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Pion Elastic and Charge-Exchange Scattering on \( p \)-Shell Nuclei by Isobar-Doorway State Model

Shinya Taniguchi, Toru Sato and Hisao Ohtsubo

Department of Physics, Graduate School of Science, Osaka University
Toyonaka 560-0043, Japan

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The purpose of the present paper is to extend the isobar-hole model (i.e., the conventional isobar-doorway state model for the pion-nucleus reaction, to open-shell nuclei) by reformulating the model, and also to study the elastic scattering of pions from \( p \)-shell nuclei, the analyzing power and the charge-exchange reaction. For this purpose, we use the projection operator formalism, and take into account the correct center-of-mass transformation and the off-shell behavior of the pion-nucleon scattering amplitude. We assume a realistic nuclear model, i.e. the \( p \)-shell nuclear model of Cohen and Kurath. By studying elastic scattering from \( ^{12,14}C \) and \( ^{15}N \), and the analyzing power and the charge-exchange reaction with \( ^{13}C \) and \( ^{15}N \), we confirm the existence of the Lane term in the spreading potential implied by Hirata et al.

§1. Introduction

Collisions of pions with nuclei produce a kind of “doorway state” with a \( \Delta \) and a hole. (A model with a simple such doorway state is called the \( \Delta \)-hole model.) The idea of isobar-doorway state was proposed by Kisslinger and Wang.\(^1\) It was developed and applied to the qualitative study of elastic pion scattering from light nuclei by Hirata, Lenz and Yazaki.\(^2\) It was also studied from the viewpoint of pion propagation in nuclear medium by Weise.\(^3\) The former authors incorporated into the theory the \( \Delta \) binding potential, the Pauli effect on the \( \Delta \) decay width, and also the spreading potential of \( \Delta \). In particular, the spreading potential is an essentially important ingredient to describe the elastic scattering and the absorption cross section. By adjusting the strength of the spreading potential, they succeeded in describing both the elastic scattering and the absorption cross section. In spite of this success they faced a new problem: They were required to introduce a spreading potential with large energy dependence. Horikawa et al.\(^4\) solved this problem by introducing a possible spin-orbit term in the \( \Delta \) spreading potential. In fact they were able to reduce the energy dependence of the spreading potential. It is a well-known fact in the conventional nuclear reaction that the spin-orbit interaction gives rise to the energy dependence of the spreading potential.

At this stage, the \( \Delta \)-hole model seems very successful, and it has been also applied to other nuclear reactions, such as the charge-exchange reaction, the inelastic scattering,\(^5,6\) the quasi-elastic scattering of pions,\(^7,8\) and neutral pion photoproduction.\(^9,10\) Similar problems have also been studied by Weise and his coworkers.\(^11\) Among these, the most serious and challenging problem for this model was to describe the charge-exchange pion-nucleus scattering. The observed energy
dependence of the integrated cross section exhibits a broad peak at the \( \Delta \) resonance position, while the conventional DWIA theories predict a minimum value at this position. Auerbach\(^{12}\) proposed a possibility of explaining the experimental data by applying the isobar-doorway state model. Hirata\(^{13}\) applied his \( \Delta \)-hole model to the pion charge-exchange reaction with \(^{13}\)C. He found that the original \( \Delta \)-hole model cannot predict the magnitude of the total cross section and that introduction of an isospin-dependent term in the spreading potential improves the agreement between theory and experiment, still preserving the strong energy dependence of the spreading potential. He, however, showed later that this energy dependence can be removed by introducing the spin-orbit term into the spreading potential.\(^{15}\) This isospin dependence of the spreading potential could be understood if the spreading potential is related to the transition between \( N\Delta \) and \( NN \) states. Similar results in the charge-exchange reaction with \(^{15}\)N were also obtained by Oset.\(^{14}\)

There exists another important item to be studied, the nuclear polarization problem in pion-nucleus scattering. Recently, much attention has been focused on this problem, since the experimental data for the analyzing power with \(^{13}\)C\(^{16}\)\(^{18}\) and \(^{15}\)N\(^{19}\),\(^{20}\) have been available. It was found that the conventional \( \Delta \)-hole model cannot explain even the sign of the measured analyzing power, while it has been successfully applied to the differential cross section and the total cross section for many nuclei. This is a serious defect of the conventional \( \Delta \)-hole model.

