Convergence of Boson Expansion Theory

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When the Holstein-Primakoff (Hermitian)-type boson expansion theory is applied for nuclear structure calculations, truncation at the fourth-order expansion of the boson Hamiltonian is often used. The Dyson-type boson mapping has no difficulty concerning boson expansion, since the Dyson boson Hamiltonian is written in up to sixth-order boson operators, and we do not have to make any truncation of the boson expansion. In the present work, we propose a new clear-cut method to obtain the normal-ordered Holstein-Primakoff boson expansion. Using this method, we test whether or not the truncation of the boson expansion is good approximation in various cases. We show that truncation at the fourth-order expansion of the normal-ordered Holstein-Primakoff boson Hamiltonian is a rather good approximation, but it is not sufficient for detailed analyses.

§1. Introduction

A many-fermion Hilbert space for even-particle systems can be mapped onto an ideal boson space, and a fermion operator is thereby exactly transformed into a corresponding boson operator. This is called the boson mapping or boson expansion theory. It is based on a Lie algebra of bilinear operators (or phonon operators) in the original fermion space. Since any shell-model problem can be formulated in terms of the Lie algebra, the problem is also worked out in the mapped boson space. If one treats all degrees of freedom in the full fermion space, the Lie algebra is of the group \( SO(2n) \), where \( n \) is the total number of the single-particle states.

If some “collective” phonons are approximately decoupled from the rest, called “non-collective” phonons, we can truncate them to retain only the collective ones and the relevant boson mapping would be based on the collective subalgebra in \( SO(2n) \). Let us call this the “closed-algebra approximation”. Under this approximation we can introduce two types of boson mappings, the Holstein-Primakoff-type mapping and the Dyson-type mapping.

As is known well, the Holstein-Primakoff-type boson mapping has the difficulty that the mapped boson operator inevitably takes the form of an infinite series boson expansion, although it has the big merit that the mapping is unitary and the boson Hamiltonian is Hermitian. In the Dyson-type mapping, a mapped operator is written as a finite series boson expansion. This is the biggest merit of the Dyson mapping. Although the non-Hermiticity of the boson Hamiltonian had long been thought to be a serious difficulty in the Dyson mapping, this problem was solved. Accordingly, there remains no difficulty in the Dyson boson mapping. The Dyson mapping for the collective fermion space (CFS), which is a multi-phonon subspace constructed by piling up many phonon operators in the full fermion Hilbert space, has turned out to be quite a powerful method to carry out shell-model calculations in the CFS. Some
realistic calculations have been done by using the Dyson mapping, demonstrating its usefulness. 5) - 8)

However, the Holstein-Primakoff-type boson expansion is often used for nuclear structure calculations, and the power series expansion of the boson Hamiltonian is usually truncated at fourth order, i.e., the order of terms consisting of four boson operators. Among these types of work, those of Kishimoto, Tamura and Sakamoto 9) - 12) have attracted particular attention because in these works this type of expansion has been applied to various kinds of nuclei, and these authors have obtained many valuable results. On the other hand, it should be noted that phenomenological boson models like the IBM (interacting boson model) 14), 15) have succeeded in describing nuclear collective phenomena over a wide range of nuclei. For the IBM, we have two big questions: (1) What physical entity is represented by the interacting "bosons"? (2) Why can the Hamiltonian in the IBM be written as a Hermitian fourth-order Hamiltonian in general? We guess that the second question relates to the convergence of the boson expansion. It is therefore very interesting to study the convergence of the Holstein-Primakoff-type boson expansion in comparison with the Dyson-type boson mapping theory. This is the aim of the present paper.

In §2 we discuss a solvable three-level model to check the convergence of the normal-ordered Holstein-Primakoff boson expansion by comparing it with the Dyson boson mapping which gives exact solutions. In §3, we apply the fourth-order normal-ordered boson expansion to the case of realistic nuclei, where the quadrupole correlation is significant, and we compare the results with the Dyson mapping to discuss the convergence. All analysis in §3 is in the quasi-particle representation. In §4, we also apply the fourth-order normal-ordered boson expansion to the case of the octupole collectivity in the particle-hole representation. Some concluding remarks are given in §5.

§2. A three-level model

In this section, we discuss a solvable three-level model to check the convergence of the Holstein-Primakoff-type boson expansion by comparing it with the Dyson boson mapping which gives exact solutions.

Let us consider a simple three-level model called the $SU(3)$ model, 16) in which each single-particle level has the same spin $j$ and thus the same degeneracy $2\Omega = 2j + 1$. A single-particle state is therefore specified by a set of quantum numbers $(im)$, where $i$ denotes the level number ($i = 0, 1, 2$) and $m$ specifies $2\Omega$ degenerate states in the level $i$. We define the generators

$$K_{ij} = \sum_{m=1}^{2\Omega} a_{im}^\dagger a_{jm}, \quad (i,j = 0,1,2) \quad (2\cdot1)$$

where $a_{im}^\dagger$ is a creation operator of a particle in a state $(im)$. These generators satisfy the Lie algebra of the group $SU(3)$

