \left( \frac{\lambda v}{H^2} \right)^{-1} H^2.$$  

(4·29)

The factor $\left( \lambda v/H^2 \right)^{-1}$ is a large number typically of $O(10^{14})$, but $(2\pi p_c(t)/H)^6$ becomes extremely small as $e^{-6H(t-t_i)} \sim e^{-360}$. Hence, $(\delta \phi)^2_{QF} \gg (\delta \phi)^2_{dec}$. Therefore even if we require that the wave packets have a peak that is sharp enough to satisfy $(\delta \phi)^2_{QF} \gg (\delta \phi)^2_{WP}$, it is still possible to choose $(\delta \phi)^2_{WP}$ so as to satisfy $(\delta \phi)^2_{WP} \gg (\delta \phi)^2_{dec}$. Furthermore, we have the inequality

$$(\delta \phi)^2_{WP} + (\delta \phi)^2_{UR} > 2(\delta \phi)^2_{WP}(\delta \phi)^2_{UR} \sim \frac{2H^2}{3} \left( \frac{2\pi p_c^6(t_i)}{H} \right)^3 (\delta \phi)^2_{dec}.$$  

(4·30)

This tells us that if the width of the wave packets, i.e., the coarse-graining scale of our view, is appropriately chosen so that the wave packets have sufficient resolution compared with $(\delta \phi)_{QF}$ throughout their evolution, two wave packets with small overlap lose their quantum coherence during inflation. Thus we conclude that, the $\nu$-term leads the quantum state of the system effectively into a decohered state without significantly distorting the shape or the peak position of each wave packet. The result can be recognized as a statistical ensemble of the states represented by these wave packets.

§5. Master equation

In the preceding section, we introduced a hand-waving interpretation of the action with noise without any proof. In this section, we more rigorously study the evolution of the density matrix and re-derive the results obtained in the preceding section.

First, we choose one solution of an approximate equation of motion of $\phi(t)$ obtained by neglecting the effect of the environment. We denote this classical trajectory
as $\tilde{\phi}(t)$. Namely, $\tilde{\phi}(t)$ satisfies

$$\ddot{\tilde{\phi}}(t) + 3H\dot{\tilde{\phi}}(t) + V'(\tilde{\phi}(t)) = 0. \quad (5.1)$$

Here we introduce the new variables $\varphi$ and $\varphi'$, which represent the deviations of $\phi$ and $\phi'$ from the classical trajectory $\tilde{\phi}$, by

$$\phi = \tilde{\phi} + \varphi, \quad \phi' = \tilde{\phi} + \varphi', \quad (5.2)$$

and we assume that $\varphi$ and $\varphi'$ are small. Then the effective action $S[\phi, \phi'] = iS_{\text{sys}}[\phi] - iS_{\text{sys}}[\phi'] + iS_{\Pi} [\phi, \phi']$ is reduced to

$$S[\phi, \phi'] = \Omega \int_{t_i}^{t} ds \ a^3(s) \left\{ \left( \frac{1}{2} \dot{\varphi}^2(s) - \frac{1}{2} V''(\tilde{\phi}(s))\varphi^2(s) + O(\varphi^3) \right) - \left( \frac{1}{2} \dot{\varphi}'^2(s) - \frac{1}{2} V''(\tilde{\phi}(s))\varphi'^2(s) + O(\varphi^3) \right) \right\}$$

$$- \int_{t_i}^{t} ds \ \Delta(s) f(s) + \int_{t_i}^{t} ds \ \Delta(s) \int_{t_i}^{s} ds' \ \nu(s, s')\Delta(s') + S^{(\mu)}[\phi, \phi']. \quad (5.3)$$

The time evolution operator for the density matrix can be obtained by constructing the Hamiltonian, recognizing $\phi$ and $\phi'$ as two different interacting fields. Neglecting cubic and higher order terms in $\varphi$ and $\varphi'$, the Hamiltonian corresponding to this action is obtained as

$$\hat{H}(t) = \frac{1}{\Omega a^3(t)} \hat{P}_+(t) \hat{P}_-(t) + \Omega a^3(t) V''(\tilde{\phi}(t))\hat{\varphi}_+(t)\hat{\varphi}_-(t) + f(t)\hat{\Delta}(t)$$

$$- \hat{\Delta}(t) \int_{t_i}^{t} ds \left[ 2 (\mu_i(t, s) + \mu_a(t, s)) \hat{\Sigma}(s) + \mu_b(t, s) \hat{\Sigma}'(s) + i\nu(t, s)\hat{\Delta}(s) \right], \quad (5.4)$$

where we have defined

$$\varphi_+ := \frac{\varphi + \varphi'}{2}, \quad \varphi_\Delta := \varphi - \varphi', \quad (5.5)$$

and

$$\hat{P}_+ = \hat{P} + \hat{P}', \quad \hat{P}_\Delta = \frac{\hat{P} - \hat{P}'}{2}, \quad (5.6)$$

are the conjugate momenta of $\varphi_+$ and $\varphi_\Delta$, respectively. We note that $\hat{P}$ and $\hat{P}'$ are the conjugate momenta of $\varphi$ and $\varphi'$. As above, hatted variables represent the Heisenberg operators corresponding to these variables.

Since the Hamiltonian (5.4) is very complicated, we simplify it by using several approximations. In the above Hamiltonian, there are Heisenberg operators at a past time. In general, the existence of such operators makes it very difficult to determine the evolution of the density matrix. Here we approximate the Heisenberg operators...
at a past time with those at the present time by using the solution of the lowest order Heisenberg equations,\(^17\) which are given by

\[
\dot{\phi} = \frac{1}{\alpha^3} \dot{P},
\]  
\[
\dot{\phi}' = \frac{1}{\alpha^3} \dot{P}'.
\] (5.7)

Approximating \(V''(\bar{\phi})\) by a constant \(v\), we can solve these equations as

\[
\phi_+(s) = T_1(s,t)\phi_+(t) + T_2(s,t)\dot{P}_\Delta(t),
\]  
\[
\dot{\phi}_\Delta(s) = T_1(s,t)\dot{\phi}_\Delta(t) + T_2(s,t)\dot{P}_+(t),
\] (5.8)

and \(T_1(s,t)\) and \(T_2(s,t)\) are calculated as

\[
T_1(s,t) = \frac{\lambda_1}{D} e^{\lambda_2(t-s)} - \frac{\lambda_2}{D} e^{\lambda_1(t-s)},
\]
\[
T_2(s,t) = \frac{1}{a^3(t)\Omega D} \left( e^{\lambda_2(t-s)} - e^{\lambda_1(t-s)} \right),
\] (5.9)

where \(\lambda_1\) and \(\lambda_2\) are constants defined in Eq. (4.24). For later purposes, we express \(\phi_+(t)\) in an alternative way as

\[
\phi_+(t) = O_2(t)e^{\lambda_2(t-s)} - O_1(t)e^{\lambda_1(t-s)},
\] (5.10)

where the operators \(O_1(t)\) and \(O_2(t)\) are defined by

\[
O_1(t) := \frac{1}{D} \left( \lambda_2\phi_+(t) + \frac{1}{a^3(t)\Omega} \dot{P}_\Delta(t) \right),
\]
\[
O_2(t) := \frac{1}{D} \left( \lambda_1\phi_+(t) + \frac{1}{a^3(t)\Omega} \dot{P}_\Delta(t) \right).
\] (5.11)

Then the \(f\)-term in the Hamiltonian becomes

\[
\hat{H}(f)(t) = \lambda(\phi(t) + \phi_+(t))\dot{\phi}_\Delta(t)f(t).
\] (5.12)