Another problem of the theory is that it assumes a very simple nuclear structure of the target nuclei (a simple jj-coupling model), although standard \( p \)-shell models, such as the Cohen-Kurath model,\(^{21}\) predict, for example, that the probability of the \( 1p_{3/2} \) closed configuration in \(^{12}\)C is only 40 percent. This fact implies that polarization problems in the pion reaction should be reinvestigated by taking into account the effects of nuclear structure. To accomplish this, we must work with the isobar-doorway states in the open-shell nuclei. Ohtsuka attempted to extend the \( \Delta \)-hole model to open-shell nuclei by replacing the hole state with the relevant nuclear state with mass number \( A - 1 \) and examining the effect of nuclear structure on the elastic scattering of pions from \(^{12}\)C.\(^{22}\) Although his treatment was not complete, he obtained almost the same results as the \( \Delta \)-hole model. This is because elastic scattering from spinless nuclei is insensitive to nuclear structure.

The purpose of the present paper is 1) to give a correct formalism for the isobar-doorway state model for the pion reaction by extending the previous \( \Delta \)-hole model so as to make it possible to study pion reactions with open-shell nuclei, 2) to study phenomenologically the spin and isospin structure of spreading potentials in connection with the charge exchange scattering of pions, the analyzing power with \(^{13}\)C and \(^{15}\)N, and the differential cross section of the pion elastic scattering from these nuclei, and 3) to study the effects of nuclear structure on the various pion reactions in which we adopt the Cohen-Kurath model.

The paper is organized as follows. In \( \S 2 \), we briefly summarize the features of the pion-nucleon scattering near the \( \Delta \) resonance and propose a model of the pion-nucleon reaction with which we can calculate the off-shell behavior of the scattering amplitude required in the pion nuclear reactions. Section 3 is devoted to a development of the isobar-doorway state model for open-shell nuclei by modifying the

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\(^{12}\) Auerbach, \(^{13}\) Hirata, \(^{14}\) Oset, \(^{15}\) Ohtsuka.
previous model. The ‘new’ content will be discussed by comparing it with previous works. In §4, we develop the formalism of the scattering $T$-matrix in the spherical coordinates. This is then applied to a realistic nuclear reaction. It is also discussed how to include the nonresonant effect in the scattering amplitude. In §5, we present calculated results for differential cross sections of elastic scattering of pions from $^{12}$C, differential cross sections and the analyzing power for elastic scattering of pions from $^{13}$C and $^{15}$N, and differential cross section of the charge-exchange reaction with $^{13}$C and $^{15}$N. We discuss various effects on these reactions, among which emphasis is put on the nuclear structure and the spin and isospin structure of the spreading potential. Finally, a summary and conclusion of the present work are given in §6.

§2. Pion-nucleon interactions

We introduce a dynamical model for describing pion-nucleon scattering in the $(3,3)$ channel, which plays an essential role in the pion-nucleus scattering studied in later sections.

We start with a model Hamiltonian of pions, nucleons and $\Delta$s:

$$H = H^{(0)} + H_I.$$  

(2.1)

Here $H^{(0)}$ is the free Hamiltonian of pions, nucleons and $\Delta$s which can be expressed as

$$H^{(0)} = H^{(0)}_{\pi} + H^{(0)}_{N} + H^{(0)}_{\Delta}.$$  

(2.2)

The interaction Hamiltonian describes the coupling of a $\Delta$ to a pion and a nucleon:

$$H_I = H_{\pi N \Delta} + H^\dagger_{\pi N \Delta}.$$  

(2.3)