$$[K_{ij}, K_{kl}] = \delta_{jk}K_{il} - \delta_{il}K_{kj}. \quad (2\cdot2)$$
We assume that the interaction Hamiltonian $H_F$ consists of operators $K_{ij}$ only and
the lowest single-particle level ($i = 0$) is completely filled in the free vacuum $|0\rangle$ in
the absence of correlation; namely, we are considering a $2\Omega$-fermion system. We
introduce the following Hamiltonian:

$$H_F = \sum_{i=1,2} \varepsilon(i) K_{ii} + V_1 \sum_{i,j=1,2} K_{i0} K_{0j}$$
$$+ \frac{1}{2} V_2 \sum_{i,j=1,2} (K_{i0} K_{j0} + K_{0j} K_{0i}) + V_3 \sum_{i,j,k=1,2} (K_{i0} K_{jk} + K_{kj} K_{0i}),$$

(2.3)

with real parameters $\varepsilon(i)$, $V_1$, $V_2$ and $V_3$.

Let us define the Tamm-Dancoff (TD) phonon operators

$$X^\dagger_\lambda \equiv \frac{1}{\sqrt{2\Omega}} \sum_i x_\lambda(i) K_{i0}. \quad (\lambda = 1, 2)$$

(2.4)

The amplitudes $x_\lambda(i)$ and the energy eigenvalues of these phonons are obtained by
solving the TD eigenvalue equation. We call the lowest energy phonon ($\lambda = 1$) the
collective phonon and the other ($\lambda = 2$) the non-collective one. The amplitudes $x_\lambda(i)$
are real, and they satisfy the orthonormality $\sum_i x_\lambda(i) x_{\lambda'}(i) = \delta_{\lambda\lambda'}$.

In this model, the exact eigenstates can be solved by diagonalizing the Hamiltonian within the subspace

$$\{|n_1, n_2\} = (n_1! n_2!)^{-1/2} (X_1^\dagger)^{n_1} (X_2^\dagger)^{n_2} |0\rangle; \quad 0 \leq n_1 + n_2 \leq 2\Omega \}.$$

(2.5)

2.1. Boson mapping for SU(3) model

Let us introduce boson operators $b_\lambda^\dagger$ and $b_\lambda$ satisfying the following commutation
relation:

$$[b_\lambda, b_\lambda^\dagger] = \delta_{\lambda\lambda'}.$$

(2.6)

We can define orthonormalized boson basis vectors as

$$\{|n_1, n_2\} = (n_1! n_2!)^{-1/2} (b_1^\dagger)^{n_1} (b_2^\dagger)^{n_2} |0\rangle; \quad 0 \leq n_1 + n_2 \leq 2\Omega,$$

(2.7)

where the orthonormality is written

$$(n_1, n_2)|n'_1, n'_2\rangle = \delta_{n_1, n'_1} \delta_{n_2, n'_2}.$$

(2.8)

Since there exists a one-to-one correspondence between the spaces (2.5) and
(2.7), we can introduce two types of boson mappings from the space (2.5) onto
the space (2.7); one is the Dyson mapping and the other is the Holstein-Primakoff
mapping.

Under the Dyson mapping, the boson images of the generators are written as

$$(K_{i0})_D = (2\Omega)^{1/2} \sum_{\lambda=1,2} x_\lambda(i) b_\lambda^\dagger \left( 1 - \frac{1}{2\Omega} n \right),$$

(2.9a)

$$(K_{0i})_D = (2\Omega)^{1/2} \sum_{\lambda=1,2} x_\lambda(i) b_\lambda,$$

(2.9b)

$$(K_{ij})_D = \sum_{\lambda, \lambda' = 1,2} x_\lambda(i) x_{\lambda'}(j) b_\lambda^\dagger b_{\lambda'},$$

(2.9c)
where \( n = \sum_{\lambda=1,2} b_{\lambda}^\dagger b_{\lambda} \).

Under the Holstein-Primakoff mapping, the generators are mapped as

\[
(K_{i0})_{\text{HP}} = (2\Omega)^{1/2} \sum_{\lambda=1,2} x_\lambda(i) b_{\lambda}^\dagger \sqrt{1 - \frac{1}{2\Omega}} n, \tag{2.10a}
\]

\[
(K_{0i})_{\text{HP}} = (2\Omega)^{1/2} \sum_{\lambda=1,2} x_\lambda(i) \sqrt{1 - \frac{1}{2\Omega}} n^2 b_{\lambda}, \tag{2.10b}
\]

\[
(K_{ij})_{\text{HP}} = \sum_{\lambda, \lambda^\prime=1,2} x_\lambda(i) x_{\lambda^\prime}(j) b_{\lambda}^\dagger b_{\lambda^\prime}. \tag{2.10c}
\]