The \(\mu\)-term is explicitly written as

\[
\hat{H}(\mu)(t) = \frac{\lambda^2 a^3(t)\Omega}{32\pi^2} \log \left( \frac{p_{\min}(t)}{H} \right) \dot{\phi}_\Delta(t) \left( \phi(t)^3 + 3\phi(t)^2\phi_+(t) \right)
\]
\[
- \lambda^2 \phi_\Delta(t) \left[ \phi(t) + \phi_+(t) \right] \int_{t_i}^t ds \left( \mu_a(t,s)\phi^2(s) + \mu_b(t,s)\phi(s)\phi_+(s) \right)
\]
\[
- \lambda^2 \phi_\Delta(t) \phi(t) \int_{t_i}^t ds \left( 2\mu_a(t,s)\phi(s)\phi_+(s) + \mu_b(t,s)\phi_+(s) + \mu_b(t,s)\phi(s)\phi_+(s) \right).
\] (5.13)

The first line of the r.h.s. corresponds to the instantaneous part, the \(\mu_i\)-term. In the second line, the ratio between the first and second term in the round brackets is given by

\[
\left| \frac{\mu_a(t,s)\phi(t)}{\mu_b(t,s)\phi(t)} \right| \sim \frac{a^2(t)H\phi(t)}{a^2(s)\phi(t)} \sim \frac{a^2(t)}{a^2(s)} \left( \frac{H^2}{v} \right).
\] (5.14)
This ratio is much greater than unity. Thus the second term can be neglected. For the same reason, the last term in the round brackets in the last line can also be neglected. Then, the last line can be rewritten by using the relation (5·10) as

$$\lambda^2 \phi_\Delta(t) \bar{\phi}(t) O_1(t) \int_{t_i}^{t} ds \{ 2 \mu_a(t, s) \bar{\phi}(s) - \lambda_1 \mu_b(t, s) \bar{\phi}(s) \} e^{\lambda_1 (t-s)}$$

$$- \lambda^2 \phi_\Delta(t) \bar{\phi}(t) O_2(t) \int_{t_i}^{t} ds \{ 2 \mu_a(t, s) \bar{\phi}(s) - \lambda_2 \mu_b(t, s) \bar{\phi}(s) \} e^{\lambda_2 (t-s)}. \quad (5·15)$$

Substituting the explicit forms of \( \mu_a(t, s) \) and \( \mu_b(t, s) \) and approximating \( \bar{\phi}(s) \) by the constant \( \bar{\phi} \), we find that the dominant contribution comes from the terms that contain \( \mu_a(t, s) \). Using the formulas given in Appendix B, we obtain

$$\hat{H}^{(\mu)}(t) = \mu_1(t) \phi_\Delta \bar{\phi} + \mu_2(t) \phi_\Delta \bar{\phi} + \mu_3(t) \phi_\Delta O_1(t) + \mu_4(t) \phi_\Delta O_2(t), \quad (5·16)$$

with

$$|\mu_1(t)|, |\mu_2(t)|, |\mu_4(t)| \lesssim \frac{\lambda^2 \alpha(t) \Omega}{16 \pi^2} \bar{\phi}^2 (H \Delta t)^2,$$

$$|\mu_3(t)| \lesssim \frac{\lambda^2 \alpha(t) \Omega}{16 \pi^2 \alpha^3(t)} \bar{\phi}^2 (H \Delta t), \quad (5·17)$$

where again we have approximated \( \bar{\phi}(t) \) by the constant \( \bar{\phi} \).

The \( \nu \)-term can be also rewritten by using the relation (5·8) as

$$\hat{H}^{(\nu)}(t) = - i \nu_1(t) \phi_\Delta^2(t) - i \nu_2(t) \phi_\Delta(t) P_+(t) + O(\phi^3). \quad (5·18)$$

The coefficients are evaluated as

$$\nu_1(t) \sim \lambda^2 \bar{\phi}(t) \int_{t_i}^{t} ds \, \nu(t, s) \bar{\phi}(s) T_1(s, t)$$

$$\sim \frac{\lambda^2 \bar{\phi}^2}{144 \pi^2 \alpha^3(t)} a^3(t) \Omega H^3 e^{-\lambda_2 (t-t_i)} = : e^{6Ht} e^{-\lambda_2 (t-t_i)} \bar{\nu},$$

$$\nu_2(t) \sim \lambda^2 \bar{\phi}(t) \int_{t_i}^{t} ds \, \nu(t, s) \bar{\phi}(s) T_2(s, t)$$

$$\sim - \frac{e^{6Ht}}{a^3(t) \Omega} \bar{\nu} \left( 1 - e^{-\lambda_2 (t-t_i)} \right), \quad (5·19)$$

where we have used \( H(t-t_i) \gg 1 \) and \( \lambda_2 (t-t_i) = O(1) \). Strictly speaking, the former is not the case for \( t \sim t_i \). However, the absolute value of the correct expression does not become much larger than that of this approximate expression. Since we shall see that the contribution from \( t \sim t_i \) does not become dominant later, this simplification is safe.

Putting all the results together, we obtain a Hamiltonian which does not contain any hereditary terms:

$$\hat{H}(t) = \frac{1}{\Omega a^3(t)} \dot{P}_\Delta \dot{P}_+ + \Omega a^3(t) v(t) \phi_\Delta \dot{\phi}_+ + u(t) \dot{\phi}_\Delta + \mu_3(t) \dot{\phi}_\Delta O_1(t)$$

$$+ \mu_4(t) \dot{\phi}_\Delta O_2(t) - i \nu_1(t) \phi_\Delta^2 - i \nu_2(t) \phi_\Delta P_+. \quad (5·20)$$
Here we have defined
\[ v(t) := V''(\bar{\phi}(t)) + \frac{1}{\Omega a^3(t)} (\lambda f(t) + \mu_2(t)), \]
\[ u(t) := \lambda f(t) \bar{\phi}(t) + \mu_1(t) \bar{\phi}(t). \]  
(5.21)

The master equation can be derived from the above Hamiltonian. In the coordinate representation, it becomes
\[ i\hbar \frac{\partial}{\partial t} \rho = H \rho, \]
where \( \hat{P}_+ \) and \( \hat{P}_\Delta \) in \( \hat{H} \) are to be replaced by \(-i(\partial/\partial \varphi_+)\) and \(-i(\partial/\partial \varphi_\Delta)\), respectively. We note that the same master equation can be derived by means of different methods.\(^ {17} \) Near Eqs. (4.17) \sim (4.19) in the preceding section, we explained that the correction caused by the \( f \)- and \( \mu \)-terms is negligibly small. For the same reason, we can show that \( v(t) \) is dominated by the first term. Hence, we approximate \( v(t) \) by a constant \( \nu \), as before. As for \( u(t) \), which also comes from the \( f \)- and \( \mu \)-terms, it is to be compared with \( a^3(t) \Omega V' \). Since we can show that \( u(t) \) is much smaller than \( a^3(t) \Omega V' \) for the same reason, the effect of \( u(t) \) is negligibly small. However, we keep this term until this fact becomes clear.

Now we discuss the initial conditions for the reduced density matrix. We set the initial conditions at a time when the size of the fluctuations of interest, \( k_c^{-1}a(t) \), becomes larger than the horizon scale, \( H^{-1} \). In our present approximation, any earlier time is inaccessible, for we used the evaluation of the effective action under the assumption that \( a(t)H \gg k_{\text{min}} \sim k_c \). Here we assume that the evolution of the fluctuations for \( t < t_i \) can be approximated by the evolution of a non-interacting field. Then the initial wave function is given by
\[ \Psi(\varphi) \propto \exp \left[ - \frac{A}{2} \varphi^2 \right], \]  
(5.22)
where
\[ A = \frac{1}{8\pi^3 H^2} \left( 1 + i \frac{H}{p_c(t_i)} \right), \]  
(5.23)
and \( p_c(t) = k_c/a(t) \).