The matrix element of $H_{\pi N \Delta}$ between the pion-nucleon state and the $\Delta$ state is written as

$$\langle \Delta | H^\dagger_{\pi N \Delta} | \pi^i(k)N (-k) \rangle = i \int_{\pi N \Delta} \frac{m_N}{m_\pi} \sqrt{\frac{m_N}{(2\pi)^3 2\omega_\pi(k) e_N(k)}} F(k) \left\langle \Delta \pi | S^\dagger \cdot k | N \nu \right\rangle T^I_i,$$  

(2.4)

where the superscript $i$ stands for the isospin of a pion, and $F(k)$ is a phenomenological vertex function. $S$ and $T^I_i$ are transition spin and isospin operators, respectively. Here, we have neglected the interaction of a nucleon going into a pion and a $\Delta$, since its effect on the $P_{33}$ channel is small. Although this approximation destroys the cross symmetry of the scattering matrix, it is not essentially important in the pion-nucleus scattering. The $T$-matrix obeys the Lippmann-Schwinger equation as

$$T = H_I + H_I \frac{1}{W - H^{(0)} + i\epsilon} T.$$  

(2.5)

This is a coupled equation for the pion-nucleon state and the $\Delta$ state. Then, we obtain the following explicit forms of the coupled equations:

$$T_{\pi N, \pi N} = H_I \frac{1}{W - H^{(0)} + i\epsilon} T_{\Delta, \pi N},$$  

(2.6)
and
\[ T_{\Delta,N} = H_I + H_I \frac{1}{W - H_N^{(0)} - H_\pi^{(0)} + i\epsilon} T_{\pi,N,N}, \] (2.7)
where we defined the matrix elements
\[ T_{\pi,N,N} = \langle \pi N | T | \pi N \rangle, \quad T_{\Delta,N} = \langle \Delta | T | \pi N \rangle. \] (2.8)
Eliminating \( T_{\Delta,N} \) from the equations and using the separable property of the effective pion-nucleon interaction, we obtain the \( T \)-matrix element of pion-nucleon scattering as
\[ T_{\pi,N,N} = H_{\pi N \Delta} \frac{1}{W - H_N^{(0)} - H_\pi^{(0)} + i\epsilon} H_{\pi N \Delta}^\dagger, \] (2.9)
where \( \hat{\Sigma} \) is the self-energy of a \( \Delta \) due to its decay into a pion and a nucleon:
\[ \hat{\Sigma} = H_{\pi N \Delta} \frac{1}{W - H_N^{(0)} - H_\pi^{(0)} + i\epsilon} H_{\pi N \Delta}. \] (2.10)
By using the explicit form of \( H_{\pi N \Delta} \) in (2.4), we obtain the off-shell \( T \)-matrix element as
\[ T_{P,33}(k',k;W) = \frac{\gamma(k')\gamma(k)}{D(W)}, \] (2.11)
where the vertex function \( \gamma(k) \) is defined as
\[ \gamma(k) = \frac{f_{\pi N \Delta}}{m_\pi} \sqrt{\frac{4\pi m_N}{(2\pi)^3\omega_\pi(k)\epsilon_N(k)}} F(k), \] (2.12)
and the full \( \Delta \) propagator is expressed as
\[ D(W) = W - m_\Delta^0 - \Sigma(W), \] (2.13)
\( m_\Delta^0 \) being the bare mass of a \( \Delta \).
The self-energy \( \Sigma(W) \) is easily obtained as
\[ \Sigma(W) = \int_0^\infty p^2 dp \frac{\gamma^2(p)}{W - \epsilon_N(p) - \omega_\pi(p) + i\epsilon} \] (2.14)
\[ = R(W) - \frac{i}{2} \Gamma(W), \] (2.15)
with
\[ R(W) = P \int_0^\infty p^2 dp \frac{\gamma^2(p)}{W - \epsilon_N(p) - \omega_\pi(p)}, \] (2.16)
and
\[ \frac{i}{2} \Gamma(W) = \left. \frac{\pi \epsilon_N(k)\omega_\pi(k)}{W} \gamma^2(k) \right|_{k=k_W}, \] (2.17)
where \( k_W \) is defined so as to satisfy the equation
\[
W = \epsilon_N(k_W) + \omega_\pi(k_W). \tag{2.18}
\]
It is noted that we are calculating the off-shell \( T \)-matrix element of the pion-nucleon scattering. The real part of the self-energy \( R(W) \) gives the mass shift of \( \Delta \) and the imaginary part gives its width.