2.2. Boson expansion

Expanding the operator \( \sqrt{1 - \frac{1}{2\Omega} n} \) in a power series in \( n \), we have a Holstein-Primakoff boson expansion. Let us call this expansion the “simple-minded” boson expansion. It has been said that this simple-minded expansion has slow convergence since it is not normal-ordered. Rearranging this, we can have a normal-ordered expansion which is written as a power series in normal-ordered boson operators, i.e.,

\[
\sqrt{1 - \frac{1}{2\Omega} n} = 1 + a^{(1)}(b_1^\dagger b_1 + b_2^\dagger b_2) + a^{(2)}(b_1^\dagger b_1 b_1 b_1 + b_2^\dagger b_1 b_1 b_1 + 2b_2^\dagger b_1 b_1 b_1 + b_2^\dagger b_2 b_2 b_2) + \cdots, \tag{2.11}
\]

where the coefficients \( a^{(1)} \) and \( a^{(2)} \) are given by

\[
a^{(1)} = \sqrt{1 - \frac{1}{2\Omega} - 1}, \tag{2.12a}
\]

\[
a^{(2)} = \frac{1}{2} \left( \sqrt{1 - \frac{2}{2\Omega} - 2\sqrt{1 - \frac{1}{2\Omega}} + 1} \right). \tag{2.12b}
\]

Putting the boson expansions (2.11) into the Holstein-Primakoff boson images (2.10), we have the boson expansions of the generators in the normal order as

\[
(K_{i0})_{\text{HP}} = (2\Omega)^{1/2} \sum_{\lambda=1,2} x_\lambda(i) b_{\lambda}^\dagger \left( 1 + a^{(1)} n + a^{(2)} \sum_{\lambda^\prime, \lambda^\prime^\prime=1,2} b_{\lambda}^\dagger b_{\lambda^\prime} b_{\lambda^\prime^\prime} b_{\lambda^\prime^\prime} + \cdots \right), \tag{2.13a}
\]

\[
(K_{0i})_{\text{HP}} = [(K_{i0})_{\text{HP}}]^\dagger, \tag{2.13b}
\]

\[
(K_{ij})_{\text{HP}} = \sum_{\lambda, \lambda^\prime=1,2} x_\lambda(i) x_{\lambda^\prime}(j) b_{\lambda}^\dagger b_{\lambda^\prime}. \tag{2.13c}
\]

These boson expansions (2.13) are equivalent to the linked-cluster expansion discussed by Kishimoto and Tamura.\(^{10}\)

The Dyson-type boson Hamiltonian is obtained by replacing the generators \( K_{ij} \) in the original fermion Hamiltonian (2.3) with the boson images (2.9).
The Holstein-Primakoff-type boson Hamiltonian is obtained by replacing the generators $K_{ij}$ in (2·3) with the boson images (2·10). We can calculate the exact eigenstates of the boson Hamiltonian in such a simple case as the present $SU(3)$ model. However, in the cases of realistic nuclei, we cannot obtain exact eigenstates of the Holstein-Primakoff-type boson Hamiltonian. We therefore expand it in a power series of boson operators and truncate it at a lower-order power, for instance, at the fourth order. Now we expand the present boson Hamiltonian obtained above in a power series to test the validity of this truncation approximation, although we can exactly diagonalize it. This is done by replacing the generators $K_{ij}$ in (2·3) with the boson expansions (2·13) and by reordering the boson creation and annihilation operators in the normal order. Usually they take only up to fourth-order terms. There are two possible cases of approximation: (1) $X_3H_4$: We take up to the third-order expansion for $(K_{ij})_{HP}$ or $(K_{ij})_{HP}$ and take the fourth-order Hamiltonian. (2) $X_5H_4$: We take up to the fifth-order expansion for $(K_{ij})_{HP}$ or $(K_{ij})_{HP}$ and take the fourth-order Hamiltonian. We compare the results of these two cases with the exact solutions which are given by the Dyson boson mapping or by diagonalizing the Holstein-Primakoff boson Hamiltonian exactly.

2.3. Numerical calculations and discussions

Assuming the spin of the single-particle level to be $j = 13/2$, i.e., $2\Omega = 14$, the single-particle energies to be $\varepsilon(1) = \varepsilon$ and $\varepsilon(2) = 2.5\varepsilon$, and the parameters in the fermion Hamiltonian (2·3) to be $V_1 = V_2 = -\chi$ and $V_3 = -\chi/2$, we obtain the exact and approximate eigenstates of the boson Hamiltonians.

The calculated excitation energies as a function of the interaction strength $\chi$ are indicated by dashed lines for the case of $X_5H_4$ in Fig. 1, where the Holstein-Primakoff boson Hamiltonian is approximated by a fourth-order boson expansion after taking up to fifth-order terms of the boson expansion of the phonon operators. These are compared with the exact excitation energies denoted by the solid lines.

The difference between the numerical results for $X_3H_4$ and $X_5H_4$ is quite small; the figure for the case of $X_3H_4$, in which the boson Hamiltonian is approximated by a fourth-order boson expansion after taking up to third-order terms of the boson expansion of the phonon operators, is almost the same as Fig. 1. We therefore omit the figure for $X_3H_4$.

In Fig. 2, we show the results for the $X_5H_6$ case, where the boson Hamiltonian is approximated by a sixth-order boson expansion after taking up to fifth-order terms of the boson expansion of the phonon operators.