As mentioned in §2, we decompose the initial wave function into a superposition of Gaussian wave packets,
\[ \Psi_\psi(\varphi) \propto \exp \left[ - \frac{F}{2} (\varphi - \psi)^2 \right], \]  
(5.24)
as
\[ \Psi(\varphi) \propto \int d\varphi \, e^{-\frac{F}{2} \psi^2} \Psi_\psi(\varphi), \]  
(5.25)
where
\[ F = \frac{A\Gamma}{\Gamma - A}. \]  
(5.26)
As before, \( \psi \) represents the initial position of the peak of wave packets, and it is real. For simplicity, we assume \( \Gamma \) is also real. The peak width of the wave packet, \( \Gamma^{-1/2} \), should be sufficiently narrow compared with the extension of the wave function, \( \sqrt{\langle \varphi^2 \rangle_{QF}} \sim H \). Here we choose \( \Gamma \) to satisfy
\[ H^2 \Gamma \gg \frac{H}{8\pi^3 p_c(t_i)} > 1. \]  
(5.27)
The latter inequality comes from the fact that the initial conditions are set after the scale of the fluctuations of interest becomes larger than the Hubble horizon scale. With this choice of $\Gamma$, we find that $F \sim A$.

Then the reduced density matrix is also decomposed as Eq. (2.6):

$$\tilde{\rho}(t) \propto \int d\psi \int d\psi' \exp \left[ -\frac{F}{2} \psi^2 + \frac{F^*}{2} \psi'^2 \right] \tilde{\rho}_{\psi\psi'}(t),$$

where we have introduced the partial reduced density matrix $\tilde{\rho}_{\psi\psi'}(t)$, whose initial conditions are given by

$$\tilde{\rho}_{\psi\psi'}(t_i) \propto \exp \left[ -\frac{\Gamma}{2} \left\{ (\varphi - \psi)^2 + (\varphi' - \psi')^2 \right\} \right].$$

The partial reduced density matrix satisfies the same evolution equation as $\tilde{\rho}(t)$, and its amplitude describes the degree of coherence between the two worlds labeled by the peaks of the wave packets $\psi$ and $\psi'$.

In order to determine the evolution of $\tilde{\rho}_{\psi\psi'}(\varphi, \varphi'; t)$, it is better to work in the $k - \varphi$ representation, which is defined by

$$\zeta(k, \varphi; t) = \int d\varphi_+ \exp \left[ -ik \varphi_+ a^3(t) \Omega \right] \tilde{\rho}_{\psi\psi'}(\varphi_+, \varphi; t).$$

Then the evolution equation for $\zeta$ becomes

$$\frac{\partial}{\partial t} \zeta = \left[ 3Hk \frac{\partial}{\partial k} - k \frac{\partial}{\partial \varphi} + v_\varphi \frac{\partial}{\partial k} - iw \varphi \right]$$

$$\left. + \mu_3(t) \varphi \tilde{O}_1 + \mu_4(t) \varphi^2 \tilde{O}_2 - v_1 \varphi^2 - a^3 \Omega \nu \varphi \right] \zeta,$$

where we have defined

$$\tilde{O}_1 := \frac{1}{a^3(t) \Omega D \left[ \lambda_2 \frac{\partial}{\partial k} - \frac{\partial}{\partial \varphi} \right]},$$

$$\tilde{O}_2 := \frac{1}{a^3(t) \Omega D \left[ \lambda_1 \frac{\partial}{\partial k} - \frac{\partial}{\partial \varphi} \right]}.$$ (5.32)

Let us introduce the notation

$$\tilde{\rho}_{\psi\psi'}(t) = N \exp \left[ -\frac{1}{2} \sum_{i,j} m_{ij} y_i y_j - \sum_i n_i y_i \right],$$

where $y_i = (\varphi_+, \varphi)$. The density matrix in the $k - \Delta$ representation is also given by the Gaussian form

$$\zeta = N_\zeta \exp \left[ -\frac{1}{2} \sum_{i,j} M_{ij} x_i x_j - \sum_i N_i x_i \right],$$

(5.34)
where \( x_i = (k, \varphi_\Delta) \). The coefficients \( m_{ij}, n_i \) and \( N \) are related to \( M_{ij}, N_i \) and \( N_\zeta \) by

\[
\begin{align*}
m_{++} &= \frac{a^6 \Omega^2}{M_{kk}}, \quad m_+ = \frac{a^3 \Omega M_{k\Delta}}{M_{kk}}, \quad m_{\Delta\Delta} = M_{\Delta\Delta} - \frac{M_{k\Delta}^2}{M_{kk}}, \\
n_+ &= \frac{ia^3 \Omega N_k}{M_{kk}}, \quad n_\Delta = N_\Delta - \frac{M_{k\Delta} N_k}{M_{kk}}, \quad N \sim N_\zeta \exp \left[ -\frac{n_+^2}{m_{++}} \right].
\end{align*}
\]

Thus we obtain

\[
\begin{align*}
M_{kk}(t_i) &= \frac{a^6 (t_i) \Omega^2}{2\Gamma}, \quad M_{k\Delta}(t_i) = 0, \quad M_{\Delta\Delta}(t_i) = \frac{\Gamma}{2}, \\
N_k(t_i) &= ia^3 (t_i) \Omega \psi_+, \quad N_\Delta(t_i) = \frac{\Gamma}{2} \psi_\Delta,
\end{align*}
\]

where

\[
\psi_+ := \frac{\psi + \psi'}{2}, \quad \psi_\Delta := \psi - \psi'.
\]

In the \( k - \varphi_\Delta \) representation, the evolution equation for the density matrix reduces to the following set of equations:

\[
\begin{align*}
\frac{d}{dt} M &= \left( \begin{array}{ccc}
6H & -2 & 0 \\
v & 3H & -1 \\
0 & 2v & 0
\end{array} \right) M + \left( \begin{array}{c}
0 \\
a^3 \Omega \nu_2 \\
2\nu_1
\end{array} \right), \\
\frac{1}{a^3(t) \Omega D} \left( \begin{array}{c}
\mu_3 (M_{k\Delta} - \lambda_2 M_{kk}) + \mu_4 (M_{k\Delta} - \lambda_1 M_{kk}) \\
2\mu_3 (M_{\Delta\Delta} - \lambda_2 M_{k\Delta}) + 2\mu_4 (M_{\Delta\Delta} - \lambda_1 M_{k\Delta})
\end{array} \right),
\end{align*}
\]

\[
\frac{d}{dt} N = \left( \begin{array}{c}
3H \\
v \\
0
\end{array} \right) N + \left( \begin{array}{c}
0 \\
iu
\end{array} \right),
\]

\[
\frac{d}{dt} N_\zeta = 0,
\]

where

\[
M = \left( \begin{array}{c}
M_{kk} \\
M_{k\Delta} \\
M_{\Delta\Delta}
\end{array} \right), \quad N = \left( \begin{array}{c}
N_k \\
N_\Delta
\end{array} \right).
\]

We first consider the evolution of \( M_{ij} \). For this purpose, we define

\[
\sigma_1 := 3H, \quad \sigma_{2,3} := 3H \pm D,
\]

\[
e_1 := \left( \begin{array}{c}
2 \\
\sigma_1 \\
2v
\end{array} \right), \quad e_2 := \left( \begin{array}{c}
2 \\
\sigma_3 \\
\sigma_3^2 / 2
\end{array} \right), \quad e_3 := \left( \begin{array}{c}
2 \\
\sigma_2 \\
\sigma_2^2 / 2
\end{array} \right).
\]

Then

\[
\left( \begin{array}{c}
0 \\
0 \\
1
\end{array} \right) = \frac{1}{2D^2} (e_2 + e_3 - 2e_1), \quad \left( \begin{array}{c}
0 \\
1 \\
0
\end{array} \right) = \frac{1}{2D^2} (2\sigma_1 e_1 - \sigma_2 e_2 - \sigma_3 e_3).
\]
follows. Introducing a new parameterization of $M_{ij}$ by

$$M = M_1 e_1 + M_2 e_2 + M_3 e_3,$$  \hspace{1cm} (5.44)

the equations for $M_j$, where $j = 1, 2$ or $3$, are decoupled as

$$\frac{dM_j}{dt} = \sigma_j M_j + S_{Mj},$$  \hspace{1cm} (5.45)

where

$$\begin{pmatrix} S_{M1} \\ S_{M2} \\ S_{M3} \end{pmatrix} = \frac{\nu_1}{D^2} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} + \frac{a^3 \Omega \nu_2}{D^2} \begin{pmatrix} \sigma_1 \\ -\sigma_2/2 \\ -\sigma_3/2 \end{pmatrix} - \frac{\mu_3}{a^3 \Omega D} \begin{pmatrix} M_1 - 2M_3 \\ -M_1 \\ 2M_3 \end{pmatrix} - \frac{\mu_4}{a^3 \Omega D} \begin{pmatrix} M_1 - 2M_2 \\ 2M_2 \\ -M_1 \end{pmatrix}. \hspace{1cm} (5.46)$$