Finally, we have an expression of the Breit-Wigner type:
\[
\mathcal{T}_{P33}(k', k; W) = \frac{\gamma(k')\gamma(k)}{W - m_\Delta^0 - R(W) + \frac{i}{2} \Gamma(W)}, \tag{2.19}
\]

The on-shell \( T \)-matrix is related to the phase shifts as
\[
\mathcal{T}_{P33}(k_W, k_W; W) = -\frac{W}{\pi \epsilon_N(k_W)\omega_\pi(k_W)} e^{i\delta_{33}} \sin \delta_{33}. \tag{2.20}
\]

Now, let us fit pion-nucleon phase shifts of the VPI analysis \(^{24}\) using the above isobar model. We adopt the following monopole form factor:
\[
F(k) = \frac{1}{1 + k^2/\alpha^2}. \tag{2.21}
\]
Here our parameters are \( m_\Delta^0, \alpha \) and \( f_{\pi N \Delta} \). The pion-nucleon phase shift \( \delta_{P33} \) reaches 90° at \( W = W_0 = 1233 \) MeV, where the width is \( \Gamma(W_0) = 110 \) MeV. If we use these as constraints, we have only one parameter left to be determined. We choose \( \alpha \) so as to reproduce the experimental phase shifts. The parameter \( f_{\pi N \Delta} \) is expressed in terms of \( \alpha \) and \( \Gamma(W_0) \) as
\[
f_{\pi N \Delta} = \sqrt{\frac{\Gamma(W_0)6\pi m_N^2W_0^2(1 + k_W^2/\alpha^2)^2}{m_N k_W^3 W_0^2}}, \tag{2.22}
\]
and then, \( m_\Delta^0 \) is given as
\[
m_\Delta^0 = W_0 - R(W) = W_0 - P\int_0^\infty p^2 dp \frac{\gamma^2(p)}{W_0 - \epsilon_N(p) - \omega_\pi(p)}. \tag{2.23}
\]
As a result we obtain the following parameter values:
\[
\alpha = 300 \text{ MeV}, \quad m_\Delta^0 = 1317 \text{ MeV}, \quad f_{\pi N \Delta} = 3.348. \tag{2.24}
\]
The calculated phase shifts are compared with the experimental data in Fig. 1, from which we can confirm that the present model works quite well.
§3. Isobar-doorway state model for open-shell nuclei

We formulate the isobar-doorway state model using Feshbach’s projection operator method, which is applicable to open-shell nuclei.

3.1. Projection operator formalism of the isobar-doorway state model

In the pion-nuclear reaction, the initial state consists of an incoming pion and a ground state nucleus. Around 180 MeV, $\Delta$ is excited in the nucleus and forms nuclear states with a $\Delta$ and $(A - 1)$ nucleons. Here, we divide the nuclear Fock space into three parts, $P$-space, $D$-space and $Q$-space, and apply Feshbach’s projection operator formalism for the nuclear reactions: $P$-space is spanned by the states with a pion and $A$ nucleons, including the initial state, $D$-space is spanned by the states with a $\Delta$ and $(A - 1)$ nucleons and $Q$-space is spanned by the states with $A$ nucleons. $Q$-space is connected to the $P$- and $D$-spaces through the pion absorption processes.

