Some Comments on the Spin of the Chern-Simons Vortices

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We compute the spin of both the topological and nontopological solitons of the Chern-Simons-Higgs model using our approach based on constrained analysis. We also propose an extension of our method to the non-relativistic Chern-Simons models. The spin formula for both the relativistic and nonrelativistic theories turn out to be structurally identical. This form invariance manifests the topological origin of the Chern-Simons term responsible for inducing fractional spin. Also, some comparisons with existing results are made.

In (2+1)-dimensional field theories, the gauge-field dynamics may be dictated by the Chern-Simons (C-S) piece instead of the Maxwell (Yang-Mills) term. Interest in the C-S field theories originated from the observation that the C-S term implements the Hopf interaction in the (2+1)-dimensional O(3) nonlinear sigma model within the framework of local gauge theory. 1) The soliton sector of the theory offers excitations (baby skyrmions) carrying fractional spin and statistics. 2), 3) Pure C-S coupled field theories can support self-dual vortex configurations — a fact exemplified by numerous models. 4) A remarkable aspect of the C-S gauge field is that it can be coupled with both Poincaré and Galileo symmetric models. 5)–9) The latter possibility is very useful in view of the applications of C-S theories in condensed matter physics. 10) There are different computations of the spin of the C-S vortices, both relativistic 3), 11)–13) and non-relativistic. 17), 18) However, the results do not always agree. 19) Recently, we have proposed a general framework for obtaining the spin of relativistic C-S vortices 14) which is found to give consistent results for a variety of C-S theories. 15), 20) In this paper an extension of our method 14), 15) applicable to the non-relativistic C-S models is presented. Consequently, a general method, equally viable in the relativistic and non-relativistic cases, is provided. To put our work in the proper perspective, let us first make a brief digression.

The usual method of defining the spin of the C-S vortices is to identify the spin with the total angular momentum in the static soliton configuration. 4), 11)–13) This angular momentum integral is constructed from the symmetric energy-momentum (E-M) tensor obtained by varying the action with respect to a background metric. Since this E-M tensor is relevant in formulating the Dirac-Schwinger conditions, 21) it will henceforth be referred to as the Schwinger E-M tensor. Correspondingly, the angular momentum following from this energy momentum tensor usually goes by the name of ‘Schwinger’. This angular momentum is both symmetric and gauge invariant, and it also occurs naturally in the context of the general theory of relativity.

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Owing to these properties it is also interpreted as the physical angular momentum. In contrast to the Noether angular momentum, however, the Schwinger angular momentum does not have a natural splitting into orbital and spin parts. Thus it is not transparent how the value of this angular momentum in the static limit may be identified with the intrinsic spin of the vortices.

In the alternative field-theoretic definition of the spin of the C-S vortices advanced in Ref. 14), one abstracts the canonical part from the physical angular momentum. The canonical part is obtained by using the conventional Noether definition. Both the canonical and the physical angular momenta are obtained from the improved versions of the corresponding E-M tensors to properly account for the constraints of the theory. Now, the Noether angular momentum contains the orbital part plus the contribution coming from the spin degrees of freedom, as appropriate for generating local transformations of the fields under Lorentz transformations. Its difference from the physical angular momentum is shown to be a total boundary containing the C-S gauge field only, the value of which depends on the asymptotic limit of the C-S field. This value vanishes for nonsingular configurations. However, for the C-S vortex configurations we obtain a nonzero contribution. This contribution is found to be independent of the origin of the coordinate system. It is therefore possible to interpret this difference as an internal angular momentum characterising the intrinsic spin of the C-S vortices. The connection between the anomalous spin and the topological C-S interaction is thus clearly revealed.

