Bethe-Salpeter Approach for Mesons in the Pion Channel within the Dual Ginzburg-Landau Theory

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We develop a formalism for the study of mesons with the Bethe-Salpeter (BS) equation using the dual Ginzburg-Landau (DGL) theory. We introduce a new method to solve the BS equation by using the expansion method with momentum square at zero momentum. We apply this formalism to mesons in the pion channel with parity \(\Pi = -(\Pi^2)\) and charge conjugation \(C = -\Pi\). Due to the confinement property of the gluon propagator in the DGL theory, we find the Regge behavior, \(M^2 \propto J\); that is, the mass square is proportional to the spin.

\section{Introduction}

It is a great challenge to describe hadrons in terms of quantum chromodynamics (QCD). Although QCD is successful in describing hard processes like deep inelastic scattering, which involve large momentum transfer, QCD becomes highly complicated for low energy phenomena due to its non-perturbative character. Particularly, in order to describe low energy phenomena, we must understand first the confinement mechanism of the colored objects, quarks and gluons. It is important also to describe spontaneous chiral symmetry breaking within the same framework.

The direct application of QCD for hadrons has been carried out by several groups using the lattice QCD framework with great success. The lattice QCD simulations provide the signatures of the color confinement and the chiral symmetry breaking.\textsuperscript{1} Hence, we strongly believe that QCD is a good theory for hadron physics. As for the clue of understanding the essential degrees of freedom for non-perturbative phenomena, the recent lattice QCD calculation provides a very interesting picture regarding color confinement and chiral symmetry breaking. Use of the maximally abelian (MA) gauge in lattice QCD simulations suggests that a color monopole field appears and the QCD vacuum becomes a dual superconductor.\textsuperscript{2} This suggests formulation of an effective theory in which the color monopole plays the essential role for low energy phenomena. The dual Ginzburg-Landau (DGL) theory was formulated following this line in a very compact form.\textsuperscript{3,4} The DGL theory was demonstrated to provide the confining potential between heavy quarks and further the spontaneous chiral symmetry breaking.\textsuperscript{4} Monopole condensation is the fundamental phenomenon to cause a linear confining potential and chiral symmetry breaking. It is the next step...
to construct mesons by using the DGL theory.

The approach using the Dyson-Schwinger (DS) equation and the Bethe-Salpeter (BS) equation formalism serves as useful means to study nonperturbative aspects of QCD. It has been applied to low-energy hadronic physics. Given a model gluon propagator, this approach can treat fundamental features of QCD, such as confinement and chiral symmetry breaking, and then extract various hadronic properties, in a closed framework. However, when we apply the BS equation formalism to hadronic bound states, we encounter a difficulty. In order to obtain a physical bound state energy directly as the solution of the BS equation, we must solve the BS equation in the time-like region, which requires information regarding the quark propagator in the time-like momentum. However, the solution of the quark DS equation is usually known only in the space-like region. Hence, it is customary to solve the DS equation for quarks in the Euclidean space and further to work with the BS equation in the Euclidean space. So far, there are two prescriptions for this problem. One is to extrapolate to obtain the quark propagator of the complex momentum, and then to use this quark propagator to solve the BS equation. Another is to extrapolate some information, such as fictitious eigenvalues of the BS equation or two-point correlation functions from the space-like region to the time-like region. These two prescriptions, however, require some assumptions with regard to the functional forms of the quark propagator or the eigenvalue. Other than these prescriptions, several methods have been developed which avoid solving the original integral equation directly, e.g. constructing a separable kernel to make solving the BS equation analytically possible, or deriving from the BS equation a mass functional of a form factor, with respect to which variation is carried out.

In this paper we construct a new approach to this problem. The idea is that we expand all the ingredients in the BS equation, i.e. the BS amplitude, two-body propagator, scattering kernel and fictitious eigenvalue, in the total momentum $\sqrt{P^2}$ around $P^2 = 0$, and solve the BS equation order by order. This is a systematic method. It is justified if the fictitious eigenvalue can be approximated with a few lower terms in powers of $\sqrt{P^2}$, at least up to the physical mass of mesons. Such a situation is expected in a confined system, because there is no two-particle breakup threshold, unlike in the Wick-Cutkosky model, in which the fictitious eigenvalue diverges at the two-particle threshold.

The $P^2$ expansion method for the BS equation is applicable to any model gluon propagator, and it can be extended even to other systems, but in this paper we would like to formulate the Bethe-Salpeter (BS) equation for mesons using the DGL theory. In particular, we apply the BS equation for the mesons in the pion channel up to high spin states.

In §2, we describe the BS equation using the ladder approximation. The effective model we use (the DGL theory) is also recapitulated. In §3, we introduce a new expansion method at zero momentum to obtain the meson masses. We adapt the formalism to the meson spectrum in the pion channel in §4 and give the numerical results using the model gluon propagator of the DGL theory in §5. Section 6 is devoted to a conclusion.
§2. Bethe-Salpeter equation

We use the Bethe-Salpeter equation to derive the meson spectra. The BS equation is graphically shown in Fig. 1. The BS amplitude \( \chi \) is defined as pole contributions to the four-point function

\[
G(x_1, x_2; y_1, y_2) \equiv \langle 0 | T \{ \psi_\alpha(x_1) \bar{\psi}_\beta(x_2) \psi_\gamma(y_2) \bar{\psi}_\delta(y_1) \} | 0 \rangle,
\]

where \( \alpha, \beta, \gamma \) and \( \delta \) are Lorentz indices. This amplitude is written as

\[
\chi_{\alpha\beta}(p, P) = \int d^4x e^{ipx} \langle 0 | T \{ \psi_\alpha(x_2) \bar{\psi}_\beta(-x_2) \} | P\lambda \rangle,
\]

where translational invariance has been used. Here \( P \) is the total 4-momentum carried by the bound state, and \( \lambda \) denotes the quantum numbers. It is convenient to define the conjugate BS amplitude with a minus sign:

\[
\bar{\chi}_{\alpha\beta}(p, P) = -\int d^4x e^{-ipx} \langle P\lambda | T \{ \psi_\alpha(-x_2) \bar{\psi}_\beta(x_2) \} | 0 \rangle.
\]

Then CPT invariance connects \( \chi \) and \( \bar{\chi} \):

\[
\bar{\chi}(p, P) = \gamma^0 \chi(p^*, P)^\dagger \gamma^0.
\]

The integral equation for the BS amplitude \( \chi \) is

\[
\chi(p, P) = iG\left( p + \frac{P}{2} \right) \Gamma(p, P) iG\left( p - \frac{P}{2} \right),
\]

where \( G(p) \) is the quark propagator. The amputated amplitude (BS vertex) \( \Gamma(p, P) \) is written as an integral of the BS amplitude \( \chi \) together with the scattering kernel. In the ladder approximation, bearing in mind the use of an effective theory for low energy non-perturbative phenomena, it is given by

\[
\Gamma(p, P) = \int \frac{d^4k}{(2\pi)^4} \gamma_\mu \chi(k, P) \gamma_\nu D^{\mu\nu}(p - k),
\]

where \( D^{\mu\nu}(q) \) is the propagator of the gauge boson. The quark propagator \( G(p) \) is generally written as

\[
G(p) = \frac{iz(p)}{\vec{p} - M(p)}.
\]
Here, $M(p)$ is the mass function and $z(p)$ is the wave function renormalization function. The quark propagator $G(p)$ can be calculated by solving the Dyson-Schwinger (DS) equation. It is important to solve the BS equation consistently with the DS equation by using the same approximation, because this procedure ensures that the pion is massless in the chiral limit.\textsuperscript{16} The DS equation in the rainbow approximation is represented as

$$-iG^{-1}(p) = -iG_0^{-1}(p) + \int \frac{d^4k}{(2\pi)^4} \gamma_\mu iG(k)\gamma_\nu D^{\mu\nu}(p-k), \quad (2.8)$$

where $G_0(p)$ is the bare quark propagator,

$$G_0(p) = \frac{i}{p - m_0}, \quad (2.9)$$

with current quark mass $m_0$.

Now we specify the effective model to be used. We use the dual Ginzburg-Landau (DGL) theory, which models the confinement based on the dual Higgs mechanism which causes squeezing of color-electric flux.\textsuperscript{3,4} The gluon propagator of the DGL theory in the Landau gauge is

$$D^{\mu\nu}(q) = (g^{\mu\nu} - \frac{q_\mu q_\nu}{q^2}) \frac{d_{\text{DGL}}(-q^2)}{-q^2}, \quad (2.10)$$

where the gauge boson function in the DGL theory, $d_{\text{DGL}}(-q^2)$, is written as

$$d_{\text{DGL}}(-q^2) = \frac{4\pi^2}{3} \left\{ 1 - \frac{2}{3} \frac{m_B^2}{m_B^2 + q^2} \left[ 1 - \frac{2(1 + q^2/a^2)}{1 + \sqrt{1 + q^2/a^2}} \right] \right\}, \quad (2.11)$$

where $A = 8/(11 - \frac{2}{3}n_f)$, $q_c$ is the infrared cutoff momentum, and $a$ is the screening cutoff parameter. This propagator will be used for the DS equation. The derivation of this gluon propagator is given in Appendix A.

