Chaotic and Chaos-Like Behavior in Continued Fractions

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(Received August 28, 1998)

Chaotic and chaos-like behavior in continued fractions is studied with respect to several types of maps, including a logistic map. Various numerical phenomena in the continued fractions are investigated, where the fractions correspond to fractal structures. Cyclic terms in the Cauchy distribution areas are introduced, including the chaos-like behavior. It is indicated that such mixed states of distributions and cycles are common in the chaotic and chaos-like behavior.

1. Fractal structures and continued fractions

Self-energies of Green’s functions can be calculated using various methods to obtain electronic densities of state in solids. In the linear combination of atomic orbitals (LCAO) method, the Green’s function can be equated to a reverse matrix of a secular equation: 1^-3)

\[ G(E) = [E I - H]^{-1}. \]  

Here \( I \) is a unit matrix, \( E \) is an energy parameter, and \( H \) is the matrix of the Hamiltonian. The Hamiltonian is represented by a symmetric matrix in the electronic interaction system. The \((i, i)\)-elements of the Green’s function can be calculated as

\[ G_{ii}(E) = \frac{1}{E - \Sigma_i(\sigma)}. \]  

where \( \Sigma \) is the selfenergy. Within a Bethe lattice (BL) approximation, the term \( \sigma \) is obtained as self-consistent continued fractions. 4) These continued fractions have fractal structures just like the trees shown in Fig. 1. In this fractal structure, the unit composition represented by a gray circle is recurrently broadened by the expansion of arms and circles including the same type of gray circles. An electron located at a site \( i \) interacts with a site \( j \) through the transfer energy \( v_{ij}(v_{ij} = v_{ji}) \). After various interactions between these sites, this electron contributes to the construction of the energy state in the system. The selfenergy...
corresponding to Fig. 1 can be expressed in terms of a continued fraction as

$$\sigma_i = \frac{v_{ik}v_{ki}}{E - \varepsilon_k - \sigma_k} + \frac{v_{ij}v_{ji}}{E - \varepsilon_j - \sigma_j} - \frac{v_{mj}v_{jm}}{E - \varepsilon_m - \sigma_m},$$

(3)

where \(\sigma_i\) is the self-consistent selfenergy,

$$\sigma_i = \sigma_j = \cdots = \sigma_m = \sigma.$$  

(4)

2. Chaotic region in continued fractions

In contrast to electronic interaction systems, the other type of directed-graph interactions can be represented by antisymmetric matrices composed of elements \(v_{ij}\) satisfying \(v_{ij} = -v_{ji}\) or asymmetric matrices with \(v_{ij} \neq v_{ji}\). In an asymmetric matrix system, the iterated values of the continued fraction selfenergies exhibit the chaotic or chaos-like behavior as a characteristic of fractal structure interactions in the BL.\(^{5,6}\)

In Eq. (3), the second term becomes the modified equation of the logistic map (LM):

$$\sigma = \frac{R}{a - \frac{b}{c - \sigma} - \frac{d}{e - \sigma}}.$$  

(5)

By fixing the parameters as \(a = 0, b = 1, c = 0, d = -1\) and \(e = 1\), we can obtain the LM equation from Eq. (5). We can also find various types of chaotic behavior in such continued fractions by using arbitrary parameters. Using Eq. (5) we obtain a map which is the modified LM displayed in Fig. 2, where \(R\) is the LM coefficient parameter. Here the convergent areas of the LM in \(0 < R < 0.5\) are broken, and the values \(\sigma\) are distributed into an infinite range as in Fig. 2(b).