The Shrödinger equation of the system

$$H|\psi\rangle = E|\psi\rangle$$

(3.1)

is decomposed into three coupled equations as

$$(E - H_P)|\psi_P\rangle = H_{PD}|\psi_D\rangle + H_{PQ}|\psi_Q\rangle,$$

(3.2)

$$(E - H_D)|\psi_D\rangle = H_{DP}|\psi_P\rangle + H_{DQ}|\psi_Q\rangle,$$

(3.3)

and

$$(E - H_Q)|\psi_Q\rangle = H_{QP}|\psi_P\rangle + H_{QD}|\psi_D\rangle,$$

(3.4)

where the projection operators $P$, $Q$ and $D$ are introduced to project the state on the $P$-, $Q$- and $D$-space, respectively. These are defined as

$$|\psi_P\rangle = P|\psi\rangle, \quad |\psi_Q\rangle = Q|\psi\rangle, \quad \text{and} \quad |\psi_D\rangle = D|\psi\rangle.$$  

(3.5)

Similarly, we define the projected Hamiltonians as

$$H_P = PHP, \quad H_{PQ} = PHQ,$$

(3.6)

and so on.

The Hamiltonian in $P$-space consists of the conventional nuclear Hamiltonian $H_A$, the free Hamiltonian $H^{(0)}_\pi$ of a pion, and the non-resonant interaction $V_{\pi N}$ between a pion and a nucleon. It is expressed as

$$H_P = H^{(0)}_P + V_{\pi N} \quad \text{with} \quad H^{(0)}_P = H_A + H^{(0)}_\pi.$$  

(3.7)

The Hamiltonian $H_D$ consists of the nuclear Hamiltonian, $H_A$, and the Hamiltonian of a $\Delta$, $H_\Delta$, includes the binding potential, and is expressed as

$$H_D = H_A + H_\Delta.$$  

(3.8)

The Hamiltonian $H_Q$ consists of only the nuclear Hamiltonian $H_A$:

$$H_Q = H_A.$$  

(3.9)
Now, we assume that the pion-nuclear reaction is dominated by the $\Delta$ resonance. This means that $P$-space couples only with $D$-space, i.e.,

$$H_{PQ} = H_{QP} = 0.$$ \hfill (3.10)

The Hamiltonians $H_{DQ}$ and $H_{QD}$ describe the pion absorption and production processes, respectively. The Hamiltonians $H_{PD}$ and $H_{DP}$ are described in terms of the $\pi N \Delta$-interaction introduced in §2.

The coupled equations (3.2)–(3.4) can be formally solved. Since no incoming waves exist in $Q$-space, the state $\ket{\psi_Q}$ is simply expressed as

$$\ket{\psi_Q} = \frac{1}{E - H_Q + i\epsilon} H_{QD} \ket{\psi_D}.$$ \hfill (3.11)

By eliminating the state $\ket{\psi_Q}$ from Eqs. (3.4) and (3.11), we obtain the following equation:

$$\left( E - H_D - H_{DQ} \frac{1}{E - H_Q + i\epsilon} H_{QD} \right) \ket{\psi_D} = H_{DP} \ket{\psi_P}. \hfill (3.12)$$

Elimination of $Q$-space introduces an additional energy-dependent effective interaction in $D$-space. By eliminating the state $\ket{\psi_D}$ from Eqs. (3.2) and (3.3) in a similar way to the above, we obtain an equation for the state $\ket{\psi_P}$ as

$$\left( E - H_P^0 - V_{\pi N} - H_{PD} \frac{1}{E - H_D - H_{DQ} \frac{1}{E - H_Q + i\epsilon} H_{QD}} \right) \ket{\psi_P} = 0.$$ \hfill (3.13)

This equation describes explicitly all the pion-nucleus reactions, such as elastic scattering, quasi-elastic scattering, charge-exchange reactions, and other inelastic scatterings. It includes the non-resonant pion-nucleon interaction $V_{\pi N}$ and also the effective interaction associated with the $\Delta$ propagation in nuclear media.