Figures 3 and 4 display the results of the boson mappings for the collective fermion subspace (CFS), which consists of the collective phonon operator only; namely, the phonon with $\lambda = 1$ in (2·4) is retained and the non-collective one is neglected. The present CFS is therefore

$$\{ |n_1, 0\rangle = (n_1!)^{-1/2}(X_1^d)^{n_1}|0\rangle; \ 0 \leq n_1 \leq 2\Omega \}.$$  \hfill (2·14)

The boson images of the generators are obtained by ignoring the non-collective ($\lambda = 2$) boson operators in (2·9) \sim (2·11).

In Fig. 3, the dashed lines indicate the excitation energies for the case of $CX_5H_4$.
where the Holstein-Primakoff boson Hamiltonian is approximated by a fourth-order boson expansion after taking the fifth-order terms of the collective boson expansion for the collective phonon operator. The solid lines in this figure and the following figures, Figs. 3 and 4, represent the results of the Dyson mapping in which no truncation of the collective boson expansions is needed. In this sense, the solid lines in Figs. 3 and 4 are, so to speak, “exact” results.

Figure 4 displays the results for the case of CX$_5$H$_6$ in which the Holstein-Primakoff boson Hamiltonian is approximated by a sixth-order boson expansion after taking the fifth-order terms of the collective boson expansion for the collective phonon operator.

Comparing the first two figures (Figs. 1 and 2) and the other two (Figs. 3 and 4), we can see that the fourth-order truncation of the expansion of the Holstein-Primakoff boson Hamiltonian gives a worse approximation in the transitional region around and beyond the RPA critical point ($\chi_c/\varepsilon = 0.02551$ in the present model). However the sixth-order truncation gives a quite good approximation in the entire region of the interaction strength and the excitation energy, and we inevitably have
to employ the sixth-order approximation of the boson expansion in order to apply the Hermitian boson mapping for the highly excited states. Accordingly, we can say that the Holstein-Primakoff boson mapping has no advantage over the Dyson boson mapping in which any truncation of boson expansion is not required.

§3. The case of quadrupole correlation

In this section, we test the convergence of the Holstein-Primakoff-type boson expansion in a realistic case, that of $^{114}$Cd. All the analyses here are done in the quasiparticle representation.

We start with a Hamiltonian consisting of the monopole pairing (P) force, the quadrupole pairing ($P_2$) force, and quadrupole-quadrupole (QQ) force. The $P_2$ force consists of the proton part and the neutron part, being denoted by $\pi$ and $\nu$, respec-
tively. After the Bogoliubov transformation determined by the P force only, we take up the normal product parts of these forces with respect to the quasiparticle operators \( a_\alpha^\dagger \) and \( a_\alpha \), where a single-particle state is characterized by a set of quantum numbers \( \alpha = (n_\alpha, l_\alpha, j_\alpha, m_\alpha, \pi \text{ or } \nu) \). We use the notation \( a = (n_\alpha, l_\alpha, j_\alpha, \pi \text{ or } \nu) \), and also \( -\alpha = (n_\alpha, l_\alpha, j_\alpha, -m_\alpha, \pi \text{ or } \nu) \).

The Hamiltonian is therefore written in the quasiparticle representation as

\[
H = H_0 + H^P + H^{P^2}(\pi) + H^{P^2}(\nu) + H^{QQ},
\]

where \( H_0 = \sum_\alpha E_\alpha a_\alpha^\dagger a_\alpha \), \( H^P = -\frac{1}{4} G : \hat{P}_0^\dagger \hat{P}_0 : \), \( H^{P^2}(\rho) = -\frac{1}{2} G_2 : \sum_M \hat{P}_2^\dagger_M(\rho) \hat{P}_2^M(\rho) : \), \( (\rho = \pi \text{ or } \nu) \), and \( H^{QQ} = -\frac{1}{2} \chi \sum_M : \hat{Q}_2^\dagger_M \hat{Q}_2^M :) \),

where \( E_\alpha \) is a single-quasiparticle energy and \( G, G_2 \) and \( \chi \) are the strengths of the P, P^2 and QQ forces, respectively. The symbol \( : : \) denotes the normal product with respect to the quasiparticle operators. The operator \( \hat{P}_0^\dagger \) denotes the \((J = 0)\) coupled nucleon pair and \( \hat{P}_2^\dagger_M(\rho) \) \((\rho = \pi \text{ or } \nu)\) the \((J = 2)\) coupled proton or neutron pair. The operator \( \hat{Q}_2^\dagger_M \) denotes the mass-quadrupole moment. Since the details of the quasiparticle representation of the Hamiltonian have been given in previous papers,\(^5\),\(^17\) we do not repeat them here.

3.1. Dyson boson mapping

We define quasiparticle pair operators by

\[
A_{JM}^\dagger(ab) = \sqrt{\frac{1}{2}} \sum_{m_\alpha m_\beta} \langle j_\alpha m_\alpha j_\beta m_\beta |JM \rangle a_\alpha^\dagger a_\beta^\dagger,
\]

\[
B_{JM}^\dagger(ab) = -\sum_{m_\alpha m_\beta} \langle j_\alpha m_\alpha j_\beta m_\beta |JM \rangle a_\alpha^\dagger \tilde{a}_\beta,
\]

where \( \tilde{a}_\beta = (-)^{j_\beta - m_\beta} a_{-\beta} \).