3. Various chaotic behaviors

Typical chaotic behavior is studied in this section, and the nature of the continued fraction chaos is discussed. In double-step continued fractions,

$$\sigma = \frac{R}{a - \frac{b}{c - \sigma}},$$  

(6)

Fig. 2. The chaotic behavior of the expanded LM in Eq. (5). The parameters are fixed as \(a = 0, b = 0.1, c = 0.1, d = -1\) and \(e = 1\).
various chaotic behavior is observed through the parameter changes, as shown in Figs. 3 (a)–(d) and Figs. 4 (a) and (b). In Fig. 3, a number of chaotic areas are observed by changing the parameter $c$ in Eq. (6). In Fig. 4(a), $R$-proportional distribution broadening occurs in the mapping, and in (b), cycle solutions reveal divergence for some value of $a$.

4. **Cycle solutions, cyclic terms and Cauchy distributions** If the systems are rep-
Fig. 5. Cyclic behavior in the continued fractions of Eq. (6) for $a = 1.0$, $c = 2.0$. (a) Cyclic terms in the Cauchy distribution area [$b = 1$. Data from the 11th through 110th iterations]. (b) Cycle solutions in the natural chaotic area [$b = -1.0$. Data from the 1001st through 1100th iterations]. This is an enlargement of Fig. 3(b).

represented by Eq. (5) or (6) with symmetric interactions ($v_{ij} = v_{ji}$), the continued fractions generate cyclic terms and Cauchy distributions, which are not principal characteristic of the chaos. Though the parameter is selected in the Cauchy distribution area, we can find several types of chaos-like behavior embedded in a fractal manner. The cyclic terms can be observed in the $R$ tracing of Eq. (6), which appear at isolated points in the area of the Cauchy distributions. An example is represented in Fig. 5(a) by using the parameters $a = 1.0$, $b = 1.0$, $c = 2.0$, where the parameter $R$ is traced by $10^{-5}$ distance division (div). The data from the first 10 iterations are discarded (cut), and the data from the subsequent 100 iterations (iter) are plotted for each $R$ value. The cyclic terms can be observed as up-down cross points in white space stripes in this figure.

Cyclic terms similar to those in the case of Fig. 5(a) are also observed in the natural chaos area, as shown in Fig. 5(b). These are different from the cycle solutions accompanied by the bifurcations denoted in Figs. 3 and 4.

5. *Simple continued fraction* In the case of simple continued fractions,

$$\sigma_{n+1} = \frac{R}{a - \sigma_n}, \quad (7)$$

detailed theoretical treatments will be given in a subsequent paper. Only the origin is explained here. The continued fraction maps possess Cauchy distribution areas denoted by complex number solutions. In contrast to such results, the Cauchy distribution areas include the innumerable cyclic points in the parameter space, as in Fig. 6. These cyclic terms are obtained as the real number solutions in any case, which are

Fig. 6. Cauchy distributions and cyclic terms for $a = 1.0$ in Eq. (7). Innumerable cycle points appear in the Cauchy distribution area. [Data from the 11th through 110th iterations, $0 < R < 5.0$ and $0.002$ div].
not the solutions of \( \sigma \) but, rather, the results of double-Markov processes.\(^8\) If Eq. (7) becomes 0 within \( k \)-time periods, the continued fraction can be rewritten as

\[
\sigma_{n+k} = \sigma_n, \quad \Lambda_k(a, R)(\sigma^2 - a\sigma + R) = 0.
\]

The cyclic terms are produced by this equation, \( \Lambda_k = 0 \).

In this paper, we have discussed chaotic and chaos-like behavior in continued fractions. We have observed chaotic behavior in continued fractions composed of selfenergies renormalized in asymmetric interaction systems under the BL approximation. In the case of symmetric interaction systems, we have observed the chaos-like behavior in the form of the mixed existence of cyclic terms and Cauchy distributions of \( \sigma \). In any case, these mapping structures are characterized by the fractal structures corresponding to the continued fractions. The cycle solutions in the chaotic mapping are simply represented as the \( \delta \)-functions of \( \sigma \). In contrast to this, the cyclic terms in the Cauchy distribution areas are determined by the parameter set composed of critical solutions of double-Markov process functions leading to the Fibonacci sequence. Detailed discussion of these points will be given in a subsequent paper.\(^8\)

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