In a later section, we investigate the “QCD-like” model in addition to the DGL model for comparison. The gauge boson function for the “QCD-like” model, $d_{\text{QCD-like}}(-q^2)$, is written

$$d_{\text{QCD-like}}(-q^2) = \frac{4\pi^2}{3} \ln \left( \frac{q^2 + q_c^2}{\Lambda_{\text{QCD}}} \right). \quad (2.12)$$

Both models reproduce the asymptotic behavior of the mass function of the quark propagator $M(p^2)$ quite well, due to the logarithmic factor which mimics the QCD running coupling constant. While the empirical value of $\Lambda_{\text{QCD}}$ is used in the DGL model,\textsuperscript{4} the parameters $q_c$ and $\Lambda_{\text{QCD}}$ are fixed to provide reasonable chiral properties in the “QCD-like” model.\textsuperscript{17,18,11,19}

To this point, we have written down the equations in the Minkowski metric. However, it should be mentioned that the gauge boson functions (2.11) and (2.12)
are reasonable only for space-like momenta. Since our knowledge of QCD is restricted mostly to the large space-like region, there is no reliable guidance to model the gauge boson function in the time-like region. It is then well-defined and convenient to write equations in a Euclidean metric $k^\mu = (k, k_4)$, where the fourth component $k^4$ of the 4-vector is related to the zeroth component of the Minkowski 4-vector as

$$k^4 \equiv ik^0, \quad p^4 \equiv ip^0, \quad P^4 \equiv iP^0.$$  \hfill (2.13)

Hereafter, we shall use the Euclidean metric, in which an on-shell particle is characterized by the 4-vector $P^\mu = (P, iE_P)$, where $E_P = \sqrt{m^2 + P^2}$, with $m$ the mass of the particle. We use the Lorentz invariant 4-momentum square $k^2 \equiv \sum_{\mu=1}^4 k^2_\mu$, so that positive $k^2$ is space-like, and negative $k^2$ is time-like. It is convenient to use the Hermitian $\gamma$-matrices, which are related to the $\gamma$-matrices in the Minkowski metric, $\gamma_M^\mu$ as $\gamma_4 \equiv \gamma_M^0$ and $\gamma_i \equiv -i\gamma_M^i$.

We can expand the BS amplitude $\chi$ and the scattering kernel $\Gamma$ in terms of the Euclidean $\gamma$-matrices as

$$\chi = S + V_\alpha \gamma_\alpha + \frac{1}{2} T_{\alpha\beta} \sigma_{\alpha\beta} + A_\alpha i\gamma_\alpha \gamma_5 + p\gamma_5$$  \hfill (2.14)

and

$$\Gamma = s + v_\alpha \gamma_\alpha + \frac{1}{2} t_{\alpha\beta} \sigma_{\alpha\beta} + a_\alpha i\gamma_\alpha \gamma_5 + p\gamma_5,$$  \hfill (2.15)

where $\sigma_{\alpha\beta} = \frac{1}{2}(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)$ and $\gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4$. Now we introduce the two-body propagator $D$, which connects the BS amplitude $\chi$ with the scattering kernel $\Gamma$. Equation (2.5) can be rewritten in Euclidean metric as

$$\chi(p, P) = z(p_+) M(p_+) - i\frac{\phi_+}{2M(p_+)} \Gamma(p, P) z(p_-) M(p_-) - i\frac{\phi_-}{2M(p_-)} \Gamma(p, P) z(p_+),$$  \hfill (2.16)

where $p_\pm = p \pm \frac{P}{2}$. Using the $\gamma$-matrix expansions (2.14) and (2.15), Eq. (2.16) can be rewritten in vector form as

$$\begin{pmatrix} S \\ V_\alpha \\ T_{\alpha\beta} \\ A_\alpha \\ p \end{pmatrix} = D(p, P) \begin{pmatrix} s \\ v_\mu \\ \frac{1}{2} t_{\mu\nu} \\ a_\mu \\ p \end{pmatrix},$$  \hfill (2.17)

where the two-body propagator matrix $D(p, P)$ is defined as

$$D(p, P) \equiv \frac{z(p_+)z(p_-)}{4} \text{Tr} \left[ \begin{pmatrix} 1 \\ \gamma_\alpha \\ \sigma_{\alpha\beta} \\ i\alpha \gamma_5 \\ \gamma_5 \end{pmatrix} M(p_+) - i\frac{\phi_+}{2M(p_+)} (1, \gamma_\mu, \sigma_{\mu\nu}, i\gamma_\mu \gamma_5, \gamma_5) M(p_-) - i\frac{\phi_-}{2M(p_-)} \right].$$  \hfill (2.18)
The explicit form of $D(p, P)$ is given in Appendix B. Furthermore, in the ladder approximation, the integrand in (2.6) (in the Euclidean metric) can be written in vector form as

$$\gamma_{\mu} \chi_{\nu} D^{\mu\nu}(q) = (1, \gamma_{\mu}, \sigma_{\mu}, i\gamma_{\mu}\gamma_{5}, \gamma_{5}) \mathcal{I}^{(lad)}(q) \left(\begin{array}{c} S \\
V_\alpha \\
\frac{1}{2} T_{\alpha\beta} \\
A_\alpha \\
P \end{array}\right),$$

(2.19)

where $q = p - k$. Here we have also defined the $\gamma$-matrix expansion of the scattering kernel $\mathcal{I}^{(lad)}(q)$ as a $5 \times 5$ matrix:

$$\mathcal{I}^{(lad)}(q) = \text{diag} \left(3, -\left(\delta_{\mu\alpha} + \frac{2q_\mu q_\alpha}{q^2}\right), -\left(\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) - \frac{4}{q^2}(q_\alpha q_\nu\delta_{\mu\beta} - q_\beta q_\nu\delta_{\mu\alpha}), \delta_{\mu\alpha} + \frac{2q_\mu q_\alpha}{q^2}, -3\right) \frac{d(q^2)}{-q^2}.$$

(2.20)

Generally, the scattering kernel $\mathcal{I}(p, k : P)$ depends on $P$, but it is independent of $P$ in the ladder approximation. Thus we can abbreviate the BS equation in vector form:

$$\chi(p, P) = \int \frac{d^4k}{(2\pi)^4} D(p, P) \mathcal{I}(p, k : P) \chi(k, P).$$

(2.21)

Its conjugate is

$$\bar{\chi}(p, P) = \int \frac{d^4k}{(2\pi)^4} \bar{\chi}(k, P) \mathcal{I}(p, k : P) D(p, P).$$

(2.22)

To solve the BS equation (2.21), it is convenient and customary to introduce a fictitious eigenvalue $\lambda(P)$ and to regard the integral equation as an eigenvalue equation:

$$\lambda(P) | \chi(p, P)\rangle = D(p, P) \int \frac{d^4k}{(2\pi)^4} \mathcal{I}(p, k : P) | \chi(k, P)\rangle.$$

(2.23)

Here, the “ket” vector $| \chi(p, P)\rangle$ should be understood as the 5-component vector in (2.17). We can also write the conjugate equation of (2.23) in “bra” form:

$$\langle \bar{\chi}(p, P) | \lambda(P)\rangle = \int \frac{d^4k}{(2\pi)^4} \langle \bar{\chi}(k, P) | \mathcal{I}(k, p : P) D(p, P).$$

(2.24)

Hereafter, the integral operation is suppressed for brevity as

$$\mathcal{I} | \chi\rangle \equiv \int \frac{d^4k}{(2\pi)^4} \mathcal{I}(p, k : P) | \chi(k, P)\rangle,$$

(2.25)

$$\langle \chi | \mathcal{I} \equiv \int \frac{d^4k}{(2\pi)^4} \langle \bar{\chi}(k, P) | \mathcal{I}(k, p : P).$$

(2.26)
§3. \( P^2 \) expansion of Bethe-Salpeter equation

Let us look at the DS equation (2.8) for the quark propagator (2.7). Since we know the gauge boson function \( d(q) \), which gives the scattering kernel \( T^{(lad)}(q) \), only for space-like momenta, we cannot freely perform the analytic continuation on the \( p^2 \) complex plane. Thus what we can do is to solve \( G(p) \) for \( p^2 \) on the positive real axis. However, to obtain bound state masses, we need \( G(p) \) on the complex plane of \( p^2 \).

Recall that the two-body propagator \( D \) is a matrix whose components are given by the mass function \( M \) with arguments \( (p±\frac{P^2}{2})^2 \). Now, \( P^2 \) must be time-like for a meson bound state, so that the argument of the mass function becomes complex, where we cannot solve the DS equation without making an artificial assumption. Thus, we cannot solve the BS equation as it is, due to the lack of knowledge regarding the quark propagator. To this time, two kinds of approaches have been made to study the meson spectrum with the BS equation. One is to calculate the eigenvalue \( \lambda(P) \) of the BS equation for space-like \( P^2 \) and then numerically extrapolate it to the time-like region. Another is to make the assumption that the mass function is known in the entire momentum region by extrapolating the mass function to the time-like region. Here we propose the third approach, where we make a systematic expansion of \( \lambda(P) \) in terms of \( P^2 \) at zero momentum. The idea is close to that of the first approach. While in the first approach the BS equation and the eigenvalue \( \lambda(P) \) were solved for finite space-like \( P^2 \), we instead solve the BS equation at \( P^2 = 0 \) and solve \( \lambda(P^2) \) as a power series in \( P^2 \) constructed from derivatives of \( \lambda(P^2) \) with respect to \( P^2 \) evaluated at \( P^2 = 0 \).

The BS equation is systematically expanded around \( P^2 = 0 \), and the eigenvalue \( \lambda(P) \) is obtained as a power series in \( P^2 \). Then the mass of the bound system is obtained from the condition

\[
\lambda(P) = 1. \tag{3.1}
\]

Expansion of the BS equation with respect to \( \sqrt{P^2} \) around \( P^2 = 0 \) has been considered by several authors in contexts different from that considered here. For example, higher order parts of the BS amplitude of the pion in the chiral limit, which is massless, were obtained and then used to calculate other physical quantities, such as the pion decay constant. Munczek and Jain obtained the bound state mass by iteratively solving the BS equation truncated up to \( O(P^2) \).