Let us consider three kinds of collective TD phonon operators, the collective quadrupole phonons, the collective monopole pairing-vibrational phonons for neutrons, and those for protons. Here, the term “collective phonon” refers to the lowest-energy solution of the TD eigenvalue equation. These collective phonons are given by the form

\[
X_{i,m_i}^{(i)} = \sum_{a,b} x_i(ab) A_{i,m_i}^\dagger(ab),
\]

where the superscript \((i)\) specifies the kind of collective TD phonon; namely, \( i = 1, 2 \) and 3 denote the quadrupole phonon, the pairing phonon for neutrons, and that
for protons, respectively, and \( l_1 = 2 \) and \( l_2 = l_3 = 0 \). We introduce three kinds of collective bosons corresponding to the above-mentioned three kinds of phonons:

\[
X_{i,t,m_i}^{(i)} \rightarrow b_{i,t,m_i}^{(i)}. \tag{3.5}
\]

As for the details of the Dyson boson mapping between the collective fermion space (CFS) constructed by piling up many-phonon operators, as mentioned above, and the collective many-boson space (CBS), one can refer to our previous papers.\(^5\)-\(^8\) According to these papers, the Dyson boson images of the phonon operators and the quasiparticle pair operators are written as

\[
(X_{i,t,m_i}^{(i)})_D = b_{i,t,m_i}^{(i)},
\]

\[
(B_{LM}(ab))_D = \tilde{L}^{-1} \sum_{ij} \tilde{t}_j D_{L}^{(ij)}(ab)(-)^{L-M} [b_{i,t}^{(i)} b_{j,t}^{(j)}]_{L-M},
\]

\[
(A_{L}^{(i,m_i)}(ab))_D = \sum_{j(l_j = l_i)} x_j(ab) (X_{j,m_j}^{(j)})_D,
\]

\[
(A_{L}^{(i,m_i)}(ab))_D = \sum_{j(l_j = l_i)} x_j(ab) (X_{j,m_j}^{(j)})_D,
\]

where \( \tilde{t} = \sqrt{2l + 1} \) and \( [\ldots] \) denotes an angular-momentum coupling as follows:

\[
[b_{i,t}^{(i)} b_{j,t}^{(j)}]_{LM} = \sum_{m_i,m_j} \langle l_i m_i l_j m_j | LM \rangle b_{i,t,m_i}^{(i)} b_{j,t,m_j}^{(j)},
\]

\[
[b_{i,t}^{(i)} \tilde{b}_{j,t}^{(j)}]_{LM} = \sum_{m_i,m_j} \langle l_i m_i l_j m_j | LM \rangle b_{i,t,m_i}^{(i)} \tilde{b}_{j,t,m_j}^{(j)},
\]

\[
[[b_{i,t}^{(i)} b_{j,t}^{(j)}]_{L} \tilde{b}_{k,t}^{(k)}]_{LM} = \sum_{M'M'k} \langle L'M' l_k m_k | LM \rangle [b_{i,t}^{(i)} b_{j,t}^{(j)}]_{L'} \tilde{b}_{k,t,m_k}^{(k)}
\]

Putting the Dyson boson images (3.6) in the Hamiltonian (3.1), we can obtain the Dyson boson Hamiltonian. Since its explicit expression is quite lengthy, we omit it here.
We obtain the energy eigenvalues of the Dyson boson Hamiltonian by solving the non-Hermitian eigenvalue equation directly or the Hermitized one according to the method shown in our previous paper. 4)

3.2. Normal-ordered Holstein-Primakoff boson expansion

The general theory of the Holstein-Primakoff-type boson expansion has been discussed by many authors (for example, Beliaev and Zelevinski, 18) Marumori, Yamamura, Tokunaga and Takada, 19, 20) Janssen, Dönsau, Fраuendorf and Jolos, 21) and Marshalek 22)). It is rather complicated to apply this general theory to obtaining a normal-ordered Holstein-Primakoff boson expansion in the collective boson space (CBS).*) This type of boson expansion has been discussed in detail by Kishimoto, Tamura and Sakamoto, 9) 12) and by other groups. 23), 24) However, all these discussions are not necessarily easy to apply to realistic cases where many kinds of phonons must be introduced to construct the collective fermion space (CFS). In the case that only one kind of phonon, e.g. the collective quadrupole phonon, is taken into account, the explicit form of the fourth-order boson Hamiltonian in the normal-ordered Holstein-Primakoff boson expansion has been given by Kishimoto and Tamura 10) and others. 23), 24) However, when many kinds of phonons are needed to construct the CFS, it is not easy to even obtain an explicit form of the fourth-order boson Hamiltonian; moreover it is quite difficult to get a higher-order Hamiltonian. We therefore propose below another way to straightforwardly construct a normal-ordered Holstein-Primakoff boson expansion.