The C-S coupled $O(3)$ nonlinear sigma model provides an ideal example for the comparison of the different methods. Schwinger’s E-M tensor for the model is given by the well-known expression,\(^{1)}\)

\[
\theta^s_{\mu\nu} = \frac{2}{f^2} (2\partial_{\mu} n^a \partial_{\nu} n^a - g_{\mu\nu} \partial_\alpha n^a \partial^\alpha n^a),
\]

where \(n^a (a = 1, 2, 3)\) are the sigma model fields satisfying

\[
n^a n^a = 1.
\]

The angular momentum integral is

\[
J = \int d^2 x \epsilon_{ij} x_i \theta^s_{0j}. \tag{3}
\]

Since the \((0, j)\) component of \(\theta^s_{\mu\nu}\) explicitly involves a time derivative of \(n^a\), \(J\) vanishes in the static configuration. The definition of Refs. 11) – 13) then predicts zero spin of the baby skyrmions. However, it has been definitely proved from quite general arguments that these excitations carry fractional spin and statistics.\(^{2)},^{3)}\) In fact it has been shown that the spin value is given by

\[
S = \frac{\theta}{2\pi}, \tag{4}
\]

where \(\theta\) is the C-S coupling strength. The connection of the baby skyrmions with the quasiparticles found in the quantum Hall state has been established, where the
anomalous spin of the excitations plays a crucial role.\textsuperscript{24} Hence the vanishing spin of the baby skyrmions predicted by the definition of Refs. 11) – 13) is clearly a contradiction. This contradiction, which was earlier noted in Ref. 16), prompted alternative definitions\textsuperscript{14,16} for obtaining the spin of the vortices. In this context it is interesting to note that the definition in Ref. 14) gives an entirely satisfactory result for this model.

In Ref. 14) we work in a gauge-independent formalism,\textsuperscript{25} and the Schwinger E-M tensor must be augmented by appropriate linear combinations of the constraints of the theory,\textsuperscript{26} so as to generate proper transformations of the fields.\textsuperscript{27} The difference of the angular momenta obtained from the Schwinger and the canonical E-M tensors is

\[ K = -\theta \int d^2 x \partial_i (x^i A^j A_j - x^j A^i A_j), \]  

where \( A^i \) is the C-S gauge field. The asymptotic form of the gauge field is only required to compute \( K \), and this form is dictated by the requirements of rotational symmetry and the Gauss constraint of the theory to be

\[ A^i(x) = \frac{Q}{2\pi} \epsilon_{ij} \frac{x^j}{|x|^2}, \]  

where \( Q \) is the topological charge. Using (5) we obtain

\[ K = \frac{Q^2}{2\pi} \theta. \]  

For the baby skyrmions, \( Q = 1 \) and the spin value agrees with Eq. (4). This example shows that the definition in Ref. 14) yields results compatible with general arguments.\textsuperscript{2}

We now extend our formalism for calculating spin of vortices in non-relativistic models with C-S coupling, which is the main thrust of the paper. Galileo invariant models cannot be made generally covariant, and so a gauge invariant E-M tensor cannot be constructed along the lines of Schwinger. Nevertheless, we can build up a gauge invariant E-M tensor using the equations of motion.\textsuperscript{5} We are then able to apply our formalism developed for the relativistic theories by substituting this tensor for the Schwinger E-M tensor. The resulting spin formula turns out to be exactly equivalent to that obtained in the relativistic case, revealing the topological connection with the origin of the fractional spin. Before discussing the non-relativistic case, a quick survey of a relativistic example is appropriate.

The Lagrangian of the C-S-H model is

\[ \mathcal{L} = (D_\mu \phi)^*(D^\mu \phi) + \frac{k}{2} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda - V(|\phi|), \]  

where the covariant derivative is defined by

\[ D_\mu = \partial_\mu + ieA_\mu. \]  

This model is known to possess topological as well as nontopological vortices. According to our definition\textsuperscript{14} the spin of the vortices is defined as

\[ K = J^S - J^N, \]
where $J^S$ and $J^N$ are, respectively, the Schwinger and Noether angular momentum. We work in a gauge independent formalism, where the constraints of the theory are weakly zero. Different E-M tensors thus must be augmented by appropriate linear combinations of the constraints of the theory to obtain proper transformations of the fields.