We solve the above equation perturbatively, taking $\lambda$ as a small parameter. In this sense, $M_j$ is also expanded in powers of $\lambda$. At the lowest order, we solve the homogeneous equation without the source term, $S_{Mj}$, in Eq. (5.45). Then to find the next order solution of Eq. (5.45), we need to solve the equation with this source term. At this stage, we can substitute the lowest order solution, $M_j^{(0)}(t)$, into $S_{Mj}$. Then $S_{Mj}$ can be considered as a given source term. Here we should keep in mind the limitation of the present analysis. In deriving Eq. (5.45), only the one loop order correction was taken into account, and the lowest order Heisenberg equation was used to remove hereditary terms in the Hamiltonian. Thus only the corrections up to $O(\lambda^2)$ are meaningful. If we carelessly solve Eq. (5.45) without regard to this limitation, many unphysical pathological features will arise.

Formally, the solution is given by

$$M_j(t) = e^{\sigma_j(t-t_i)} M_j(t_i) + \delta M_j(t),$$  \hspace{1cm} (5.47)

where

$$\delta M_j(t) = \int_{t_i}^t ds e^{\sigma_j(t-s)} S_{Mj}(s).$$  \hspace{1cm} (5.48)

Noting that

$$\begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \frac{1}{D^2} \begin{pmatrix} -\nu \\ \sigma_2/8 \\ \sigma_3/8 \end{pmatrix} \begin{pmatrix} \sigma_1 \\ -\sigma_2/2 \\ -\sigma_3/2 \end{pmatrix} - \frac{\mu_3}{\Omega D} \begin{pmatrix} M_1 - 2M_3 \\ -M_1 \\ 2M_3 \end{pmatrix} - \frac{\mu_4}{\Omega D} \begin{pmatrix} M_1 - 2M_2 \\ 2M_2 \\ -M_1 \end{pmatrix},$$  \hspace{1cm} (5.49)

we can calculate the initial value of $M_j(t)$ as

$$\begin{pmatrix} M_1(t_i) \\ M_2(t_i) \\ M_3(t_i) \end{pmatrix} \sim \frac{a^6(t_i) \Omega^2}{2 \Gamma D^2} \begin{pmatrix} -\nu \\ \sigma_2/8 \\ \sigma_3/8 \end{pmatrix} + \frac{\Gamma}{2D^2} \begin{pmatrix} -1 \\ 1/2 \\ 1/2 \end{pmatrix}. \hspace{1cm} (5.50)$$
Then the lowest order solution in $\lambda$ is given by $M_j^{(0)}(t) = e^{\sigma_j(t-t_i)}M_j(t_i)$. For later convenience, we introduce

$$\tilde{\Gamma} := \left( \frac{1}{\Gamma} + \frac{\Gamma}{\Gamma_c^2} \right)^{-1},$$

where we have defined a critical value for $\Gamma$ by

$$\Gamma_c = Da^3(t_i)\Omega \sim \frac{3}{H^2} \left( \frac{H}{2\pi p_c(t_i)} \right)^3.$$  \hfill (5.52)

Then it follows that $M_2(t_i) \sim a^6(t_i)\Omega^2/2D^2\tilde{\Gamma}$. We note that the order of magnitude of the initial value of the other components is the same or smaller than that of $M_2(t_i)$.

To obtain the next order correction, first we need to evaluate the source term $S_{Mj}$. The order of magnitude of $\nu$-term is estimated as

$$S^{(\nu)}_{Mj} = O \left( \frac{\lambda^2 \phi^2 a^6(t)\Omega^2 H}{144\pi^2 \alpha'} \right),$$

where $\alpha' = \Omega k_{\text{min}}^3$, while that of $\mu$-term is estimated as

$$\begin{pmatrix} S^{(\mu)}_{M1} \\ S^{(\mu)}_{M2} \\ S^{(\mu)}_{M3} \end{pmatrix} \lesssim \frac{\lambda^2 \phi^2 a^6(t)\Omega^2 H(H\Delta t)^2}{16\pi^2 \tilde{\Gamma} H^2} \begin{pmatrix} 1 \\ 1 \\ a^3(t_i)/a^3(t) \end{pmatrix}.$$  \hfill (5.54)

Now it is clear that the contribution from the $\mu$-term is smaller than $4\alpha'(H\Delta t)^2/(\tilde{\Gamma} H^2)$ times that from the $\nu$-term. Here, we restrict our attention to the case

$$\tilde{\Gamma} \gg \frac{1}{H^2}.$$  \hfill (5.55)

Then the contribution from $\nu$-terms dominates, and it can be evaluated as

$$\begin{pmatrix} \delta M_1(t) \\ \delta M_2(t) \\ \delta M_3(t) \end{pmatrix} \sim \frac{\nu_1(t)}{9H^2} \begin{pmatrix} (1 - e^{-\lambda_2(t-t_i)})/\lambda_2 \\ (3H/2\lambda_2^2)(1 - e^{-\lambda_2(t-t_i)})^2 \\ 1/(6H) \end{pmatrix},$$

It is not trivial that the contribution from the $\mu$-term is suppressed in comparison with that from the $\nu$-term. The time dependence of $\mu_3(t)$ is approximately proportional to $a^6(t)$. Hence, if there appears the combination $a^{-3}(t_i)\mu_3(t)M_2(t)$ in $S^{(\mu)}_{Mj}$, it behaves as $a^6(t)$ and dominates the source term when $a^3(t)/a^3(t_i)$ becomes exponentially large. The disappearance of this kind of dangerous terms is not manifest in Eqs. (5.45) and (5.46). Here we should note that the contribution of the $\mu$-term to $\delta M_3$ is much more suppressed by the existence of the factor $a^3(t_i)/a^3(t)$ compared with that of the $\nu$-term. Thus, $\delta M_1$ and $\delta M_2$ might be dominated by the $\mu$-term, but $\delta M_3$ is not. Here we note that, in the following evaluations of $m_{ij}(t)$ and $n_j(t)$, there is no relevant contribution from $\delta M_1$ and $\delta M_2$. Among the $\delta M_j$,
a relevant contribution is provided only by $\delta M_3$, which is almost insensitive to the effect of the $\mu$-term.

We are interested in the late time behavior of the partial reduced density matrix. When

$$e^{3H(t-t_i)} \gg \frac{1}{\Gamma H^2},$$

(5.57)

$M_1(t)$ and $M_3(t)$ are dominated by the inhomogeneous solution, $\delta M_j(t)$. Then from Eq. (5.47), we find

$$M_1(t) \sim \frac{\lambda^2 \phi^2 a^3(t) \Omega}{1296 \pi^2 p_{\text{min}}^3(t)} H(1-e^{-\lambda_2(t-t_i)}),$$

$$M_2(t) \sim e^{\sigma_2(t-t_i)} M_2(t_i) + \delta M_2(t) \sim \frac{a^6(t) \Omega^2}{4\Gamma} e^{-\sigma_3(t-t_i)} + \delta M_2(t) = \frac{a^6(t) \Omega^2}{4\Gamma'},$$

$$M_3(t) \sim \frac{\lambda^2 \phi^2 a^3(t) \Omega}{7776 \pi^2 p_{\text{min}}^3(t)},$$

(5.58)

where we have defined

$$\Gamma' : = \left( \frac{1}{\Gamma} + \frac{\Gamma}{\Gamma_e} + \frac{1}{\Gamma_\nu} \right)^{-1}$$

(5.59)

and

$$\Gamma_\nu = \frac{a^6(t) \Omega^2}{4 \delta M_2(t)}.$$  

(5.60)

Note that the $M_i(t)$ are all real.