Before proceeding to details, it is illuminating to have a look at the eigenvalue \( \lambda(P^2) \) of a solvable system as a function of \( P^2 \). The Wick-Cutkosky model is the most well-known relativistic two-body bound system. It describes scalar-scalar bound states via massless scalar exchange. Note that a method to solve the BS equation for scalars directly in Minkowski space has been developed and tested for a few cases recently. The BS equation for the Wick-Cutkosky model in the Minkowski metric is written as

\[
\lambda(P^2)\chi(p, P) = D \left(p + \frac{P^2}{2}ight) D \left(p - \frac{P^2}{2}\right) \int \frac{d^4k}{(2\pi)^4} \cdot \frac{g^2}{-(p-k)^2 - i\epsilon} \chi(k, P), \tag{3.2}
\]

with \( D(p) = \frac{1}{m_0^2-p^2-i\epsilon} \). Here, \( g \) is the coupling constant. The eigenvalue \( \lambda(P^2) \)
Fig. 2. The eigenvalue $\lambda(P^2)$ as a function of $P^2$ in the time-like region for various values of the coupling strength $g$ in the Wick-Cutkosky model. The solution of $\lambda(P^2) = 1$ provides the mass of the composite particle. The function $\lambda(P^2)$ curves suddenly near the two-particle threshold, $P^2 = (2m_0)^2$.

can be obtained for the $S$-wave bound state for any value of $P^2$, since we know the propagator $D(p)$ on the entire complex plane of $p^2$. We give the numerical results for this BS equation with the Wick-Cutkosky model in Fig. 2, where the eigenvalue $\lambda(P^2)$ is plotted as a function of $P^2$. Around $P^2 = 0$, the eigenvalue $\lambda(P^2)$ behaves like a straight line with a positive slope, while it curves toward infinity when $P^2$ approaches the threshold $P^2 = (2m_0)^2$. This suggests that the eigenvalue of the BS equation, $\lambda(P^2)$, can be expressed as a power series in $P^2$ around $P^2 = 0$. From Fig. 2, it is obvious that the eigenvalue $\lambda(P^2)$ cannot be expressed as power series in $P^2$ near the two-particle threshold $P^2 = 4m_0^2$. This comes from the fact that the two-body propagator has a singularity at $P^2 = 4m_0^2$. The basic assumptions of our treatment are, thus, as follows:

1. The quark propagator is regular on the non-negative real axis of $p^2$.
2. The eigenvalue of the BS equation $\lambda(P^2)$ is regular near $P^2 = 0$.

These assumptions imply the following:

1. We are allowed to expand the two-body propagator $D(p, P)$ in powers of $P^2$,
   and we do not have to solve the DS equation on the complex plane.
2. We are allowed to perform the analytic continuation of $\lambda(P^2)$ from space-like
   $P^2$ to time-like $P^2$ by changing its sign.

Since we are interested in the bound states of quarks, which are confined, these
assumptions are reasonable. The remaining question is then the order of truncation
in the powers of $P^2$. For the Wick-Cutkosky model, the actual expansion parameter
should be $P^2/4m_0^2$, the ratio of the bound state mass to the value of the two-particle
threshold. For a strongly bound system, $P^2/4m_0^2$ is small, so that the first few terms
of the power series reproduce the exact eigenvalue quite well. However, this power series expansion fails for a weakly bound system. In Fig. 2, we show the eigenvalue \( \lambda(P^2) \) for three cases of the strength of the coupling constant. We can clearly see how the expansion parameter \( P^2/4m_0^2 \) affects the validity of the power series expansion of \( \lambda(P^2) \).

On the other hand, for a quark-antiquark bound state, we believe that there is no threshold due to the confinement. Although we do not have well-developed field theoretical tools to describe confined particles, it is quite natural and plausible to assume the absence of a free quark and antiquark threshold in our model. A possible implement of the “confinement” in our model is then the assumption that the position of the two-particle threshold is located at infinity of time-like momenta.

This consideration leads us to the conjecture that the eigenvalue of the BS equation is closely approximated by a straight line. If this is true, we can solve the eigenvalue \( \lambda(P^2) \) as

\[
\lambda(P^2) = \lambda^0 + c^{(2)}P^2 + c^{(4)}P^4 + \cdots,
\]

(3.3)

with \( c^{(2)}P^2 \gg c^{(4)}P^4 \) up to the momentum of \( \lambda(P^2) = 1 \). Note that the zeroth order eigenvalue \( \lambda^0 \) can be calculated from the quark propagator on space-like momenta. \( c^{(2)} \) is also calculable with the derivative of the space-like quark propagator with respect to the momentum. Thus we do not need any assumption regarding the functional forms of the quark propagator and the model gluon propagator (in general, the \( q\bar{q} \) scattering kernel on complex momentum plane).

Let us now proceed to the details of the \( P^2 \) expansion method. First, we expand the relevant values as a power series in \( P \):

\[
\lambda(P) = \lambda^0 + \lambda^1 + \lambda^2 + \cdots,
\]

(3.4)

\[
|\chi\rangle = |\chi^0\rangle + |\chi^1\rangle + |\chi^2\rangle + \cdots,
\]

(3.5)

\[
D = D^0 + D^1 + D^2 + \cdots,
\]

(3.6)

\[
I = I^0 + I^1 + I^2 + \cdots,
\]

(3.7)

where the superscripts 0, 1, 2, \( \cdots \) denote the order of \( \sqrt{P^2} \). Then the BS equation

\[
\lambda |\chi\rangle = DI |\chi\rangle
\]

is expanded in the order of \( \sqrt{P^2} \) as the following series of equations:

\[
(\lambda^0 - D^0I^0) |\chi^0\rangle = 0,
\]

(3.9)

\[
(\lambda^0 - D^0I^0) |\chi^1\rangle = -\lambda^1 |\chi^0\rangle + D^1I^0 |\chi^0\rangle + D^0I^1 |\chi^0\rangle,
\]

(3.10)

\[
(\lambda^0 - D^0I^0) |\chi^2\rangle = -\lambda^2 |\chi^0\rangle + (D^2I^0 + D^1I^1 + D^0I^2) |\chi^0\rangle
- \lambda^1 |\chi^1\rangle + (D^1I^0 + D^0I^1) |\chi^1\rangle,
\]

(3.11)

\[
\cdots.
\]

Here, the abbreviated forms (2.25) and (2.26) are used. We can calculate \( \lambda^n \) by solving the expanded BS equation up to \( n \)-th order, starting from the zeroth order BS equation (3.9).
The first order eigenvalue $\lambda_1$ can be obtained by multiplying the first order BS equation (3.10) by $\langle \chi^0 \mid (D^0)^{-1} \rangle$:

$$\langle \chi^0 \mid (\lambda^0 - D^0 I^0) (D^0)^{-1} \mid \chi^1 \rangle = -\lambda_1 \langle \chi^0 \mid (D^0)^{-1} \mid \chi^0 \rangle + \langle \chi^0 \mid (D^0)^{-1} D^1 I^0 \mid \chi^0 \rangle + \langle \chi^0 \mid I^1 \mid \chi^0 \rangle. \quad (3.12)$$

The left-hand side vanishes using the conjugate of (3.9), and we obtain

$$\lambda_1 = \frac{\langle \chi^0 \mid (D^0)^{-1} D^1 I^0 \mid \chi^0 \rangle - \lambda_0\langle \chi^0 \mid I^0 \mid \chi^0 \rangle + \langle \chi^0 \mid I^1 \mid \chi^0 \rangle}{\langle \chi^0 \mid (D^0)^{-1} \mid \chi^0 \rangle}. \quad (3.13)$$

Then the first order BS amplitude is given by the integral equation whose inhomogeneous term is given by the zeroth order amplitude $|\chi^0\rangle$:

$$\langle \chi^0 \mid (\lambda^0 - D^0 I^0) \mid \chi^1 \rangle = \{ \lambda^0 D^1 (D^0)^{-1} - \lambda_1 \} \langle \chi^0 \mid \chi^0 \rangle + D^0 I^1 \langle \chi^0 \mid \chi^0 \rangle. \quad (3.14)$$

In deriving this equation, the relation

$$I^0 \langle \chi^0 \mid \chi^0 \rangle = \lambda_0 (D^0)^{-1} \langle \chi^0 \mid \chi^0 \rangle \quad (3.15)$$

derived from (3.9) has been substituted in (3.10). The second order eigenvalue $\lambda_2$ is then obtained as

$$\lambda_2 = \frac{1}{\langle \chi^0 \mid (D^0)^{-1} \mid \chi^0 \rangle} [ \langle \chi^0 \mid (D^0)^{-1} D^2 I^0 + (D^0)^{-1} D^1 I^1 + I^2 \mid \chi^0 \rangle - \lambda_1 \langle \chi^0 \mid (D^0)^{-1} \mid \chi^1 \rangle + \langle \chi^0 \mid (D^0)^{-1} D^1 I^0 + I^1 \mid \chi^1 \rangle ] \quad (3.16)$$

by multiplying (3.11) by $\langle \chi^0 \mid (D^0)^{-1} \rangle$. Thus the power series of the eigenvalue $\lambda(P)$ can be calculated systematically.