The normal-ordered Holstein-Primakoff boson expansion of a phonon operator \((X^{(i)\dagger}_{m_i})_{HP}\) is in general written as

\[
(X^{(i)\dagger}_{m_i})_{HP} = A^{(1)} b^{(i)\dagger}_{m_i} + \sum_{jkl} A^{(3)} (jk(L); i) [b^{(j)\dagger}_{lj} b^{(k)\dagger}_{lk}] L b^{(-l)}_{jl} l_{m_i} + \text{(the 5th order term)} + \cdots .
\]  

Orthonormalized boson basis states are written

\[
|0\rangle, \\
|i; JM\rangle = b^{(i)\dagger}_{m_i} |0\rangle, \quad J = l_i, \ M = m_i, \\
|jk; JM\rangle = \frac{1}{\sqrt{1 + \delta_{jk}}} [b^{(j)\dagger}_{lj} b^{(k)\dagger}_{lk}]_{JM} |0\rangle, \\
\cdots \cdots \cdots .
\]

*) The boson mapping of a collective fermion space (CFS) onto the corresponding collective boson space (CBS) is sometimes called a “modified Marumori mapping” (for example, see Ref. 13)). With this idea, there are still two possible ways of boson mapping, one is of the Holstein-Primakoff (Hermitian)-type and the other is of the Dyson-type. Some people use the term “modified MYT mapping”, 1) whose basic idea is just the same as the “normal-ordered Holstein-Primakoff boson expansion” discussed here. The aim of the present work is to study the convergence of the Holstein-Primakoff-type boson expansion by comparing it with the “Dyson-type” mapping. To demonstrate this clearly and to avoid confusions in terminology, we use the expression “normal-ordered Holstein-Primakoff boson expansion”.

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Let us symbolically represent these basis states by \(|n\rangle\). This represents \(n\)-boson states. Let \(|n\rangle\) be the orthonormalized fermion basis vector corresponding to \(|n\rangle\). It is, in principle, obtained by inversely transforming \(|n\rangle\) into the fermion space (CFS). The explicit form of \(|n\rangle\) is not necessary here.

According to a theorem discussed in our previous papers, \(^4\), \(^7\), \(^8\) which is not exactly but practically valid in such realistic cases as treated in this section, we have

\[
\langle n| X^{(i)\dagger}_{l,m_i} | n'\rangle = \langle n| (X^{(i)\dagger}_{l,m_i})_{\text{HP}} | n'\rangle = \langle n| (X^{(i)\dagger}_{l,m_i})_{\text{D}} | n'\rangle \left[ \frac{(n')^\dagger (X^{(i)}_{l,m_i})_{\text{D}} | n\rangle}{(n'| (X^{(i)\dagger}_{l,m_i})_{\text{D}} | n'\rangle} \right]^{1/2}. \tag{3·11}
\]

Using the explicit forms of the Dyson images of the phonon operators, \((X^{(i)\dagger}_{l,m_i})_{\text{D}}\) and \((X^{(i)}_{l,m_i})_{\text{D}}\), given by \((3·6a)\) and \((3·6b)\), we have the matrix elements between \(|0\rangle\) and \(|\uparrow\rangle\) as

\[
\langle 1| (X^{(i)\dagger}_{l,m_i})_{\text{D}} | 0\rangle = (\uparrow; l_m l_i | (X^{(i)\dagger}_{l,m_i})_{\text{D}} | 0\rangle = 1, \tag{3·12a}
\]
\[
\langle 0| (X^{(i)}_{l,m_i})_{\text{D}} | 1\rangle = (0 | (X^{(i)}_{l,m_i})_{\text{D}} | \uparrow; l_m l_i ) = 1. \tag{3·12b}
\]

Putting these into \((3·11)\), we have

\[
\langle 1| X^{(i)\dagger}_{l,m_i} | 0\rangle = \langle 1| (X^{(i)\dagger}_{l,m_i})_{\text{HP}} | 0\rangle = A^{(1)} = 1. \tag{3·13}
\]

In a similar way, we can obtain the coefficient \(A^{(3)}(jk(L_1)l; i)\) in the expansion \((3·9)\) by calculating the matrix elements of the Dyson boson images \((X^{(i)\dagger}_{l,m_i})_{\text{D}}\) and \((X^{(i)}_{l,m_i})_{\text{D}}\) between \(|1\rangle\) and \(|2\rangle\). The result is

\[
A^{(3)}(jk(L_1)l; i) = (-)^{l_1+L-l} \frac{\hat{L}}{L} \Delta^{(jkil)} L \left( \sqrt{1 - C^{(jkil)}_L} / \Delta^{(jkil)}_L - 1 \right), \tag{3·14}
\]

where \(C^{(jkil)}_L\) is given by \((3·8b)\) and \(\Delta^{(jkil)}_L\) is defined by

\[
\Delta^{(jkil)}_L = \frac{1}{2} \{ \delta_{ij} \delta_{kl} + (-)^{l_1+L-l} \delta_{ik} \delta_{jl} \}. \tag{3·15}
\]