From a detailed analysis of the model (8) we arrive at the following augmented expressions for $J^S$ and $J^N$:

\begin{equation}
J^S = \int d^2x e^{ij} x_i [\pi \partial_j \phi + \pi^* \partial_j \phi^* - k \epsilon^{lm} A_j \partial_l A_m],
\end{equation}

\begin{equation}
J^N = \int d^2x \left[ e^{ij} x_i \left( \pi \partial_j \phi + \pi^* \partial_j \phi^* - \frac{k}{2} \epsilon^{lm} A_i \partial_j A_m \right) + \frac{k}{2} A_i^l A_j \right].
\end{equation}

Here $\pi$ ($\pi^*$) is the momentum conjugate to $\phi$ ($\phi^*$). Using (11) and (12) in (10) we obtain

\begin{equation}
K = -\frac{k^2}{2} \int d^2 x \partial_i \left[ x_i A_j A^j - A_i x_j A^j \right],
\end{equation}

which is the same formula as that obtained for the $O(3)$ nonlinear sigma model (see Eq. (5)). Note that the integrand is a boundary term, so that only the asymptotic form of the gauge field $A_i$ is required for the computation of $K$.

For topological vortices, the matter field $\phi$ at infinity bears a representation of the broken $U(1)$ symmetry:

\begin{equation}
\phi \approx v e^{i n \theta},
\end{equation}

where $n$ is the topological charge. The requirement of finite energy of the configuration dictates that asymptotically

\begin{equation}
e A_i = n \epsilon_{ij} \frac{x^j}{|x|^2}.
\end{equation}

The above form is rotationally symmetric and satisfies Gauss’s law. As a consequence, the magnetic flux $\Phi$ is quantised,

\begin{equation}
\Phi = \frac{2\pi n}{e}.
\end{equation}

After a straightforward calculation using (13) and (15), we obtain

\begin{equation}
K = \frac{\pi k n^2}{e^2}.
\end{equation}

The nontopological vortices lie at the threshold of stability. For these the magnetic flux $\Phi$ is arbitrary. The asymptotic form of the gauge field is now expressed as

\begin{equation}
A_i = \frac{\Phi}{2\pi} \epsilon_{ij} \frac{x^j}{|x|^2},
\end{equation}

and the spin computed from (13) is

\begin{equation}
K = \frac{k \Phi^2}{4\pi}.
\end{equation}
Equations (17) and (19) give the spin of the topological and nontopological vortices of the C-S-H model, respectively. Note that the sign of the spin is + in both cases. This again is the same as in the case of the elementary excitations of the model. In earlier analysis\(^1\) there was a difference in sign which was explained by the introduction of a new interaction. This is not necessary in the present discussion.

Now we apply the same general method to non-relativistic models. Consider the Lagrangian

$$L = i \phi^* D_t \phi - \frac{1}{2m} \left( D_k \phi \right)^* D_k \phi + \frac{k}{2} \epsilon_{\mu \nu \lambda} A_\mu \partial_\nu A_\lambda,$$

(20)

where \(\phi\) is a bosonic Schrödinger field. The model (20) is invariant under Galilean transformations but not under the transformations of the Poincaré group. Note that the Galilean transformations take time and space on an unequal footing. Thus the space-time metric is not defined. In writing (20) we have adopted a spatial Euclidean metric. Covariant and contravariant components are thus not to be distinguished.

The action of the model (20) cannot be made generally covariant. The powerful method of constructing a gauge-invariant energy-momentum (E-M) tensor, as formulated by Schwinger, is thus not available. Nonetheless, it is possible to construct a gauge-invariant E-M tensor by appealing to the equations of motion.\(^5\) Our program is then clear. We will find a gauge-invariant momentum density from the matter current obtained by using the equations of motion. These equations will then be exploited to demonstrate the conservation of the corresponding momentum. We work in the gauge independent formalism in contrast to the gauge-fixed approach of Ref. 5). A suitable linear combination of the Gauss constraint is to be added to the gauge-invariant momentum operator, in order to generate the correct transformation of the fields under spatial translation. A gauge-invariant angular momentum is then constructed using this momentum density. The canonical angular momentum obtained by Noether’s prescription is now subtracted from it. The spin of the vortices is, as usual, defined by

$$K = J - J^N,$$

(21)

which has the same form as Eq. (10), but now \(J\) is the gauge invariant angular momentum constructed by using the equations of motion.