Since we consider mesons which consist of quarks and antiquarks of equal mass, the two-body propagator matrix $D(p, P)$ has the structure

$$D = D^{\text{even}} + D^{\text{odd}}, \quad (3.17)$$

where $D^{\text{even}}$ and $D^{\text{odd}}$ are the even and odd order parts in $\sqrt{P^2}$, and their structures are

$$D^{\text{even}} = \begin{pmatrix} D^{SV:SV} & 0 \\ 0 & D^{TA:TA} \\ 0 & 0 & D^{P:P} \end{pmatrix}, \quad (3.18)$$

$$D^{\text{odd}} = \begin{pmatrix} 0 & D^{SV:TA} \\ D^{TA:SV} & 0 \\ 0 & D^{P:TA} \end{pmatrix}. \quad (3.19)$$

Here $D^{SV:SV}$, $D^{TA:TA}$, $D^{SV:TA}$ and $D^{TA:SV}$ are $2 \times 2$ matrices, $D^{TA:P}$ is a $2 \times 1$ matrix, $D^{P:TA}$ is a $1 \times 2$ matrix, and $D^{P:P}$ is a $1 \times 1$ matrix. Similarly, we can decompose the scattering kernel matrix $I$ into the even part $I^{\text{even}}$ and the odd part $I^{\text{odd}}$ in $\sqrt{P^2}$.

For any reasonable scattering kernel, its odd part vanishes:

$$I^{\text{odd}} = 0. \quad (3.20)$$
If we utilize this structure of $D$ and $I$, we can simplify the series of the BS equation (3.9)–(3.11). Since $D^0$ has a block diagonal structure and $D^1$ has a block off-diagonal structure, and since $I^1 = 0$, we can see

$$\lambda^1 = 0$$

from (3.13). Then, from (3.14), (3.20) and (3.21), we can obtain a simpler equation for $|\chi_1\rangle$,

$$(\lambda^0 - D^0 I^0) |\chi_1\rangle = \lambda^0 D^1 (D^0)^{-1} |\chi_0\rangle,$$

and from (3.16) with (3.15), (3.20)–(3.22), we can obtain a simpler equation for $\lambda^2$,

$$\lambda^2 = \frac{1}{\langle \chi_0 | (D^0)^{-1} \chi_0 \rangle} \left[ \lambda^0 \langle \chi_0 | (D^0)^{-1} D^2 (D^0)^{-1} |\chi_0\rangle + \langle \chi_0 | I^0 |\chi_0\rangle \right] + \frac{1}{\lambda^0} \langle \chi_0 | (D^0)^{-1} D^1 I^0 D^0 I^0 |\chi_1\rangle].$$

(3.23)

In our treatment, the eigenvalue $\lambda(P^2)$ is given as a power series in $P^2$. We assume that $\lambda(P^2)$ can be analytically continued from space-like $P^2$ to time-like $P^2$ in this series. Then the masses of mesons $m$ are determined by the equation

$$\lambda(P^2) = -m^2 = 1.$$  

(3.24)

We confine the present calculation to the terms up to $O(P^2)$, although extension to higher-order terms can be performed systematically, provided that we know the quark propagator and its derivatives of necessary order (in the space-like region). Then the mass of the meson $m$ is obtained as

$$m^2 = -\frac{1}{c^{(2)}} (1 - \lambda^0),$$

(3.25)

where $c^{(2)}$ is the coefficient of the order $P^2$ term in the power series of $\lambda(P^2)$, as in (3.3). Thus the sign of $c^{(2)}$ is crucial for modes to be physical.

The zeroth-order BS equation, which determines the zeroth-order BS amplitude $|\chi_0\rangle$ and the zeroth-order eigenvalue $\lambda^0$, is invariant under the Euclidean $O(4)$ rotation. Therefore, $|\chi_0\rangle$ can be decomposed into radial part and angle-spinor part. Euclidean (4-dimensional) bispinor harmonics have been studied by several authors. Here we follow the study developed by Delbourgo et al. 26 and Ladányi. 27 We have already decomposed the bispinor BS amplitude in terms of Euclidean $\gamma$-matrices (see Eq. (2.14)), so that the remaining task is the decomposition of these five components into radial and angular parts. This decomposition is made in Appendix C.

§4. $\pi$-family meson mass spectrum

As an application of the formalism developed in this paper, we study here the meson mass spectrum in the pion channel. The transformation properties of 4-dimensional bispinor under the parity $\Pi$ and the charge conjugation $C$ suggest that
the zeroth order amplitude for $\Pi = -(-1)^l$ and $C = -\Pi$ mesons ($\pi$-family mesons) is \[\chi^0_{NJM}(p) = f_N(p)Y_{NJM}(\hat{p})\gamma_5,\] (4-1)

where $Y_{NJM}(\hat{p})$ represents the $O(4)$ spherical harmonics defined in Appendix C and $\hat{p}$ is the unit vector in the direction of the four-vector $p$. Here $N - J$ is assumed to be an even, non-negative integer, such that the norm of the meson state is positive definite. Other bispinor components are of higher order in the $P^2$ expansion and can be generated from the zeroth order amplitude (4-1) through Eq. (3.22). From the structures of the two-body propagator matrix $D(p, P)$ and the scattering kernel matrix $T^{(lad)}(q)$ ((3-18) and (2-20)), we see that the pseudoscalar sector of the zeroth order BS equation (3.9) is self-contained:

\[\chi^0_{NJM}(p)Y_{NJM}(\hat{p}) = \frac{3z(p)^2}{M^2(p) + p^2} \int \frac{d^4k}{(2\pi)^4} \frac{d(p-k)}{(p-k)^2} f_N(k)Y_{NJM}(\hat{k}).\] (4-2)

Here the wave function renormalization function $z(p)$ and the mass function $M(p)$ of the quark propagator satisfy the coupled equations

\[\frac{1}{z(p)} = 1 + \frac{1}{p^2} \int \frac{d^4k}{(2\pi)^4} \frac{d(p-k)}{(p-k)^2} \frac{z(k)}{k^2 + M^2(k)} \times \left\{ p \cdot k + \frac{2p \cdot (p-k)k \cdot (p-k)}{(p-k)^2} \right\},\] (4-3)

\[M(p) = z(p)m_0 + 3z(p) \int \frac{d^4k}{(2\pi)^4} \frac{d(p-k)}{(p-k)^2} \frac{z(k)M(k)}{k^2 + M^2(k)}.\] (4-4)

Our task is then to solve Eq. (4-2) for $\chi^0$ and $f_N(p)$, using the solutions $M(k)$ and $z(k)$ of the simultaneous equations (4-3) and (4-4).

Although we could solve Eqs. (4-2)–(4-4) as they are, we introduce a couple of approximations to simplify expressions in this section. However, as mentioned in §2, the same approximation must be applied to all of these equations.

The first approximation we make is related to the angular part of the loop integral. To perform the integral over the angular part of $k$, we decompose the gauge boson function $d(p^2 + k^2 - 2pkx)$ into $O(4)$ partial waves:

\[d(p^2 + k^2 - 2pkx) = \sum_{l=0}^{\infty} d_l(p, k)C^l_0(x),\] (4-5)

where $C^l_0(x)$ is the Gegenbauer polynomial, and the partial wave component is defined as

\[d_l(p, k) \equiv \frac{2}{\pi} \int_{-1}^{1} dx (1 - x^2)^{l/2} C^l_0(x)d(p^2 + k^2 - 2pkx).\] (4-6)

Here, $x = \cos \theta$, where $\theta$ is the angle between $\hat{p}$ and $\hat{k}$. We can now rewrite Eqs. (4-3) and (4-4) using $d_l(p, k)$ as

\[\frac{1}{z(p)} = 1 + \frac{1}{(4\pi)^2} \int d^2k^2 \frac{k^2}{k^2 + M^2} \frac{z(k)}{p^2 + M^2(k)}.\]
\begin{equation}
\times \frac{1}{2} \sum_{l=1}^{\infty} d_l(p, k) \left[ (l+2)X^l + lX^{l+2} \right],
\end{equation}

\begin{equation}
M(p) = z(p)m_0 + \frac{3}{(4\pi)^2} z(p) \int dk^2 \frac{k}{p} \frac{z(k)M(k)}{k^2 + M^2(k)} \sum_{l=0}^{\infty} d_l(p, k)X^{l+1},
\end{equation}

where

\begin{equation}
X \equiv \begin{cases} 
\frac{k}{p} & \text{if } k < p, \\
\frac{p}{k} & \text{if } k > p.
\end{cases}
\end{equation}

Similarly, the zeroth order BS equation (4.2) becomes

\begin{equation}
\lambda_0 f_N(p) = \frac{3}{M^2(p) + p^2} \frac{1}{(4\pi)^2} \frac{1}{N+1} \int dk^2 \frac{k}{p} f_N(k) \sum_{n=0}^{\infty} X^{n+1} d^n_N(p, k),
\end{equation}

where

\begin{equation}
d^n_N(p, k) \equiv \frac{2}{\pi} \int_{-1}^{1} C^n_n(x) C^1_N(x)d(p^2 + k^2 - 2pkx)(1 - x^2)^{1/2} dx.
\end{equation}

The first approximation is to neglect the nonzero \(O(4)\) angular momentum components of the gauge boson function \(d_l(p^2 + k^2 - 2pkx):\)

\begin{equation}
d_l(p, k) = 0 \quad \text{for } l \geq 1.
\end{equation}

This is a good approximation if the gauge boson function is sufficiently smooth. Since the kernel of the integral equation (4.7) contains \(d_l(p, k)\) with \(l \geq 1\) only, Eq. (4.7) becomes trivial under the approximation (4.12) so that, setting \(z(p) \equiv 1\), we have only to solve Eq. (4.8) as a functional of \(M(p):\)

\begin{equation}
M(p) = m_0 + \frac{3}{4\pi^2} \int dk^2 \frac{k}{p} \frac{M(k)}{k^2 + M^2(k)} X d_0(p, k).
\end{equation}