Using the results \((3·13)\) and \((3·14)\), we have the normal-ordered Holstein-Primakoff boson expansion for the phonon operator as

\[
(X^{(i)\dagger}_{l,m_i})_{\text{HP}} = b^{(i)\dagger}_{l,m_i} + \sum_{jkl} \sum_L (-)^{l_1+L-l} \frac{\hat{L}}{L} \Delta^{(jkil)} L \left( \sqrt{1 - C^{(jkil)}_L} / \Delta^{(jkil)}_L - 1 \right) [[b^{(j)\dagger}_{l_k} b^{(k)\dagger}_{l_k}]_L b^{(l)}_{l_k}]_{l,m_i} + (\text{the 5th order term}) + \cdots, \tag{3·16a}
\]
\[
(B_{LM}(ab))_{\text{HP}} = (B_{LM}(ab))_{\text{D}}, \tag{3·16b}
\]
\[ (B^\dagger_{LM}(ab))_{\text{HP}} = (B_{LM}(ab))^\dagger_{\text{HP}}, \]  
\[ (A^\dagger_{l,m}(ab))_{\text{HP}} = \sum_{j(l_j=l_i)} x_j(ab)(X^{(j)}_{l,m})_{\text{HP}}, \]  
\[ (A_{l,m}(ab))_{\text{HP}} = \sum_{j(l_j=l_i)} x_j(ab)(X^{(j)}_{l,m})_{\text{HP}}. \]

These results contain the \textit{linked-cluster expansion} proposed by Kishimoto and Tamura;\(^\text{10}\) if only one kind of phonon, e.g. only the collective quadrupole phonon, is taken into account, then our results are the same as theirs. The present derivation of the normal-ordered Holstein-Primakoff boson expansion is more general and clear-cut than theirs.

Inserting these results into the original quasiparticle Hamiltonian (3·1) and (3·2), we can obtain a normal-ordered Holstein-Primakoff boson Hamiltonian in the form of a power series in boson operators.

3.3. \textit{Numerical calculations and discussion}

We carried out numerical calculations to study the convergence of the above-mentioned normal-ordered Holstein-Primakoff boson expansion (referred to as NHP hereafter). We did for the realistic case of \(^{114}\text{Cd}\) and compare the results with those of the Dyson mapping which can be considered an “exact” theory in the sense that the boson mapping is performed under the closed-algebra approximation\(^{25,5 - 8}\) without using any truncation of the boson expansion.

We take only up to the fourth-order terms of the boson Hamiltonian of NHP and neglect the higher-order terms. One of the reasons for this is that we are especially interested in the convergence of the fourth-order boson Hamiltonian of NHP. Another reason is that, as discussed in the previous subsection, we can straightforwardly obtain the higher-order terms using our new method, but it is still quite tedious to get the sixth-order boson Hamiltonian of NHP, particularly in the case when many kinds of phonons are taken into account. This situation is quite different from that of the Dyson mapping, in which we can have the explicit form of the whole boson Hamiltonian, as stated in \S 3.1. Therefore we leave the analysis of the higher-order terms as a future problem. We numerically calculated the energy eigenvalues and relating \(B(E2)\) values, and compared them with the results of the Dyson mapping.

The active single-particle orbits are taken to be \(1d_{5/2}, 0g_{7/2}, 2s_{1/2}, 0h_{11/2}\) and \(1d_{3/2}\) for neutrons, and \(1p_{3/2}, 0f_{5/2}, 1p_{1/2}\) and \(0g_{9/2}\) for protons. The single-particle wave functions are chosen to be the harmonic-oscillator wave function and the single-particle energies are the same as those used by Uher and Sorensen.\(^{26}\)

The value of the parameter \(G_0\), which is related to the pairing-force strength \(G\) through \(G = G_0 A^{-1}\) with the mass number \(A\), is taken to be \(G_0 = 15\) MeV. The value of the parameter \(\chi_0\), which is related to the QQ-force strength \(\chi\) through \(\chi_0 = \chi b^4 A^{3/5}\) with the harmonic-oscillator constant \(b^2 = 1.0 A^{1/3}\) fm\(^2\), is chosen to be \(\chi_0 = 230\) MeV. The \(P_2\)-force strength is specified by the ratio \(g_2 = G_2/\chi\), whose value is taken to be \(g_2 = 0.1\). These parameter values can reproduce the experimental data of the excitation energy of the first \(2^+\) state in \(^{114}\text{Cd}\). The effective charge,
Fig. 5. Calculated low-energy levels for $^{114}$Cd under (b) the Dyson boson mapping and (c) NHP-4 are shown together with (a) the experimental energy levels. These calculated levels are nothing other than the results at $k = 1$ in Fig. 6.

$e_{\text{eff}} = e + \delta e$ for protons and $e_{\text{eff}} = \delta e$ for neutrons, is taken so as to roughly reproduce the experimental $B(E2; 2^+_1 \rightarrow 0^+_1)$; the value is taken to be $\delta e = 1.47e$.

The numerical results are shown in Figs. 5 and 6, in which the symbol "NHP-4" means that we take into account up to the fourth-order terms of the boson Hamiltonian of NHP (normal-ordered Holstein-Primakoff boson expansion).

In Fig. 5, the calculated low-energy levels for $^{114}$Cd under the Dyson boson mapping and NHP-4 are shown together with the experimental energy levels.

Comparing these numerical results with experiments, one can see that the model space adopted and the interaction parameters used are not unreasonable.

In Fig. 6, the calculated excitation energies of $^{114}$Cd are shown as a function of the QQ-force strength which is parametrized by the ratio $k$ to the optimal value.