By using a standard technique for two-potential problems, the pion-nucleus scattering $T$-matrix is expressed as

$$T_{fi} = \langle \phi_f^{(-)} | V_{\pi N} | \phi_i^{(+)} \rangle + \langle \phi_f^{(-)} | H_{PD} G_\Delta(E) H_{DP} | \phi_i^{(+)} \rangle,$$ \hfill (3.14)

where we have defined the $\Delta$-nucleus Green’s function $G_\Delta$ in the $D$-space as

$$G_\Delta(E)^{-1} = E - H_D - H_{DP} \frac{1}{E - H_P + i\epsilon} H_{PD} - H_{DQ} \frac{1}{E - H_Q + i\epsilon} H_{QD},$$ \hfill (3.15)

and the states $\ket{\phi_{f,i}^{(\pm)}}$ in $P$-space obey an equation with the nonresonant $V_{\pi N}$ given by

$$(E - H_P^0 - V_{\pi N}) \ket{\phi_{f,i}^{(\pm)}} = 0.$$ \hfill (3.16)
This non-resonant interaction also appears in the effective interaction of the \( \Delta \)-Green’s function (3.15). In evaluating the matrix element of this interaction, we use the distorted waves \( \phi_{f,i}^{(\pm)} \) described by the optical potential, whose details will be discussed in the next section. In what follows, we will drop, for simplicity, this interaction from the \( \Delta \)-Green’s function.

All the information on the dynamics of the \( \Delta \) interaction in nuclei is included in the \( \Delta \)-nucleus Green’s function. Here, the effective interactions due to the pion absorption and pion rescattering are generated by the coupling of \( D \)-space with \( Q \)- and \( P \)-spaces, respectively. Since the effective interaction due to this coupling of \( Q \)-space is very complicated and is difficult to evaluate microscopically, we replace this by a phenomenological potential, called the spreading potential, \( W_{SP} \) of a \( \Delta \), as follows:

\[
H_{DQ} \frac{1}{E - H_Q - i\epsilon} H_{QD} \rightarrow W_{SP}.
\]

The pion rescattering term \( W^{(\pi)} \) is defined as

\[
W^{(\pi)} = H_{DP} \frac{1}{E - H_P + i\epsilon} H_{PD},
\]

which includes the self-energy of \( \Delta \) and rescattering and Pauli forbidden terms of the previous \( \Delta \)-hole model. \( H_{DP} \) and its Hermitian conjugate \( H_{PD} \) are single particle operators for nucleons and \( \Delta \)s, expressed as

\[
H_{DP} = \sum_{\alpha,\beta,i} \int d\mathbf{k} F_{i,\alpha,\beta}(\mathbf{k}) \Delta_{\alpha}^\dagger N_\beta a(\mathbf{k},i),
\]

with

\[
F_{i,\alpha,\beta}(\mathbf{k}) = \langle \alpha | H_{\pi N \Delta}^\dagger | \beta \rangle,
\]

where \( \Delta_{\alpha}^\dagger, N_\beta \) and \( a(\mathbf{k},i) \) are creation and annihilation operator of \( \Delta \)s, nucleons and pions, respectively. Then, the operator form of \( W^{(\pi)} \) is expressed as

\[
W^{(\pi)} = \sum_{\alpha,\beta,\alpha',\beta'} \Delta_{\alpha}^\dagger N_{\beta'} \int d\mathbf{k} \frac{F_{i,\alpha',\beta'}(\mathbf{k}) F_{i,\alpha,\beta}(\mathbf{k})}{E - H_A - \omega_\pi(k) + i\epsilon} N_{\beta}^\dagger \Delta_{\alpha}.
\]

3.2. Comparison with the conventional \( \Delta \)-hole model

To understand the physical meaning of the term \( W^{(\pi)} \) more clearly, we assume the independent-particle model for the nuclear Hamiltonian as\(^{\text{*)}}\)

\[
H_A = \sum_{\beta} \epsilon_{\beta} N_{\beta}^\dagger N_{\beta}.
\]

Then, we divide the term \( W^{(\pi)} \) as follows:

\(^{\text{*)}}\) Strictly speaking this is an approximation. In real situations, this has also been shown to be a good approximation in evaluating the isobar propagator.
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\[
\begin{align*}
\Delta & \quad \pi^i \\
\Delta_\alpha & \quad N_\beta \\
\Delta_{\alpha'} & \quad \pi^i \\
N_{\beta'} & \quad \Delta_{\alpha'}
\end{align*}
\]