Under this approximation, the partial-wave component of the BS kernel (4.11) becomes

\begin{equation}
d^n_N(p, k) = d_0(p, k) \delta_{nN}.
\end{equation}

The zeroth order BS equation (4.10) is then written as

\begin{equation}
\lambda_0 f_N(p) = \frac{3}{M^2(p) + p^2} \frac{1}{(4\pi)^2} \frac{1}{N+1} \int dk^2 \frac{k}{p} f_N(k)X^{N+1} d_0(p, k).
\end{equation}

Note that Eq. (4.15) for \(N = 0\) possesses the eigenvalue \(\lambda_0 = 1\) in the chiral limit \((m_0 = 0)\) with the solution

\begin{equation}
f_0(p) \propto \frac{M(p)}{M^2(p) + p^2},
\end{equation}

which corresponds to the massless pion. Thus our approximation (4.12) is consistent with the spontaneous breaking of the chiral symmetry.\(^{16}\)
The next approximation is the (generalized) Higashijima-Miransky approximation, \(31), 32\) which puts
\[
d_0(p,k) = d(\max[p^2, k^2]). \tag{4.17}
\]
This approximation allows us to rewrite the integral equation (4.15) in a differential form. The zeroth order BS equation (4.15) finally becomes, in integral form,
\[
\lambda_0 f_N(p) = \frac{3}{M^2(p) + p^2} \frac{1}{(4\pi)^2} \frac{1}{N+1} \left\{ d(p^2) \int_0^{\infty} dk \left( \frac{k}{p} \right)^N f_N(k) \right. \\

\left. + \int_{p^2}^{\infty} dk' \left( \frac{p}{k'} \right)^N f_N(k') d(k^2) \right\}. \tag{4.18}
\]
The differential form of this equation is discussed in Appendix D, where we study the asymptotic behavior of the zeroth order BS amplitude \(f_N(p)\). The asymptotic form derived there will be used as an initial guess for a numerical study of the integral equation (4.18) in the next section.

Let us now turn to the higher order component of the BS amplitude and the eigenvalue. Since we have obtained the zeroth order eigenvalue \(\lambda_0\) and the BS amplitude \(\chi_0\), we can calculate the second order eigenvalue \(\lambda_2\) using (3.23) together with the first order BS amplitude \(\chi_1\). Although the first order BS amplitude \(\chi_1\) can be determined by solving the integral equation (3.22) without any approximation, we here introduce the approximation which gives the Pagels-Stoker formula for the pion decay constant in the chiral limit. \(25)\) We set
\[
D_0 I_0 |\chi_1\rangle = 0 \tag{4.19}
\]
in the integral equation (3.22) for the first order amplitude \(\chi_1\). This approximation is rather ad hoc, but it simplifies the calculation greatly. Under this approximation, the first order BS equation (3.22) becomes just an algebraic relation between \(|\chi_1\rangle\) and \(|\chi_0\rangle\),
\[
|\chi_1\rangle = D^1 (D^0)^{-1} |\chi_0\rangle, \tag{4.20}
\]
and the second order eigenvalue \(\lambda_2\) becomes
\[
\lambda_2 = \frac{\lambda_0 (\langle \chi_0 | (D^0)^{-1} D^2 (D^0)^{-1} | \chi_0 \rangle + \langle \chi_0 | (D^0)^{-1} D^1 I_0 D^1 (D^0)^{-1} | \chi_0 \rangle + \langle \chi_0 | I^2 | \chi_0 \rangle) \langle \chi_0 | (D^0)^{-1} | \chi_0 \rangle}{\langle \chi_0 | (D^0)^{-1} | \chi_0 \rangle}, \tag{4.21}
\]
in the notation of §3. Recall that \(I^{(lad)} = 0\) in the ladder approximation and
\[
(D^0)_{P,P} = M^2(p) + p^2. \tag{4.22}
\]
Since there is only one non-vanishing component which connects \(\chi_1\) and \(\chi_0\) in the first order two-body propagator matrix \(D\) in the \(\pi\)-family channel, that is,
\[
D^{1A,P} = \frac{1}{M^2(p) + p^2} \left( \frac{i\epsilon_{\alpha\beta\delta\kappa} p_\delta P_\kappa}{2M'(p)p_\alpha P_\alpha - M(p)P_\alpha} \right), \tag{4.23}
\]
the first order BS amplitude $\chi^1$ has only tensor and axial-vector components of the form:

$$|\chi^1| = \frac{1}{M^2(p) + p^2} \left( \frac{i\epsilon_{\alpha\beta\delta\kappa} p_\delta P_\kappa}{2M'(p)p_\alpha p_\alpha - M(p)P_\alpha} \right) f_N(p)Y_{NJ}\langle \hat{p} \rangle. \quad (4.24)$$

Therefore, the BS amplitude up to the order $\sqrt{p^2}$ under this approximation is

$$|\chi| = f_N(p)Y_{NJ}(\hat{p}) \left[ \gamma_5 + \frac{i}{M^2(p) + p^2} \left\{ \frac{1}{2} \epsilon_{\alpha\beta\delta\kappa} p_\delta \sigma_{\alpha\beta} + \left\{ 2M'(p)p_\alpha p_\alpha - M(p)P_\alpha \right\} \gamma_\alpha \gamma_5 \right\} \right]. \quad (4.25)$$

The expressions for the building blocks of $\lambda^2$ in (4.21) are

$$\langle \chi^0 \mid (\mathcal{D}^0)^{-1} \mathcal{D}^2 (\mathcal{D}^0)^{-1} \mid \chi^0 \rangle$$

$$= \frac{1}{4} P^2 - \frac{1}{(2\pi)^4} \int_0^\infty dp^2 f_N(p)^2 \left[ \frac{2M(p)\partial M(p)}{\partial p^2} + 3 + p^2 \right] \left\{ \frac{1}{M^2(p) + p^2} \left( 2M(p)\frac{\partial^2 M(p)}{\partial p^2} + 1 \right) \right\}, \quad (4.26)$$

$$\langle \chi^0 \mid (\mathcal{D}^0)^{-1} \mathcal{D}^1 \mathcal{I}^0 \mathcal{D}^1 (\mathcal{D}^0)^{-1} \mid \chi^0 \rangle$$

$$= \frac{1}{4} \delta_{N0} + \frac{2N + 3}{4(N + 1)} X^{N+2} - \frac{2N + 1}{4(N + 1)} X^N \frac{1}{4M'(p)M'(k)p k} \left\{ \left( X + \frac{1}{X} \right) \delta_{N0} - \delta_{N1} \right\}$$

$$+ \frac{1}{4} \left( \frac{2p}{k} - \frac{k}{p} \right) X^{N+1} + \left( 1 - X^4 \right) \delta_{N0}$$

$$= \frac{1}{4} \left( \frac{2p}{k} - \frac{k}{p} \right) X^{N+1} + \left( 1 - X^4 \right) \delta_{N0} \quad (4.27)$$

and

$$\langle \chi^0 \mid (\mathcal{D}^0)^{-1} \mid \chi^0 \rangle = \frac{1}{4} \left( \frac{2p}{k} - \frac{k}{p} \right) X^{N+1} + \left( 1 - X^4 \right) \delta_{N0}. \quad (4.28)$$

§5. Numerical results

In this section, we give the numerical results obtained from (3.25) with the gauge boson functions of the DGL theory (2.11) in the chiral limit. We fix the QCD scale
Fig. 3. The mass function $M(p^2)$ of the quark propagator as a function of $p^2$ in the space-like region for various values of the dual gluon mass $m_B$ obtained by solving the DSE in the DGL theory.

Parameter as $\Lambda_{QCD} = 200$ MeV, and all the other parameters in the DGL theory are fixed by the linear potential and the chiral properties: $q_c^2 = 10A_{QCD}^2$, $m_B = 0.5$ GeV and $a = 85$ MeV. 4)

We first calculate the mass function of quarks by solving the DS equation (4.13) with $m_0 = 0$. We show in Fig. 3 the mass function, $M(p^2)$, as a function of the momentum squared $p^2$ for various dual gluon mass, $m_B$. The mass function drops suddenly around $q_c$ and increases with the dual gluon mass, which implies that the monopole condensation plays an important role in producing the dynamical quark mass. 4) The values of $M(0)$ are 1.10, 1.88 and 2.77 in units of $A_{QCD}$ for $m_B = 2.0$, 2.5 and 3.0 $A_{QCD}$, respectively. In the following, we use $m_B = 2.5A_{QCD}$. Note that, for this value, $M(0) = 376$ MeV, which is comparable to the constituent quark mass. The ultraviolet behavior of the mass function in the chiral limit is the same as expected from the renormalization group speculation. 33)

\[
M(p^2) \sim p^2 \rightarrow \infty \left( \frac{\langle \bar{\psi}\psi \rangle}{\ln(p^2/A_{QCD}^2)} \right)^{A^2/2}.
\] (5.1)

We list the results of the BS equation for zeroth order eigenvalue, $\lambda^0$, and $c^{(2)}$, which is the coefficient of the $O(P^2)$ term of $\lambda(P^2)$ defined in (3.25), for various $O(4)$ angular momenta, $N$, in Table I. In the present calculations, we truncate the expansion of $\lambda(P^2)$ up to $O(P^2)$, so that we approximate $\lambda(P^2)$ by a line. The meson masses are obtained by (3.25) as the crossing point with $\lambda(P^2) = 1$. This situation is depicted in Fig. 4.