If we again substitute this solution into $S^{(\mu)}_{M_j}$, its time dependence becomes proportional to $a^9(t)$, while that of $S^{(\nu)}_{M_j}$ is given by $a^6(t)$. Thus it seems that $S^{(\mu)}_{M_j}$ becomes dominant when $a^3(t)/a^3(t_i)$ becomes large. However, this does not imply the breakdown of the present perturbation scheme. In the present calculation, we used the lowest order Heisenberg equation in deriving Eq. (5.45). Thus as mentioned above, such a substitution of $\delta M_j$ into $S^{(\mu)}_{M_j}$ does not give a valid result. If we perform the next order calculation, we should see perfect cancellation between such dangerous terms that might break the perturbation scheme.

Using Eqs. (5.35), (5.44), (5.49) and (5.58), we finally obtain

$$m_{++}(t) \sim \frac{a^6(t) \Omega^2}{2M_2(t)} e^{\sigma_3(t-t_i)} \frac{a^6(t_i) \Omega^2}{2M_2(t_i)} \sim 2 \bar{\Gamma} e^{\sigma_3(t-t_i)},$$

$$m_{+\Delta}(t) \sim \frac{iv}{3H} a^3(t) \Omega,$$

$$m_{\Delta\Delta}(t) = \left( 4D^2 M_2(t) M_3(t) - D^2 M_1^2(t) \right) /2M_2(t) \sim 18H^2 M_3(t).$$

(5.61)

It should be mentioned that $m_{+\Delta}$ remains purely imaginary. Thus it does not contribute to the absolute magnitude of the density matrix.

Next we consider the $N$-part. In a similar manner, we can obtain the solution of Eq. (5.39) as

$$\begin{pmatrix} N_k(t) \\ N_\Delta(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_2 & \lambda_1 \end{pmatrix} \begin{pmatrix} N_1(t) \\ N_2(t) \end{pmatrix},$$

(5.62)
Here $\delta N_j(t)$ is an inhomogeneous solution related to the $u$-term. In later discussion, we see that the effect of these terms can be neglected. Thus, for the present purpose, we have only to know their order of magnitude. Hence, we roughly approximate the time dependence of $u(t)$ by $\propto a^3(t)$. Then we obtain

$$
\begin{pmatrix}
\delta N_1(t) \\
\delta N_2(t)
\end{pmatrix}
\sim
\frac{i u(t)}{9H^2}
\begin{pmatrix}
-3H(1-e^{-\lambda_2(t-t_i)})/\lambda_2 \\
1
\end{pmatrix}.
$$

By using the relation

$$
\begin{pmatrix}
N_1 \\
N_2
\end{pmatrix}
= \frac{1}{D}
\begin{pmatrix}
\lambda_1 & -1 \\
-\lambda_2 & 1
\end{pmatrix}
\begin{pmatrix}
N_k \\
N_{\Delta}
\end{pmatrix},
$$

we obtain $N_1(t_i)$ and $N_2(t_i)$ from (5·36). After evolution described by Eq. (5·63), and under transformation by Eqs. (5·62) and (5·35), we finally obtain

$$
n_+(t) \sim \frac{i a^3(t)\Omega^2 N_1(t)}{2M_2(t)}
\sim 2\tilde{\Gamma}'
\left[ -e^{\lambda_2(t-t_i)}\psi_+ + e^{\sigma_3(t-t_i)} \left( \frac{u(t)}{3H^2a^3(t)\Omega} \right) \frac{H(1-e^{-\lambda_2(t-t_i)})}{\lambda_2}
-\frac{i e^{\lambda_2(t-t_i)} \Gamma\psi_{\Delta}}{6Ha^3(t_i)\Omega} \right],
$$

$$
n_{\Delta}(t)= \frac{D}{2M_2(t)} (2M_2(t)N_2(t) - 2M_3(t)N_1(t) + M_1(t)(N_2(t) - N_1(t)))
\sim -i \left[ -\frac{u(t)}{3H} + e^{\lambda_2(t-t_i)} \frac{12H\tilde{\Gamma}'M_1(t)}{a^3(t)\Omega^2} \psi_+ 
+ e^{\lambda_2(t-t_i)} \frac{2\tilde{\Gamma}'M_1(t)}{a^3(t)\Omega^2} \psi_{\Delta} \right].
$$

We have solved for the evolution of the partial reduced density matrix. By using this solution, we examine the classicality condition. Let us begin with considering the first condition. To see the broadening of wave packets, we consider the expectation value of $(\varphi_+)^2$, which is evaluated as

$$
\langle(\varphi_+)^2\rangle := \left[ \int \tilde{\rho}_{\varphi_+}(\varphi_+)^2 d\varphi_+ d\varphi_{\Delta} \right] / \left[ \int \tilde{\rho}_{\varphi_+}(\varphi_+)^2 d\varphi_+ d\varphi_{\Delta} \right] = \frac{2}{m_{++}} \sim \frac{1}{\tilde{\Gamma}'}
$$

at late time.

When the initial width of wave packets is large, i.e., when $\Gamma \ll \Gamma_c$, the first term in (5·59) will dominate. Then we find that $\langle(\varphi_+)^2\rangle$ stays almost constant. On the other hand, if we assume that the initial width of wave packets is small, i.e., when
\( \Gamma \gg \Gamma_c \), the second term in (5·59) will dominate. Then the width of wave packets at late time is evaluated as

\[
\langle (\varphi_+)^2 \rangle = \frac{1}{\Gamma} \left( \frac{\Gamma}{\Gamma_c} \right)^2 \frac{H^2}{9} (\Gamma H^2) \left( \frac{2\pi p_c(t_i)}{H} \right)^6,
\]  

(5·68)

and it becomes much larger than the initial width of wave packets, \( \Gamma^{-1} \). As is known from the fact that this broadening exists even in the case when the interaction is turned off, it is due to the uncertainty relation. We can see that Eq. (5·68) exactly corresponds to Eq. (4·26).

It is easy to see that the minimum width of wave packets at late time is achieved when \( \Gamma \sim \Gamma_c \) and is evaluated as

\[
\langle (\varphi_+)^2 \rangle_{\text{min}} \sim 2H^2 \Gamma_c + \frac{2 \delta M_2(t_i)}{a^6(t) \Omega^2} \sim \frac{2H^2}{3} \left( \frac{8\pi^3 p_c^3(t_i)}{H^3} \right) + \frac{\lambda}{216 \alpha' \pi^2} \left( \frac{H(1 - e^{-\lambda_2(t-t_i)})}{\lambda_2} \right)^2 \left( \frac{v}{H^2} \right) H^2.
\]  

(5·69)

If

\[
\frac{\Gamma_c}{\Gamma_\nu} \sim \frac{\lambda}{4 \alpha' \pi^2} \left( \frac{H}{2\pi p_c(t_i)} \right)^3 \left( \frac{H^2}{v} \right) \ll 1,
\]  

(5·70)

we can neglect the second term in Eq. (5·69), which is the effect due to the \( \nu \)-term. However, when the initial conditions are set at late time to satisfy \( \Gamma_c \gg \Gamma_\nu \), the broadening effect due to the uncertainty relation becomes relatively small. In this case, the broadening mechanism that is responsible for the \( \nu \)-term is important. In the previous section, we estimated the fluctuations in \( \delta \phi \) due to this effect in Eq. (4·25), and we left the task to justify our interpretation of \( \langle (\delta \phi)^2 \rangle_{EA} \). Now we can easily see that the second term in Eq. (5·69) corresponds to the expression previously derived in Eq. (4·25). Hence, we have confirmed that \( \langle (\delta \phi)^2 \rangle_{EA} \) represents the broadening caused by the environment.