(a) \(\hat{\Sigma}\) diagrams

(b) \(\hat{W}\) diagrams

Fig. 2. Diagrams of the \(\hat{\Sigma}\) and \(\hat{W}\) operators. The double solid line represents \(\Delta\), and the solid line and the dashed line represent a nucleon and a pion, respectively.

\[
W^{(\pi)} = \hat{\Sigma} + \hat{W}
\] (3.23)

with

\[
\hat{\Sigma} = \sum_{\alpha,\alpha',h,h'} \Delta^\dagger_{\alpha'} \Delta_\alpha \int dk \sum_{\beta,\beta'} \frac{F^{i\dagger}_{\alpha',\beta}(k) F^{i}_{\alpha,\beta}(k)}{E - (H_A + \epsilon_\beta) - \omega_\pi(k) + i\epsilon}.
\] (3.24)

and

\[
\hat{W} = - \sum_{\alpha,\alpha',\beta,\beta'} \Delta^\dagger_{\alpha'} N^\dagger_{\beta} \int dk \sum_{i} \frac{F^{i\dagger}_{\alpha',\beta'}(k) F^{i}_{\alpha,\beta}(k)}{E - (H_A + \epsilon_\beta + \epsilon_{\beta'}) - \omega_\pi(k) + i\epsilon} N_{\beta'} \Delta_\alpha.
\] (3.25)

The term \(\hat{\Sigma}\) is regarded as the self-energy of a \(\Delta\), as shown in Fig. 2(a), and the term \(\hat{W}\) is the two-body \(N\Delta\) rescattering operator, as shown in Fig. 2(b).

The decay of a \(\Delta\) into a pion and a nucleon is modified inside the nucleus. This modification is expressed by the rescattering term \(\hat{W}\), which is similar to the renormalization of the magnetic moment of a nucleon inside the nucleus by the pion-exchange current.

3.2.1. The conventional isobar-hole model

Now we shall show that the term \(W^{(\pi)}\) is reduced to the term in the original \(\Delta\)-hole model for the closed shell target. The term \(W^{(\pi)}\) can be rewritten in terms of annihilation operators \(\Delta_\alpha\) and \(N_h\) of isobar and nuclear hole, and their conjugates, as

\[
W^{(\pi)} = \hat{\Sigma} + \hat{W}_\pi + \delta \hat{W}_\pi,
\] (3.26)

with

\[
\hat{W}_\pi = \sum_{\alpha,\alpha',h,h'} \Delta^\dagger_{\alpha'} N^\dagger_{h'} \int dk \sum_{i} \frac{F^{i\dagger}_{\alpha',h'}(k) F^{i}_{\alpha,h}(k)}{E - H_A - \omega_k + i\epsilon} N_h \Delta_\alpha.
\] (3.27)
\[ \delta \hat{W}_\pi = - \sum_{\alpha, \alpha'} \Delta_{\alpha'}^{\dagger} \Delta_{\alpha} \int dk \sum_{h, i} \frac{F_{\alpha', h}^{\dagger}(k) F_{\alpha, h}^{\dagger}(k)}{E - (H_A + \epsilon_k) - \omega_k + i\epsilon}. \]  

These expressions of \( \hat{W}_\pi \) and \( \delta \hat{W}_\pi \) agree with the rescattering and the Pauli correction terms given in the previous section.

3.2.2. Extended \( \Delta \)-hole model

Ohtsuka\(^{22}\) extended the \( \Delta \)-hole model to \( p \)-shell nuclei and studied pion elastic scattering from \( ^{12}\text{C} \). Now we discuss the difference between his model and the present model. He gave the pion-nucleus scattering \( T \)-matrix as

\[ T = H_{PD} \frac{1}{D(E - H_\Delta) - W - \delta W - W_{SP}} H_{DP} \]  

with

\[ H_{DP} = \sum_{i=1}^{A} H_{\pi N \Delta}^{\dagger}(i). \]