The eigenvalue $\lambda(P^2)$, which satisfies the condition $\lambda(P^2 = -m^2) = 1$, in-
Fig. 4. The eigenvalues $\lambda(P^2)$ for various values of the $O(4)$ angular momentum $N$ as functions of $P^2$. The solution of $\lambda(P^2) = 1$ provides the mass of the meson. Mesons with total spin $J$ which satisfy the condition $0 \leq J, J - 2, \cdots \leq N$ are degenerated for each $N$.

The mass square $m^2$ increases monotonically with the quantum number $N$. The resulting mass square $m^2$ is provided in the last column of Table I. These results can be plotted for the mass square $m^2$ as a function of the spin $J$, as shown in Fig. 5. The experimental masses with the corresponding widths\(^{(34)}\) are indicated by crosses with vertical bars. Here we have not changed the parameters of the DGL theory, which have been determined by the chiral properties.\(^{(4)}\) The calculated results lie nearly on the plotted line. The dashed lines are obtained by connecting the two lowest points for each mass group. Hence, the Regge behavior is nearly reproduced in this framework.

We show the radial part of the zeroth order BS amplitudes, $f_N(p)$, in Fig. 6. The radial part of the BS amplitudes in Fig. 6 are scaled so that $f_N(p)/p^N$ becomes unity at $p = 0$. For small $p$, $f_N(p)$ behaves as $p^N$, and the behavior of $f_N(p)$ for large $p$ is given in (D.10), derived in Appendix D. The peaks of $f_N(p)$ shift gradually towards higher $p^2$ values with $N$. It is worth noting that our result for $N = 0$ seems to be consistent with the pion BS amplitude in Refs. 12) and 6).

<table>
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<th>$N$</th>
<th>$\lambda^0$</th>
<th>$c^{(2)}$</th>
<th>$m^2$ (GeV$^2$)</th>
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<td>0.000</td>
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</table>
Fig. 5. The square meson masses in the pion channel are plotted as functions of spin $J$ in units of GeV$^2$. The calculated results are indicated by black dots, and the experimental masses are denoted by crosses with vertical bars indicating their widths. The dashed lines go through the masses at $J = 0$ and 1 for each “Regge group”.

Fig. 6. Functions $f_N(p)/p^N$ are shown, where $f_N(p)$ is the radial part of the zeroth-order BS amplitude defined in (4.1). The $f_N(p)/p^N$ are scaled so as to become unity at $p = 0$. To clearly display the smaller $p$ behavior, we plot in the smaller figure $f_N(p)/p^N$ as a function of $p^2$ for small momenta.

It is interesting to consider the mass square as a function of the spin $J$ for various $m_B$. It is shown in Fig. 7. As $m_B$ increases, the mass square increases. For comparison with other models, shown here are the results of the BS calculation using the QCD-like model gluon propagator. For the parameters of the QCD-like model, we use $\Lambda_{QCD} = 750$ MeV and $q^2_c = 1.15\Lambda_{QCD}^2$. The mass square in the QCD-like model does not increase as rapidly as a linear function. The string
tension extracted from the slope of the lowest Regge trajectory for the DGL case ($m_B = 2.5\Lambda_{\text{QCD}}$) is 0.81 GeV/fm, which is consistent with the generally accepted value of the string tension $\sim 1$ GeV/fm.\(^{35}\) Thus, the model gluon propagator of the DGL theory approximately reproduces the Regge behavior for the pion family meson, while the model gluon propagator of the QCD-like model seems to fail. Furthermore, the choice of the value of the dual gluon mass $m_B = 2.5\Lambda_{\text{QCD}}$, which is determined in Ref. 4) from the chiral properties of QCD, gives the most reasonable result.

§6. Conclusion

We have studied the meson spectra in the pion channel within the dual Ginzburg-Landau (DGL) theory using the Bethe-Salpeter (BS) equation. First, we have developed a new method to solve the BS equation by using the expansion method with momentum square around zero momentum. This method is expected to be valid in the case that the mass of the bound state is located near zero in the sense that it is located in the range where the eigenvalue of the BS equation does not change rapidly, and therefore the eigenvalue can be approximated with fewer powers of $P^2$. A confined system is the best case. It is expected to have this feature because it has no two-particle breakup threshold. This method does not need any extrapolation into the complex plane for the integrand in the BS equation. Another merit of this method is that it is a systematic method, in which we can calculate higher powers of $P^2$ step by step. Here, however, we have calculated the eigenvalue $\lambda(P^2)$ only up to $O(P^2)$ for simplicity.
We have applied this expansion method to the pseudoscalar meson with parity $\Pi = -(-)^J$ and charge conjugation $C = -\Pi$ (the pion channel). We have found Regge behavior for the mass square as a function of the spin (i.e. $M^2 \propto J$), with the string tension, which is consistent with the experimental value. This is attributed to the confining nature of the gluon propagator in the DGL theory. In the case of the QCD-like model, the mass square does not exhibit Regge behavior. In addition to the result in Ref. 4), this result with the BS equation seems to support the assumption that the DGL theory describes the non-perturbative feature of QCD quite well.

It will be interesting to investigate other channels of the mesons, such as scalar or vector mesons, mesons with strangeness, and so on, with this method in order to increase our knowledge on the non-perturbative features of QCD as well as to confirm the validity of this method. We are planning to perform these studies in the future.

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Appendix A

Gluon Propagator of the Dual Ginzburg-Landau Theory

The DGL Lagrangian is written as

$$\mathcal{L}_{\text{DGL}} = \mathcal{L}_{\text{dual}} + \bar{q}(i\gamma_{\mu}\partial^{\mu} - m_0 + e\gamma_{\mu}A^{\mu} \cdot \mathbf{H})q + \sum_{\alpha=1}^{3} \left| (i\partial_{\mu} + g\epsilon_{\alpha} \cdot \mathbf{B}_{\mu})\chi_{\alpha} \right|^2 - \lambda \sum_{\alpha=1}^{3} \left( \left| \chi_{\alpha} \right|^2 - v^2 \right)^2, \quad (A.1)$$

where

$$\mathcal{L}_{\text{dual}} = -\frac{1}{2n^2} [n \cdot (\partial \wedge \mathbf{A})] [n \cdot (\partial \wedge \mathbf{B})] + \frac{1}{2n^2} [n \cdot (\partial \wedge \mathbf{B})] [n \cdot (\partial \wedge \mathbf{A})]$$

$$-\frac{1}{2n^2} [n \cdot (\partial \wedge \mathbf{A})]^2 - \frac{1}{2n^2} [n \cdot (\partial \wedge \mathbf{B})]^2. \quad (A.2)$$

Here, $\mathcal{L}_{\text{dual}}$ denotes the Zwanziger form of the gauge field Lagrangian in order to treat the color charge and monopole currents, where $A_{\mu}$ denotes the gauge fields and $B_{\mu}$ the dual gauge fields in the abelian space.\(^{36}\) Also, $\chi_{\alpha}$ denotes the color monopole and $\epsilon_{\alpha}$ the root vectors of $SU(3)$, with $\mathbf{H}$ representing the diagonal $SU(3)$ generators, $\mathbf{H} = \left( \frac{\lambda_3}{2}, \frac{\lambda_8}{2} \right)$. The color charge $e$ and the monopole charge $g$ satisfy the Dirac condition, $eg = 4\pi$. The last term in (A.1) is the Higgs term to cause monopole condensation, where $\lambda$ and $v$ are the coupling constant and the strength of the monopole condensation, the parameters of the DGL theory. $(\partial \wedge \mathbf{A})_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, and $n_{\mu}$ is the Dirac string vector, which is a constant 4-vector.
This Lagrangian was used to calculate the linear potential between a heavy quark-antiquark pair. The chiral symmetry breaking was investigated by using the DS equation.

We briefly review the derivation of the gluon propagator within the DGL theory.

We take the mean field approximation for the monopole field \( \chi_\alpha = v \) and obtain the DGL Lagrangian under this approximation as

\[
L_{\text{DGL}}^{\text{MF}} = L_{\text{dual}} + \bar{q}i\gamma_\mu \partial^\mu q - m_0 + e\gamma_\mu A^\mu \cdot H q + \frac{1}{2}m_B^2 B^2_{\mu}.
\] (A.3)

Here, the dual gluon mass \( m_B \) is related to the monopole condensate:

\[
m_B = \sqrt{3} g v.
\]

We can then eliminate the dual gluon field \( B_{\mu} \) in favor of the gluon field to obtain the behavior of the quark field in the monopole condensed vacuum. The monopole condensed gluon propagator is obtained in the Lorentz gauge as

\[
D_{\mu\nu}(q) = \frac{1}{q^2 - m_B^2} \left\{ \frac{1}{q^2 q^2 - m_B^2 (n \cdot q)^2 X_{\mu\nu}} \right\},
\] (A.4)

where \( \alpha \) is the gauge parameter, and \( X_{\mu\nu} \) is the second order tensor composed of \( n_\mu \) and \( q_\nu \):

\[
X_{\mu\nu} = \frac{1}{n^2} \epsilon^{\lambda\mu\alpha\beta} \epsilon_{\lambda\nu\gamma\delta} n^\alpha n^\gamma q^\beta q^\delta.
\]