Before discussing the second condition of classicality, we briefly consider the peak location of wave packets, which is calculated as

\[
\langle \varphi_+(t) \rangle = \frac{\mathcal{R}(n_+)}{m_{++}} \sim e^{-\lambda_2(t-t_i)} \psi_+ + \frac{u(t)}{3H^2 a^3(t) \Omega} \frac{H(1 - e^{-\lambda_2(t-t_i)})}{\lambda_2}.
\]  

(5·71)

The first term represents the change of the separation of the different trajectories labeled by the initial position of the peak, \( \psi_+ \). In the present model, congruence of the classical trajectories converges gradually. The second term is independent of \( \psi_+ \). This term arises because the interaction with the environment was not taken into account when we determined \( \tilde{\varphi}(t) \). Thus it can be interpreted as the correction to \( \tilde{\varphi}(t) \) due to the effect of the environment. Recall that \( u(t) \) comes from the \( f \)- and \( \mu \)-terms. By the same reasoning given near Eqs. (4·17) \~ (4·19), we can show that this correction to the motion of \( \tilde{\varphi}(t) \) is negligibly small.
Now, we turn to the second condition. As a measure of decoherence, we use the ratio (2.10),

\[ R := \frac{\max|\hat{\rho}_{\psi\psi'}(t)|}{\max|\hat{\rho}_{\psi\psi'}(t_i)|}, \]

where "max" stands for the maximum value when \( \varphi_+ \) and \( \varphi_\Delta \) are varied. As was discussed in §2, \( R \) is the quantity that represents how efficiently the coherence between the two worlds (two wave packets) labeled by \( \psi \) and \( \psi' \) gets lost. With this definition, \( R \) is evaluated as

\[ R = \exp (K(t) - K(t_i)), \]

with

\[ K := \frac{1}{2} \left( \frac{\Re(n_\Delta)^2}{m\Delta\Delta} + \frac{\Im(n_+)^2}{m_{++}} \right). \]

In deriving this formula, we have used the fact that \( N_c \) is a constant and we have neglected the small logarithmic correction that comes from the Gaussian-integral in Eq. (5.30). Then, after a straightforward calculation, we obtain

\[ K(t_i) = \frac{\Gamma}{4} \psi_\Delta^2, \]

and

\[ K(t) \sim \left( \frac{M_1(t)}{M_2(t)M_3(t)} + 1 \right) \frac{\Gamma' \Gamma^2}{4\Gamma_c^2} \psi_\Delta^2 \]

for \( H(t - t_i) \gg 1 \). In Eq. (5.75), the first and the second terms in the round brackets represent the contributions from the corresponding terms in Eq. (5.73). It is clear that the second term dominates \( K(t) \). Neglecting the first term, we obtain

\[ R = \exp \left( -\frac{\Gamma'}{4} \left[ 1 + \frac{\Gamma}{\Gamma'} \right] \psi_\Delta^2 \right). \]

This means that the coherence between two wave packets labeled by \( \psi \) and \( \psi' \) is exponentially suppressed for large \( \psi_\Delta \), and the typical scale of the decoherence between two different wave packets, \( (\delta\psi)_\text{dec}^2 \), is given by

\[ (\delta\psi)_\text{dec}^2 \sim \frac{2}{\Gamma' \left[ 1 + (\Gamma/\Gamma') \right]}. \]

From this, we find that

\[ (\delta\psi)_\text{dec}^2 < 2((\delta\varphi_+)^2) \]

holds in general.

Let us consider several limiting cases.

1) When \( \Gamma \ll \Gamma' \), we have \( (\delta\psi)_\text{dec}^2 \approx 2((\delta\varphi_+)^2) \). In this case, since the broadening of wave packets is responsible for the uncertainty relation, it is completed rather quickly. Once the overlap of two wave packets becomes large, their coherence is
maintained. Hence, the typical scale of decoherence is determined by the final width of wave packets.

2) When \(1/\Gamma_\nu \gg 1/\Gamma, \Gamma/\Gamma_c^2\), we have \((\delta \psi)^2_{\text{dec}} \approx 2\Gamma^{-1}\). In this case, the broadening of wave packets is responsible for the effect of the \(\nu\)-term. It proceeds rather slowly. Therefore, the decoherence occurs before the initial width of wave packets is broadened. Thus, the typical scale of decoherence is determined by the initial width of wave packets that we choose.

Here we find that the typical scale of the decoherence is not directly related to \((\delta \phi)^2_{\text{dec}}\), which was evaluated in Eq. (4·29). This scale appears in \(m^{-1}_{\Delta \Delta}\), which gives exactly the same expression as \((\delta \phi)^2_{\text{dec}}\). To understand the meaning of this fact, we focus on one diagonal component of the partial reduced density matrix, \(\tilde{\rho}_{\psi \psi}(\varphi, \varphi'; t)\). \(\tilde{\rho}_{\psi \psi}(\varphi, \varphi'; t)\) also has two continuous arguments, \(\varphi\) and \(\varphi'\). The suppression of the off-diagonal elements of \(\tilde{\rho}_{\psi \psi}(\varphi, \varphi'; t)\) is determined by \(m_{\Delta \Delta}\), and \(\tilde{\rho}_{\psi \psi}(\varphi, \varphi'; t)\) becomes exponentially small for large \(\varphi_{\Delta}\) which satisfies \(\varphi_{\Delta} \gg (\delta \phi)_{\text{dec}}\). Thus we can say that decoherence occurs within a wave packet on the scale of \((\delta \phi)_{\text{dec}}\). However, for narrower wave packets, the classicality of the evolution of the system cannot be maintained through the entire duration of interest. In the present case, this condition of the classicality of the evolution of the system completely determines the minimum width of the wave packet.

§6. Summary and discussion

We investigated the evolution of the perturbation of the inflaton field with a \(\lambda \phi^4\) potential after the scale of the perturbation exceeds the Hubble horizon scale during inflation. The effect of the coupling to the smaller scale modes through the self-interaction was taken into consideration by using the closed-time path formalism. That is, the smaller scale modes are considered as the environment. The initial conditions for the quantum state of the fluctuations of the inflaton field were set well after the time of the horizon crossing of the considered mode. The initial state was assumed to be given by a pure state density matrix composed of a direct product of the usual Euclidean vacuum state, which has a variance of \(O(H^2)\). This initial quantum state can be recognized as a quantum mechanical superposition of wave packets with a sharp peak. In the present model, we found that the influence of the environment does not strongly distort these wave packets, but it extinguishes the coherence between the wave packets with different peak positions. The efficiency of the decoherence is so great that different wave packets can be recognized as consisting of completely independent worlds. Hence, we can conclude that the initial pure state evolves into a mixed state which can be interpreted as a statistical ensemble.

In the context of the inflationary universe scenario, the primordial fluctuations of the universe are evaluated by using an ad hoc classicalization ansatz such that the expectation value of the squared field operator can be interpreted as the amplitude of the variance of the statistical ensemble. In this paper we have shown that this ansatz can be verified in a simple model. Thus the result obtained here gives a partial justification of the standard calculation of the primordial fluctuations.

However, an important issue is whether the inflaton field behaves classically
during the reheating process that occurs after inflation. This is because the previous study on reheating is based mostly on the assumption that the fluctuations have already become classical before the reheating occurs. In order to prove that the standard calculation works, we need not show that the fluctuations of the inflaton become classical throughout the entire duration of inflation. We need only show this to be the case at the end of the inflation. In this sense, the conditions for the standard calculation to be justified might be weaker than those required in the present paper.

In this paper, we restricted our consideration to a specific model, and the effect of the metric perturbation and the process of reheating were not taken into account at all. For this reason, further study is required.

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Appendix A

--- Evaluation of $f(s)$, $\mu(s, s')$ and $\nu(s, s')$ ---

In this appendix, we give the details of the calculation to obtain the approximate expressions for $f(s)$, $\mu(s, s')$ and $\nu(s, s')$ given in Eqs. (3·18) and (3·20).