From the Lagrangian (20), we write the Euler-Lagrange equation corresponding to \(A_\mu\),

$$k \epsilon_{\mu \nu \lambda} \partial_\nu A_\lambda = j_\mu,$$

(22)

where \(j_\mu\) is given by

$$j_0 = \phi^* \phi,$$

(23)

$$j_i = \frac{1}{2im} [\phi^* (D_i \phi) - \phi (D_i \phi)^*].$$

(24)

Observe that (22) leads to the continuity equation

$$\partial_0 j_0 + \partial_i j_i = 0,$$

(25)

so that \(j_0\) and \(j_i\) can be interpreted as the matter density and current density, respectively.
From the E-L equation corresponding to $A_0$, we obtain the Gauss constraint of the theory,
\[ G = \phi^* \phi - k \epsilon_{ij} \partial_i A_j \approx 0. \] (26)

Now we come to the construction of the gauge-invariant momentum operator. The $(0,i)$th component of the E-M tensor $T_{0i}$ (i.e. the momentum density) is obtained from the matter current:
\[ T_{0i} = \frac{i}{2} [\phi^*(D_i \phi) - \phi(D_i \phi)^*]. \] (27)

We verify by a straightforward calculation using the equations of motion that $T_{0i}$ indeed satisfies the appropriate continuity equation,
\[ \partial_0 T_{0i} + \partial_k T_{ki} = 0, \] (28)
where $T_{ki}$ is the stress tensor,
\[ T_{ki} = \frac{1}{2m} [(D_k \phi)^*(D_i \phi) + (D_k \phi)(D_i \phi)^* - \partial_i (\phi^* D_k \phi + \phi(D_k \phi)^*)]. \] (29)

Using the expression (27) of $T_{0i}$ we construct a gauge invariant momentum operator,
\[ P_i = \int d^2 x T_{0i}. \] (30)

Exploiting (28) and neglecting the boundary term we find that $P_i$ is indeed conserved:
\[ \frac{dP_i}{dt} = 0. \] (31)

The boundary term vanishes due to the condition that the covariant derivative $D_i \phi$ is zero on the boundary. This is required to keep the energy finite.

The Lagrangian (20) is first order in time derivatives. It is then easy to read off the basic brackets from (20) by symplectic arguments. The nontrivial brackets are
\[ \{ \phi(x), \phi^*(y) \} = -i \delta(x - y), \]
\[ \{ A_i(x), A_j(y) \} = \frac{1}{k} \epsilon_{ij} \delta(x - y). \] (32)

Using these basic brackets we obtain,
\[ \{ \phi(x), P_i \} = \partial_i \phi(x) + iA_i \phi(x). \] (33)

Hence the transformation of $\phi$ deviates from the expected canonical structure. For proper transformation of the fields under spatial translation we are required to supplement $T_{0i}$ with the Gauss constraint:
\[ T_{0i}^G = T_{0i} + A_i G. \] (34)

The corresponding momentum operator
\[ P_i = \int d^2 x \left[ \frac{i}{2} (\phi^* D_i \phi - \phi(D_i \phi)^*) + A_i G \right] \] (35)
turns out to be an appropriate generator of spatial translation. The term containing
the Gauss operator in (35) exactly generates a piece in \{\phi(x), P_1\} which cancels the
anomalous term in (33).