With this preparation, we work out the DS equation for light quarks. Since light quarks are expected to move around rapidly within the confined region, we take the average over directions of Dirac strings. We consider also the confinement property of gluons; the area law indicates that the propagation of gluons should be hindered due to the non-perturbative effect, and we therefore introduce the screening cutoff \( a \). Hence, the gluon propagator is approximated in the Landau gauge as

\[
D_{\mu\nu}(q) = \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{d_{\text{DGL}}(-q^2)}{-q^2},
\] (A.5)

where the gauge boson function in the DGL theory, \( d_{\text{DGL}}(-q^2) \), is written as

\[
d_{\text{DGL}}(-q^2) = \frac{4\pi^2}{3} \frac{2A}{\ln \left( \frac{q^2 + q_c^2}{m_B^2 + q_c^2} \right)} \left\{ 1 - \frac{2}{3} \frac{m_B^2}{m_B^2 + q_c^2} \left( 1 - \frac{2(1 + q^2/a_c^2)}{1 + \sqrt{1 + q^2/a_c^2}} \right) \right\}. \] (A.6)

Here, we have used the Higashijima-Miransky prescription\(^{31},32 \) to take into account the ultraviolet behavior of the QCD coupling with \( A = 8/(11 - 3\alpha) \), and \( q_c \) is the infrared cutoff momentum. All the parameters in this expression are fixed from the linear potential and the chiral properties of the QCD vacuum.\(^4\)

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**Appendix B**

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**Two-Body Propagator Matrix**

Here, the expression for the two-body propagator matrix \( D(p, P) \) defined in (2.18) is given. The 5 \( \times \) 5 matrix \( D(p, P) \), whose components have Lorentz indices, has the form

\[
D(p, P) = \frac{z(p_+)z(p_-)}{M^2(p_+) + p_+^2} \frac{1}{M^2(p_-) + p_-^2} \bar{D}(p, P).
\] (B.1)
The matrix elements of $\tilde{D}(p, P)$ are given below. In the following list, $S$, $V$, $T$, $A$ and $P$ stand for the scalar, vector, tensor, axial vector and pseudoscalar components, respectively. In the subscripts of $\tilde{D}_{C_1,C_2}(p, P)$, where $C_1$ and $C_2$ are one of $S$, $V$, $T$, $A$ or $P$, $C_1$ ($C_2$) specifies the row (column) of the matrix $\tilde{D}(p, P)$. In the right-hand sides of the list (B.2)–(B.21), the indices $\alpha$ and $\beta$ ($\mu$ and $\nu$) are used for the Lorentz indices belonging to $C_1$ ($C_2$), and the indices $\kappa$ and $\rho$ are used for taking summations.

$$
\tilde{D}_{S:S}(p, P) = (M(p_+)M(p_-) - p_+^\alpha p_-^\alpha), \tag{B.2}
$$

$$
\tilde{D}_{S:V}(p, P) = -i\{p_+^\alpha M(p_-) + p_-^\alpha M(p_+)\}, \tag{B.3}
$$

$$
\tilde{D}_{V:S}(p, P) = -i\{p_+^\alpha M(p_-) + p_-^\alpha M(p_+)\}, \tag{B.4}
$$

$$
\tilde{D}_{S:T}(p, P) = i(p_+^\alpha p_-^\nu - p_+^\nu p_-^\alpha), \tag{B.5}
$$

$$
\tilde{D}_{T:S}(p, P) = -i(p_+^\alpha p_-^\beta - p_+^\beta p_-^\alpha), \tag{B.6}
$$

$$
\tilde{D}_{V:V}(p, P) = \{M(p_+)M(p_-) + p_+^\alpha p_-^\alpha\}\delta_{\alpha\nu} - \{p_+^\alpha p_-^\nu + p_-^\nu p_+^\alpha\}, \tag{B.7}
$$

$$
\tilde{D}_{V:T}(p, P) = (\delta_{\alpha\nu}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\nu})\{M(p_+)p_-^\nu - M(p_-)p_+^\nu\}, \tag{B.8}
$$

$$
\tilde{D}_{T:V}(p, P) = (\delta_{\alpha\mu}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\mu})\{M(p_+)p_-^\nu - M(p_-)p_+^\nu\}, \tag{B.9}
$$

$$
\tilde{D}_{V:A}(p, P) = -i\epsilon_{\alpha\beta\mu\nu}p_+^\mu p_-^\nu, \tag{B.10}
$$

$$
\tilde{D}_{A:V}(p, P) = -i\epsilon_{\alpha\beta\mu\nu}p_+^\mu p_-^\nu, \tag{B.11}
$$

$$
\tilde{D}_{T:T}(p, P) = \{M(p_+)M(p_-) - p_+^\alpha p_-^\alpha\}\delta_{\alpha\nu} - \{p_+^\alpha p_-^\nu + p_-^\nu p_+^\alpha\}\delta_{\beta\nu} + \{p_+^\alpha p_-^\nu + p_-^\nu p_+^\alpha\}\delta_{\alpha\beta} - \{p_+^\alpha p_-^\nu + p_-^\nu p_+^\alpha\}\delta_{\beta\mu}, \tag{B.12}
$$

$$
\tilde{D}_{A:A}(p, P) = \{M(p_+)M(p_-) - p_+^\alpha p_-^\alpha\}\epsilon_{\alpha\beta\mu\nu}, \tag{B.13}
$$

$$
\tilde{D}_{T:A}(p, P) = i\{M(p_+)p_-^\nu + M(p_-)p_+^\nu\}\epsilon_{\alpha\beta\mu\nu}, \tag{B.14}
$$

$$
\tilde{D}_{A:T}(p, P) = -i\epsilon_{\alpha\beta\mu\nu}p_+^\mu p_-^\nu, \tag{B.15}
$$

$$
\tilde{D}_{P:T}(p, P) = -i\epsilon_{\mu\nu\rho\sigma}p_+^\rho p_-^\sigma, \tag{B.16}
$$

$$
\tilde{D}_{A:P}(p, P) = M(p_+)p_-^\alpha - M(p_-)p_+^\alpha, \tag{B.17}
$$

$$
\tilde{D}_{P:A}(p, P) = M(p_+)p_-^\alpha + M(p_-)p_+^\alpha, \tag{B.18}
$$

$$
\tilde{D}_{P:P}(p, P) = M(p_+)M(p_-) + p_+^\alpha p_-^\alpha, \tag{B.19}
$$

$$
\tilde{D}_{S:0}(p, P) = \tilde{D}_{\bar{A}:S}(p, P) = \tilde{D}_{S:P}(p, P) = \tilde{D}_{P:S}(p, P) = \tilde{D}_{V:P}(p, P) = \tilde{D}_{P,V}(p, P) = 0. \tag{B.21}
$$

### Appendix C

**Bispinor Basis**

Since $V_\alpha$, $T_{\alpha\beta}$ and $A_\alpha$ in (2.14) carry Lorentz indices, it is convenient to introduce scalar, vector, and tensor ultraspherical ($O(4)$) harmonics. Since irreducible representations of $O(4)$ can be labeled by two half-integers $(j_1, j_2)$ ($O(4) \sim SU(2) \times SU(2)$), the ordinary $O(3)$ angular momentum $J$ and $M$, and the parity $\Pi$,
for $S$, $V_\alpha$, $T_{\alpha\beta}$, $A_\alpha$ and $P$ in the BS amplitude $\chi$ (2.14), only two types of $O(4)$ representation occur:

$$(j_1, j_2) = \left( \frac{N}{2}, \frac{N}{2} \right), \quad \left( \frac{N+1}{2}, \frac{N-1}{2} \right), \quad \text{or} \quad \left( \frac{N-1}{2}, \frac{N+1}{2} \right). \quad (C.1)$$

We here list all relevant harmonics:

- **Scalar**
  
  \[ \left( \frac{N}{2}, \frac{N}{2} \right) \text{ representation: } Y_{NJM}(\hat{x}), \]

- **Vector**
  
  \[ \left( \frac{N+1}{2}, \frac{N-1}{2} \right) \text{ representation: } Y_{NJM,\mu}(\hat{x}), \]
  
  \[ \left( \frac{N-1}{2}, \frac{N+1}{2} \right) \text{ representation: } Y_{NJM,\mu}(\hat{x}), \]

- **Tensor**
  
  \[ \left( \frac{N}{2}, \frac{N}{2} \right) \text{ representation: } Y_{NJM,\mu\nu}(\hat{x}), \]
  
  \[ \left( \frac{N+1}{2}, \frac{N-1}{2} \right) \text{ representation: } Y_{NJM,\mu\nu}(\hat{x}), \]

Here, $Y_{NJM}(\hat{x})$ is an $N$-th surface harmonics; i.e. it satisfies

$$\Delta R^N Y_{NJM}(\hat{x}) = 0, \quad (C.2)$$

where $R = (x_1^2 + x_2^2 + x_3^2 + x_4^2)^{1/2}$, and $\Delta$ is the 4-dimensional Laplacian. For others, the orders of the surface harmonics are indicated in the parentheses of the superscripts: $Y_{NJM,\mu}(\hat{x})$ and $Y_{NJM,\mu\nu}(\hat{x})$ are $N$-th surface harmonics, and $Y_{NJM,\mu}^{(N\pm1)}(\hat{x})$ and $Y_{NJM,\mu\nu}^{(N\pm1,\pm)}(\hat{x})$ are $N \pm 1$-th surface harmonics.