A.1. $f(s)$

We begin with the evaluation of $f(s)$. The explicit expression for $f(s)$ is written as

$$f(s) = \frac{a(s) \Omega}{4\pi^2} \int_{k_{\text{min}}}^{\infty} dk \frac{k}{2} \left[ 1 + \frac{1}{k^2 \eta^2} \right].$$

(A.1)

This expression is divergent and requires renormalization. For this purpose, we use the point splitting technique. The function $f(s)$ is basically given by using the Weightman function as $f(s) \propto \int d^3 x G^{(+)}(x, s; x, s)$. Now we regularize the expression as $\int d^3 x G^{(+)}(x, s, x', s)$, where $x'$ is chosen to satisfy $|x - x'|^2 = z^2$. After this regularization, $f(s)$ is calculated straightforwardly as

$$f(s) = \frac{a^3(s) \Omega}{16\pi^2} \int_{p_{\text{min}}(s)}^{\infty} dp \int_{1}^{\frac{1}{p^2}} d(\cos \theta) p \left( 1 + \frac{H^2}{p^2} \right) e^{ip \cos \theta z}$$

$$= \frac{a^3(s) \Omega}{16\pi^2} \left\{ \frac{2}{z^2} - \frac{p^2_{\text{min}}(s)}{z^2} + 2H^2 - H^2 \left[ \text{Ei}(-ip_{\text{min}}(s)z) + \text{Ei}(ip_{\text{min}}(s)z) \right] + O(z) \right\}$$

$$\sim \frac{a^3(s) \Omega}{8\pi^2} \left\{ \frac{1}{z^2} - H^2 \log z - H^2(\gamma - 1) - \frac{p^2_{\text{min}}(s)}{2} - H^2 \log(p_{\text{min}}(s)) + O(z) \right\},$$

(A.2)

where $\text{Ei}$ is the exponential integral function. The divergent first term in the last line exists even in the limiting case where the background curvature can be neglected. Hence, it should be subtracted in the renormalization procedure. The second term
is attributed to the renormalization of the curvature coupling term, $\xi R\phi^2$, where $R$ is the scalar curvature of the spacetime. Setting an appropriate renormalization condition, we obtain the renormalized $f(s)$ as

$$ f(s) = \frac{a^3(s)\Omega}{8\pi^2} \left[ -\frac{1}{2}H^2 p_{\text{min}}(s) + H^2 \log \left( \frac{H}{p_{\text{min}}(s)} \right) \right]. \quad (A\cdot3) $$

A.2. $\mu(s, s')$

The function $\mu(s, s')$ is explicitly written as

$$ \mu(s, s') = -i \frac{a^3(s)a^3(s')\Omega}{16\pi^2} \int_{k_{\text{min}}}^\infty dk k^2 \left( u^*(s)^2 u(s')^2 - u(s)^2 u^*(s')^2 \right) $$

$$ = -i \frac{a(s)a(s')\Omega}{16\pi^2} \int_{k_{\text{min}}}^\infty dk \left( e^{2ik(\eta - \eta')} \left( 1 + \frac{i}{k\eta'} \right)^2 \left( 1 - \frac{i}{k\eta} \right)^2 - (\text{c.c.}) \right), \quad (A\cdot4) $$

where $\eta$ and $\eta'$ are the conformal time corresponding to $s$ and $s'$. We divide $\mu(s, s')$ into two pieces, $\mu_r(s, s')$ and $\mu_s(s, s')$. We have

$$ \mu_s(s, s') := -i \frac{a(s)a(s')\Omega}{16\pi^2} \int_{k_{\text{min}}}^\infty dk \left( e^{2ik(\eta - \eta')} - (\text{c.c.}) \right). \quad (A\cdot5) $$

This is the portion that contains the ultraviolet divergence corresponding to the coupling constant renormalization. Then, $\mu_r(s, s')$ consists of the remaining regular terms defined by $\mu_r(s, s') := \mu(s, s') - \mu_s(s, s')$.

The expression for $\mu_r(s, s')$ is slightly complicated, but there is no technical difficulty in its evaluation. After a straightforward calculation, we obtain

$$ \mu_r(s, s') \sim \frac{a(s)a(s')\Omega}{\pi^2} \left\{ \frac{1}{12} \left( \frac{1}{\eta} - \frac{1}{\eta'} \right) + \frac{1}{12} \left[ \frac{\eta'}{\eta^2} - \frac{\eta}{\eta'^2} \right] \right\} \times \left( -\frac{7}{3} + \gamma + \log [2k_{\text{min}}(\eta - \eta')] \right). \quad (A\cdot6) $$

Here we take into account (3·17) and pick up only the dominant terms in the $k_{\text{min}} \to 0$ limit. However, this expression is still complicated. In this paper, we only aim to show that the effect of the $\mu$-term is negligibly small. For this purpose, we can simplify the expression by taking the $|\eta| \ll |\eta'|$ limit as

$$ \mu_r(s, s') \sim -\frac{a^3(s)\Omega H}{12\pi^2} \log (k_{\text{min}}(\eta - \eta')). \quad (A\cdot7) $$

When $s \sim s'$, this simplified expression is not correct. However, since the expression for $\mu(s, s')$ given in (A·6) vanishes in the coincidence limit, $s' \to s$, the present simplification does not underestimate the effect of the $\mu$-term. Hence, this simplification is justified.
The singular part $\mu(s, s')$ needs regularization as above. Recalling that $\mu(s, s')$ is essentially given by the product of the Weightman function as

$$
\mu(s, s') \propto \int d^3x \left[ G^{(+)}(x, t; 0, t') \right]^2 + (\text{c.c.}),
$$

we can introduce the point splitting regularization by replacing $[G^{(+)}(x, t; 0, t')]^2$ by $G^{(+)}(x, t; 0, t')G^{(+)}(x + \epsilon, t; 0, t')$, where $|\epsilon|^2 = z^2$. Then the regularized expression for $\mu(s, s')$ becomes

$$
\mu(s, s') = -i \frac{a(s)a(s')\Omega}{32\pi^2} \int_{k_{\text{min}}}^{\infty} dk \int_{-1}^{1} d(\cos \theta) \left( e^{2ik(\eta - \eta')} - e^{-2ik(\eta - \eta')} \right) e^{-ikz\cos \theta/a(s)}.
$$

In order to subtract the singular term which needs renormalization, it is necessary to consider the expression including the $s'$ integral,

$$
I := \int_t^s ds' \Sigma(s')\mu(s, s').
$$

Integrating by parts, this integral can be rewritten as

$$
I = \left[ a^2(s')U(s, s')\Sigma(s') \right]_t^s - \int_t^s ds'U(s, s') \frac{d(\Sigma(s')a^2(s'))}{ds'},
$$

where

$$
U(s, s') := \frac{a(s)\Omega}{64\pi^2} \int_{k_{\text{min}}}^{\infty} dk \int_{-1}^{1} d(\cos \theta) \left[ e^{2ik(\eta - \eta')} + e^{-2ik(\eta - \eta')} \right] e^{-ikz\cos \theta/a(s)}
$$

$$
= \int_{s'}^{s''} \frac{ds''}{a^2(s'')} \mu(s, s'').
$$

In evaluating $U(s, s')$, when $s \neq s'$, we can take the $z \to 0$ limit without any trouble, and it is easily evaluated as

$$
U(s, s') \sim -\frac{a(s)\Omega}{16\pi^2} \left[ \log(2k_{\text{min}}(\eta - \eta')) + \gamma \right].
$$

On the other hand, when $s = s'$, $U(s, s)$ becomes

$$
U(s, s) \sim \frac{a(s)\Omega}{16\pi^2} \left[ (1 - \gamma - \log z) - \log(p_{\text{min}}(s)) \right],
$$

which contains the logarithmic divergence corresponding to the coupling constant renormalization. After the renormalization, we obtain

$$
U(s, s) \sim -\frac{a(s)\Omega}{16\pi^2} \log \left( \frac{p_{\text{min}}(s)}{H} \right).
$$

Combining all the results, we finally obtain the expression given in Eq. (3.19).
A.3. $\nu(s, s')$

In the same way, the dominant terms in $\nu(s, s')$ are evaluated as

$$\frac{a(s)^3 a(s')^3 \Omega}{8\pi^2} \int_{k_{\text{min}}}^{k_{\text{max}}} k^2 dk \left( u^*(s)^2 u(t')^2 + u(s)^2 u^*(t')^2 \right)$$

$$\sim \frac{a(s)a(s')\Omega}{32\pi^2} \left( \frac{2}{3} \frac{1}{\eta^2 \eta'^2 k_{\text{min}}^3} + \frac{\sin[2k_{\text{max}}(\eta - \eta')]}{\eta - \eta'} \right). \quad (A.16)$$

The second term becomes proportional to $\delta(\eta - \eta')$ in the $k_{\text{max}} \to \infty$ limit. Since the time dependence of the second term is different from that of the first one, we cannot simply discard the second term. However, the important quantity for the discussion in the present paper is the integral over $s'$ of the product of $\nu(s, s')$ and some function $F(s, s')$ which is always a smooth function of $s - s'$. Thus we can conclude that the first term gives the dominant contribution, and the second term can be neglected.