We now come to the construction of \(J\), the gauge invariant angular momentum,
from the momentum density (35):
\[
J = \int d^2x \epsilon_{ij} \phi_i \left[ \frac{i}{2} (\phi^* D_j \phi - \phi (D_j \phi)^*) + A_j G \right].
\tag{36}
\]
The canonical angular momentum \(J^N\) is obtained from Noether’s theorem as\(^7\)
\[
J^N = \int d^2x \epsilon_{ij} \left( \frac{i}{2} \phi^* \partial_j \phi - \phi (\partial_j \phi)^* \right) - \frac{k}{2} \epsilon_{mn} A_m \partial_j A_n \bigg] + \frac{k}{2} A_j A_j \bigg].
\tag{37}
\]
Substituting (36) and (37) into (21) we obtain
\[
K = -\frac{k}{2} \int d^2x \partial_i \left[ x_i A^2 - x_j A_j A_i \right].
\tag{38}
\]
Observe that the master formula for the calculation of the spin (38) is identical to
Eq. (13). The asymptotic form of \(A_i\) following from general considerations already
elaborated leads to the same structure as in (15). Inserting this into (38) exactly
reproduces (17) as the spin of the vortices.

We note in passing that self-dual soliton solutions can be obtained by including a
quartic self-interaction in (20),\(^3\) which are interpreted as the non-relativistic limit of
the nontopological vortices of the relativistic Chern-Simons-Higgs model considered
previously. The spin of these solitons can be calculated by (38) using the asymptotic
form (18). The result turns out to be identical with (19). This is expected, because
the existence of the fractional spin is connected to the Chern-Simons piece, which
is a topological term. The spin of the vortices of the model (20) with the quartic
self-interaction and an external magnetic field was calculated previously.\(^17\),\(^18\) The
method used in these calculations is in spirit akin to that of Refs. 11)–13), but in
this case the background contribution must be subtracted to get the exact spin. The
result of Refs. 17) and 18) scales with the topological number as in our case, but with
opposite signature. The same comments apply to this comparison as made earlier in
connection with the C-S-H model.

To conclude, we have found that the usual method of defining the spin as the
static limit of the physical angular momentum yields contradictory results when applied
to compute the spin of the solitons of the Chern-Simons (C-S) coupled \(O(3)\)
nonlinear sigma model.\(^16\) In this connection we have observed that a consistent re-
sult is obtained when we apply our general formalism, which exploits the constraints
of the theory for computing the spin in the C-S theories.\(^14\) Here, the canonical part
of the physical angular momentum is abstracted by subtracting the canonical (Noether)
angular momentum from the angular momentum obtained from Schwinger’s E-M tensor.\(^21\) The difference was found to be nonzero for singular configurations. In
particular, for C-S vortices this difference was shown to be independent of the origin
of the coordinate system. Consequently we interpret it as the intrinsic spin of the
vortices. The formula for the spin turns out to be model independent, and contrary to other approaches, where detailed field configurations are necessary, only the asymptotic form of the gauge field is required for its evaluation.

The spin of the topological and nontopological vortices of the C-S-H model was reviewed by the general formalism mentioned above. The spins of both types of vortices of the model turn out to have the same sign. We also find that the sign of the spin of the topological vortices is the same as that of the elementary excitations of the model.\(^{28}\) This is a satisfactory result because the spin-statistics connection is then respected with the usual Aharonov-Bohm phase. Notably, in Ref. 13) an opposite sign was found so that a new interaction was required to properly account for this phase.\(^{29}\)

Our formalism is directly applicable to the relativistic theories, but the Chern-Simons interaction enjoys the rare distinction of being suitable to be coupled to both Poincaré and Galilean symmetric models.\(^5\) - \(^9\) We were thus motivated to extend our formalism to non-relativistic theories. Moreover, a systematic discussion of the spin in such theories is nonexistent. Although an explicit calculation exists,\(^{17},^{18}\) the connection between that method and the corresponding ones used for the relativistic models is not quite clear. Our extended formalism treats the non-relativistic models within the same general framework used for the relativistic case. The resulting spin formula is identical to that obtained for the relativistic theories. This points to the topological origin of the C-S term, responsible for the induction of the fractional spin, either in relativistic or non-relativistic models.

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References

19) Significantly, the values reported in Ref. 11) and 13) also differ in sign among themselves notwithstanding the fact that they use the same definition of spin.