The 4-dimensional scalar spherical harmonics, $Y_{NJM}(\hat{x})$, (39), (27), (38) involve the Gegenbauer polynomials $C_N^\nu(\cos \theta)$ (29) and the 3-dimensional scalar spherical harmonics $Y_{JM}(\theta, \varphi)$, and are written as

$$Y_{NJM}(\hat{x}) = Y_{NJM}(\theta, \vartheta, \varphi) = G_N^{(J)}(\theta) Y_{JM}(\theta, \varphi), \quad (C.3)$$

$$G_N^{(J)}(\theta) = \left[ \frac{2N(N+1)(N-J)}{\pi(N+J+1)!} \right]^{1/2} J! C_{N-J}^{J+1}(\cos \theta) \sin^J \theta, \quad (C.4)$$

where the angular variables, $\theta$, $\vartheta$ and $\varphi$, are defined by

$$x_1 = r \sin \vartheta \sin \varphi, \quad x_2 = r \sin \vartheta \cos \varphi, \quad x_3 = r \cos \vartheta, \quad x_4 = R \cos \theta, \quad \theta = (x_1^2 + x_2^2 + x_3^2)^{1/2} = R \sin \theta, \quad (C.5)$$

are used. The vector spherical harmonics in the $\left( \frac{N}{2}, \frac{N}{2} \right)$ representation are defined as \(^{(27)}\)

$$Y_{NJM,\mu}(\hat{x}) = \sqrt{\frac{N}{2(N+1)}} \hat{x}_\mu Y_{NJM}(\hat{x}) + \sqrt{\frac{1}{2N(N+1)}} R \frac{\partial}{\partial x_\mu} Y_{NJM}(\hat{x}), \quad (C.6)$$

$$Y_{NJM,\mu\nu}(\hat{x}) = -\sqrt{\frac{N+2}{2(N+1)}} \hat{x}_\mu \hat{x}_\nu Y_{NJM}(\hat{x}) + \sqrt{\frac{1}{2N(N+1)(N+2)}} R \frac{\partial}{\partial x_\mu} Y_{NJM}(\hat{x}), \quad (C.7)$$

\(^{(*)}\) Note that $\pm$ in the superscript is independent of the power $N \mp 1$. 

\[ \]
and the tensor spherical harmonics in the \( \left( \frac{N}{2}, \frac{N}{2} \right) \) representation as\(^{27}\)

\[
Y^{(N)}_{NJM, \mu\nu}(\hat{x}) = -\frac{i}{N(N+2)} \left( x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \right) Y_{NJM}(\hat{x}), \tag{C.8}
\]

and equivalently,

\[
\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} Y^{(N)}_{NJM, \alpha\beta}(\hat{x}). \tag{C.9}
\]

\(Y^{(N, \pm)}_{NJM, \mu}(\hat{x})\) and \(Y^{(N\pm1, \pm)}_{NJM, \mu\nu}(\hat{x})\) are constructed as follows.\(^{27}\) First, the 4-dimensional three-vector spherical harmonics are defined as

\[
Y_{(NJLM), \mu}(\hat{x}) = G^{(L)}_{N}(\theta) Y_{(JLM), \mu}(\vartheta, \varphi), \tag{C.10}
\]

where \(Y_{(JLM), \mu}(\vartheta, \varphi)\) is the \(\mu\)th component of the usual 3-dimensional vector spherical harmonics.\(^{40}\) Then, the vector spherical harmonics in \( \left( \frac{N\pm1}{2}, \frac{N\pm1}{2} \right) \) representation, \(Y^{(N, \pm)}_{NJM, \mu}(\hat{x})\), are\(^{27}\)

\[
Y^{(N, -)}_{NJM, \mu}(\hat{x}) = Y_{NJM, \mu}(\hat{x}), \quad (i = 1, 2, 3) \tag{C.11}
\]

\[
Y^{(N, -)}_{NJM, 4}(\hat{x}) = 0, \tag{C.12}
\]

\[
Y^{(N, +)}_{NJM, \mu}(\hat{x}) = \frac{i}{2(N+1)} \epsilon_{\mu\nu\rho\sigma} (x_\rho \partial_\sigma - x_\sigma \partial_\rho) Y^{(N, -)}_{NJM, \nu}(\hat{x}). \tag{C.13}
\]

Next, the tensor spherical harmonics in the \( \left( \frac{N\pm1}{2}, \frac{N\pm1}{2} \right) \) representation \(Y^{(N\pm1, \pm)}_{NJM, \mu\nu}(\hat{x})\) are\(^{27}\)

\[
Y^{(N\pm1, \pm)}_{NJM, \mu\nu}(\hat{x}) = U_{\pm, \sigma} Y^{(N\pm1, \sigma)}_{NJM, \mu\nu}(\hat{x}), \tag{C.14}
\]

where the summation for \(\sigma\) is taken over \(\pm\). Here, \(U_{\pm, \sigma}\) is a 2 \(\times\) 2 matrix defined as

\[
U_{\pm, \sigma} = \frac{i}{\sqrt{2}} \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right), \tag{C.15}
\]

and

\[
Y^{(N\pm1, \pm)}_{NJM, \mu\nu}(\hat{x}) = \hat{x}_\mu Y^{(N\pm)}_{NJM, \nu}(\hat{x}) - \hat{x}_\nu Y^{(N\pm)}_{NJM, \mu}(\hat{x}), \quad (i = 1, 2, 3) \tag{C.16}
\]

\[
Y^{(N\pm1, \pm)}_{NJM, 4}(\hat{x}) = \frac{1}{N + 1} \left\{ \hat{x}_\mu Y^{(N\pm)}_{NJM, \nu}(\hat{x}) - \hat{x}_\nu Y^{(N\pm)}_{NJM, \mu}(\hat{x}) \right\} - R \left[ \partial_\mu Y^{(N\pm)}_{NJM, \nu}(\hat{x}) - \partial_\nu Y^{(N\pm)}_{NJM, \mu}(\hat{x}) \right]. \tag{C.17}
\]

Combining the Euclidean \(\gamma\) matrices with these \(O(4)\) spherical harmonics, we have a complete bispinor basis:\(^{27),37}\)
The Asymptotic Behavior of the Zeroth Order BS Amplitude

Here we consider the asymptotic behavior of the zeroth order BS amplitude $f_N(p)$ in order to solve the zeroth order BSE starting from a plausible initial solution with correct asymptotic form.

If we introduce $\Sigma^N(p) = \frac{M^2(p^2) + p^2}{p^2} f_N(p)$, $\lambda(p^2) = \frac{3}{4\pi^2} d(p^2)$ and $\lambda^N(p^2) = \frac{\lambda(p^2)}{p^2}$, then the zeroth order BS equation (4.18) can be rewritten as

\[
(N + 1)^0 \lambda^0 \Sigma^N(p) = \frac{1}{4} \int_0^\infty dk^2 \lambda^N(k^2) \frac{k^{2N}}{M^2(k) + k^2} \Sigma^N(k) + \frac{1}{4p^2} \lambda^N(p^2) \int_0^{p^2} dk^2 \frac{k^{2N+2}}{M^2(k) + k^2} \Sigma^N(k). \tag{D.1}
\]

Differentiating (D.1) with respect to $p^2$, we obtain

\[
(N + 1)^0 \lambda^0 \Sigma'^N(p) = \left( \frac{\lambda^N(p^2)}{4p^2} \right)' \int_0^{p^2} dk^2 \frac{k^{2N+2}}{M^2(k) + k^2} \Sigma^N(k). \tag{D.2}
\]

Here, $'$ denotes $\frac{d}{dp^2}$. Repeated differentiation gives

\[
(N + 1)^0 \lambda^0 \left( \frac{\Sigma'^N(p)}{(\lambda^N(p^2)/4p^2)} \right)' = \frac{p^{2N+2}}{M^2(p) + p^2} \Sigma^N(p). \tag{D.3}
\]
We write the asymptotic form of $\Sigma^N(p)$ as

$$\Sigma^N(p) \sim c x^\alpha (\ln x)^\beta,$$  \hspace{1cm} (D.4)

where $c$ is an arbitrary constant and $x \equiv p^2/\Lambda^2_{\text{QCD}}$. Substituting (D.4) and the asymptotic forms of $d(p^2)$ and $M(p)$ in the chiral limit, \cite{31,30,33,41}

$$d(p^2) \sim \frac{4\pi^2}{3} \frac{2A}{\ln(p^2/\Lambda^2_{\text{QCD}})}$$  \hspace{1cm} (D.5)

and

$$M(p^2) \sim c' \frac{\Lambda_{\text{QCD}}}{x} (\ln x)^{\frac{A}{2}-1}$$  \hspace{1cm} (D.6)

into (D.3), we obtain

$$\frac{1}{(N+1)\lambda^0} = \frac{\left[\frac{\Sigma^{N'}(p)}{(\Lambda^N(p^2)/4p^2)}\right]'}{p^{N+2}/p^{2+N} + \Sigma^N(p)}$$

$$\sim \frac{2}{AN+1} \left[ \frac{1}{N+1} \ln x + \frac{\alpha (\alpha + N + 1) \ln x + \beta (\alpha + N + 1)}{\alpha (\beta - \frac{\alpha}{N+1}) + O\left(\frac{1}{\ln x}\right)} \right].$$  \hspace{1cm} (D.7)

The differential equation (D.3) is supplemented with the boundary condition at $p^2 \to \infty$, derived from (D.1):

$$\left[\frac{\Sigma^N(p)}{(\Lambda^N(p^2)/4p^2)}\right]' = 0.$$  \hspace{1cm} (D.8)

$\alpha$ and $\beta$, which simultaneously satisfy (D.7) and the condition (D.8), are

$$\alpha = -(N+1), \quad \beta = \frac{A}{2(N+1)\lambda^0} - 1.$$  \hspace{1cm} (D.9)

The asymptotic form of $f_N(p)$ finally becomes

$$f_N(p) \sim c p^N \left( \frac{p^2}{\Lambda^2_{\text{QCD}}} \right)^{-(N+1)} \left( \ln \frac{p^2}{\Lambda^2_{\text{QCD}}} \right)^{2(N+1)\lambda^0-1}.$$  \hspace{1cm} (D.10)

References

24) R. E. Cutkosky, ibid. 1135.
34) K. Kusaka, H. Toki and S. Umisedo, to be published.