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In Fig. 6, the calculated excitation energies of $^{114}$Cd are shown as a function of the QQ-force strength which is parametrized by the ratio $k$ to the optimal value.

Table I. Calculated and experimental $B(E2)$ values of $^{114}$Cd in units of $10^3 e^2 fm^4$. The polarization charge is chosen as $\delta e = 1.47e$. The experimental data are taken from Blachot and Marguier. 27

<table>
<thead>
<tr>
<th></th>
<th>Dyson</th>
<th>NHP-4</th>
<th>Experiment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B(E2; 2^+_1 \rightarrow 0^+_1)$</td>
<td>1.016</td>
<td>0.969</td>
<td>1.018</td>
</tr>
<tr>
<td>$B(E2; 2^+_2 \rightarrow 2^+_1)$</td>
<td>0.826</td>
<td>0.805</td>
<td>0.932</td>
</tr>
<tr>
<td>$B(E2; 2^+_3 \rightarrow 0^+_1)$</td>
<td>0.010</td>
<td>0.014</td>
<td>0.016</td>
</tr>
<tr>
<td>$B(E2; 4^+_1 \rightarrow 2^+_1)$</td>
<td>1.357</td>
<td>1.463</td>
<td>2.019</td>
</tr>
</tbody>
</table>
explained above; namely, \( k = 1.0 \) denotes the optimal strengths \( \chi_0 = 230 \text{ MeV} \). Here the ratio \( g_2 \) is not changed while the value of \( k \) varies. In Fig. 6, the solid lines denote the results of the Dyson mapping and the dashed lines those of NHP-4. It is quite noticeable that these two kinds of lines (Dyson mapping and NHP-4) are rather close to each other especially in the region of weak QQ force and low-excited state. There appears some difference between the results of the Dyson mapping and NHP-4 in the strong QQ-force region and for the highly-excited state.

Table I, calculated \( B(E2) \) values are shown together with experimental data. The effective charge used is \( e = 1.47e \). We can see from these data that there is some slight (< 10%) difference between the results of the Dyson mapping and NHP-4. This means that there must be about 10% difference in the wave functions. However we can still say that the NHP-4 is a fairly good approximation for the case of low-excited states.

§4. The case of octupole correlation

In this section, we test the convergence of the normal-ordered Holstein-Primakoff boson expansion (NHP) for the case of octupole correlation. We have published an analysis of the octupole correlation in \(^{208}\text{Pb}\) by using the Dyson boson mapping in a previous paper. \(^7\) We apply the fourth-order truncation approximation of NHP to this case, \(^{208}\text{Pb}\). We give discussion very similar to that in the preceding section. The main difference is that the particle-hole representation is used in the present section, while everything in the preceding section was done in the quasiparticle representation.

In this figure, the solid lines denote the results of the Dyson mapping and the dashed lines those of NHP-4. The present calculation was done within the boson space spanned by the octupole bosons and the monopole pairing-vibrational bosons only neglecting the quadrupole bosons. All parameters used are identical to those used in the previous paper. \(^7\)

We can see that the NHP-4 provides a fairly good approximation for such low-
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excited states as those treated in the present section.

§5. Concluding remarks

In the present paper, we have proposed a new clear-cut method to obtain the normal-ordered Holstein-Primakoff boson expansion. Applying this method to a simple three-level model (SU(3) model) and realistic cases of $^{114}$Cd and $^{208}$Pb, we have tested the convergence of the normal-ordered Holstein-Primakoff (Hermitian) boson Hamiltonian by comparing it with the Dyson boson mapping theory which has no difficulty concerning boson expansion. The case of $^{114}$Cd is a typical example of so-called vibrational nuclei, in which the pairing correlation and the quadrupole correlation are important, and everything is described in the quasiparticle representation. In the case of $^{208}$Pb, the octupole correlation is significant, and the particle-hole representation is used.

We can summarize the important points of our investigation as follows:

1. Our new method is quite convenient to obtain an explicit expression of the fourth-order truncation of the normal-ordered Holstein-Primakoff boson Hamiltonian. It is straightforward to extend this to higher orders. The result obtained gives a general and explicit expression of the linked-cluster expansion discussed by Kishimoto and Tamura.\(^{10}\)

2. The truncation at the fourth-order expansion of the normal-ordered Holstein-Primakoff boson Hamiltonian is a rather good approximation for the region of low-excited states or for weak correlation strength. Therefore, if one wishes to have a fourth-order boson Hamiltonian, the new method discussed above should be used.

3. It is most desirable to construct the sixth-order boson Hamiltonian. But this is much more troublesome than the Dyson boson mapping. Who needs such a complicated theory? The Dyson mapping is more reasonable if one wishes to obtain the sixth-order boson Hamiltonian.

As seen in the discussion above, the fourth-order expansion of the normal-ordered Holstein-Primakoff (Hermitian) boson Hamiltonian obtained by using our new method can give a very good description of the low-excited collective states. It is very desirable to extend our discussion here to other transitional and rotational nuclei such as Sm isotopes, where probably the sixth-order expansion would play a more important role. This is now in progress.

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