Appendix B

--- Approximate Formulas for Integrals ---

Here we explain the details of the approximation used in evaluating several integrals. In this appendix, we set $\eta = -e^{-H s}$, $\eta' = -e^{-H s'}$ and $\eta_i = -e^{-H t_i}$.

First we evaluate the integral

$$I_1 := \int_{t_i}^{s} ds' H \log(k_{\text{min}}(\eta - \eta'))$$

$$= - \int_{\eta_i}^{\eta} \frac{dn'}{n'} \log(k_{\text{min}}(\eta - n')). \quad (B.1)$$

Changing the variable of integration to $x := (\eta' - \eta)/\eta$, the integral becomes

$$I_1 = \int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} \log x + \int_{(\eta_i - \eta)/\eta}^{1} \frac{dx}{1 + x} \log(-k_{\text{min}}\eta). \quad (B.2)$$

The second term can be explicitly calculated to give $H(s - t_i) \log(-k_{\text{min}}\eta)$. When $(\eta_i - \eta)/\eta < 1$, the first term is bounded by

$$\left| \int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} \log x \right| < \left| \int_0^{1} \frac{dx}{1 + x} \log x \right| = \frac{\pi^2}{12}. \quad (B.3)$$

Contrastingly, when $(\eta_i - \eta)/\eta > 1$, the first term is evaluated by

$$\int_0^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} \log x = -\frac{\pi^2}{12} + \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} \log x \quad (B.4)$$

and

$$\left| \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} \log x \right| < \log \frac{\eta_i - \eta}{\eta} \int_1^{(\eta_i - \eta)/\eta} \frac{dx}{1 + x} = \log \frac{\eta_i - \eta}{\eta} \log \frac{\eta_i}{2\eta}. \quad (B.5)$$
Thus we find
\[ |I_1| \lesssim (H \Delta t)^2. \] (B-6)

Next we consider
\[ I_2 := \int_{t_i}^{s} ds' H \frac{a^2(s')}{a^2(s)} \log (k_{\text{min}}(\eta - \eta')) \]
\[ = -\eta^2 \int_{\eta_i}^{\eta} \frac{d\eta'}{\eta^3} \log (k_{\text{min}}(\eta - \eta')). \] (B-7)

After a straightforward calculation by using the same change of variables as above, we obtain
\[ I_2 = -\frac{1}{2} \left[ \frac{\eta^2}{\eta_i^2} \log \left( \frac{\eta_i}{\eta} - 1 \right) + \log \left( \frac{\eta_i}{\eta_i - \eta} \right) + 2 \frac{\eta - \eta_i}{\eta_i} \right] + \frac{1}{2} \left( 1 - \frac{\eta^2}{\eta_i^2} \right) \log (-k_{\text{min}} \eta). \] (B-8)

Thus we find
\[ |I_2| \lesssim H \Delta t. \] (B-9)

Finally, we consider
\[ I_3 := \int_{t_i}^{s} ds' H \frac{a^3(t_i)}{a^3(s')} \log (k_{\text{min}}(\eta - \eta')) \]
\[ = \eta_i^{-3} \int_{\eta_i}^{\eta} d\eta' \eta' \log (k_{\text{min}}(\eta - \eta')). \] (B-10)

In the same way, after a straightforward calculation, we obtain
\[ I_3 = \eta_i^{-3} \left[ \frac{1}{3} \left( \frac{\eta_i^3}{\eta^3} - 1 \right) \log \left( \frac{\eta_i}{\eta} - 1 \right) - \frac{1}{9} \left( \frac{\eta_i}{\eta} - 1 \right)^3 \right. \]
\[ - \frac{1}{2} \left( \frac{\eta_i}{\eta} - 1 \right) \left. - \left( \frac{\eta_i}{\eta} - 1 \right) + \frac{1}{3} \left( \frac{\eta_i^3}{\eta^3} - 1 \right) \log (-k_{\text{min}} \eta) \right]. \] (B-11)

This implies
\[ |I_3| \lesssim H \Delta t. \] (B-12)

**Appendix C**

**Effect of Finite \textit{k}_c**

In this paper, we have clarified the behavior of the mean field averaged over a fixed comoving volume \( \Omega = 1/(2\pi k_c)^3 \) after the corresponding physical scale exceeds the horizon scale during inflation. Intuitively, it is natural to consider that the fluctuations of this mean field represent those of a perturbation mode with the comoving wave number \( k_c \). To show this, let us begin by considering a much larger volume \( \Omega' \gg \Omega \) as the total system. In this case, the mode of interest is no longer the largest scale of the system. Then \( \Omega \) in Eq. (3.5) should be replaced by \( \Omega' \). Now
we need to keep the next order term in Eq. (3-6). Neglecting modes with \( k < k_{\text{min}} \) other than that specified by \( k^\sigma = k^+_c \), where \( k_{\text{min}} > |k_c| \), we have

\[
S \sim \tilde{\Omega} \int_{t_i}^t dt \ a^3(s) \left[ \frac{1}{2} \dot{\phi}^2(s) - V(\phi(s)) \right] \\
+ \frac{1}{2} \int_{t_i}^t dt \ a^3(s) \left[ \left( \frac{q^+_{kc} k^+_{kc} a^2(s)}{a^2(s) + V''(\phi(s))} \right) \left( q^+_{kc} \right)^2 \right] \\
- \frac{1}{2\sqrt{\tilde{\Omega}}} \sum_{\sigma} \sum_{|k| < k_{\text{min}}} \int_{t_i}^t dt \ a^3(s) V''(\phi(s)) q^+_{kc} q^\sigma_{kc} k^2_a \\
+ \frac{1}{2} \sum_{\sigma} \sum_{|k| < k_{\text{min}}} \int_{t_i}^t dt \ a^3(s) \left[ \left( \frac{q^\sigma_{kc}}{a^2(s)} + V''(\phi(s)) \right) \left( q^\sigma_{kc} \right)^2 \right], \quad (C.1)
\]

in place of Eq. (3-7). Expanding \( \phi(s) \) as given in Eq. (5-1), the action (3-7) takes a form similar to that given in (C.1) with the replacement \( q^+_{kc} / \sqrt{\tilde{\Omega}} \rightarrow \varphi \). The only difference is in the interaction term,

\[
- \frac{1}{2} \sum_{\sigma} \sum_{|k| < k_{\text{min}}} \int_{t_i}^t dt \ a^3(s) V''(\phi(s)) \varphi q^\sigma_{kc} q^\sigma_{kc} \\
\leftrightarrow - \frac{1}{2} \sum_{\sigma} \sum_{|k| < k_{\text{min}}} \int_{t_i}^t dt \ a^3(s) V''(\phi(s)) \varphi (q^\sigma_{kc})^2. \quad (C.2)
\]

This small difference cause a tiny change in the estimate of the coefficients, \( f, \mu \) and \( \nu \), but the effective action (5-3) is essentially unchanged. Hence the behavior of the fluctuations of \( \varphi \) discussed in the present paper can be considered as those of a mode with the comoving wave number \( k_c \).

References


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