Although this looks very similar to the isobar-doorway state model for closed shell nuclei, the quantities \( W \) and \( \delta W \) were defined as

\[ W = \left( \sum_{i=1}^{A} H_{\pi N \Delta}^{\dagger}(i) \right) \frac{P_N}{E - H_P^0 + i\epsilon} \left( \sum_{j=1}^{A} H_{\pi N \Delta}(j) \right), \]

and

\[ \delta W = - \sum_{i=1}^{A} H_{\pi N \Delta}^{\dagger}(i) \frac{P_N}{E - H_P^0 + i\epsilon} H_{\pi N \Delta}(i). \]

The operator \( P_N \) is the projection operator for a nuclear model space. In Ref. 22, the model space is assumed to be configuration spaces with \( (0s)^n(0p)^{4-n}, n \) being 3 or 4. It is noted that the intermediate nuclear states in \( W \) are antisymmetric states, while those in \( \delta W \) are not necessarily antisymmetric states, because of the non-symmetric operator \( H_{\pi N \Delta}(i) \). In our model, the corresponding term \( W^{(\pi)} \) is given as

\[ W^{(\pi)} = \left( \sum_{i=1}^{A} H_{\pi N \Delta}^{\dagger}(i) \right) \frac{P}{E - H_P^0 + i\epsilon} \left( \sum_{j=1}^{A} H_{\pi N \Delta}(j) \right), \]

which can be rewritten as

\[ W^{(\pi)} = W + \left( \sum_{i=1}^{A} H_{\pi N \Delta}^{\dagger}(i) \right) \frac{P - P_N}{E - H_P^0 + i\epsilon} \left( \sum_{j=1}^{A} H_{\pi N \Delta}(j) \right). \]
The second term on the right-hand side can also be expressed as

\[
(\text{r.h.s.}) = \Sigma + \delta W + \sum_{i \neq j} H^\dagger_{\pi N \Delta}(i) \frac{P - P_N}{E - H^0_P + i\epsilon} H_{\pi N \Delta}(j), \tag{3.35}
\]

with

\[
\Sigma = \sum_i H^\dagger_{\pi N \Delta}(i) \frac{P}{E - H^0_P + i\epsilon} H_{\pi N \Delta}(i). \tag{3.36}
\]

Therefore, the formalism of Ref. 22) agrees with ours, if the contribution of the last term is small. Fortunately, this is the case for $^{12}\text{C}$.

§4. Matrix elements of the pion-nucleus $T$-matrix

The $T$-matrix of the pion-nucleus scattering can be divided into two terms as

\[
T = T_{bg} + T_{\Delta}, \tag{4.1}
\]

where $T_{\Delta}$ and $T_{bg}$ describe resonant and non-resonant pion-nucleus scattering. $T_{bg}$ is assumed to be described in terms of the nuclear optical potential. The $\Delta$ resonant part $T_{\Delta}$ is expressed as

\[
T_{\Delta} = (1 + T_{bg} G^{(+)\dagger}_{0}) H_{\pi N \Delta}(E) H^\dagger_{\pi N \Delta}(1 + G^{(+)\dagger}_{0} T_{bg}), \tag{4.2}
\]

with

\[
G^{(+)} = \frac{1}{E - H^0_P + i\epsilon}. \tag{4.3}
\]

Here the $\pi N \Delta$ interaction is dressed by pion distorted waves, and the pion-rescattering term in $G_{\Delta}$ is expressed as

\[
W^{(\pi)} = \sum_{\alpha, \beta, \alpha', \beta'} \Delta^\dagger_{\alpha' N \beta} \int d\hat{k} F^{\dagger}_{\alpha', \beta'}(\hat{k}) G^{(+)}_{0}(1 + T_{bg} G^{(+)\dagger}_{0}) F_{\alpha \beta}(\hat{k}) N^\dagger_{\beta \alpha}. \tag{4.4}
\]

In this section we describe the partial wave expansion of the $T$-matrix, and give the formalism for $T_{\Delta}$ in the angular momentum representation. Then, we describe the optical potential, which is similar to the standard pion-nucleus optical potential in PIPIT.25)

4.1. Partial wave expansion of the $T$-matrix

The partial wave expansion of the $T$-matrix element of pion-nucleus scattering between nuclear states with angular momenta $(J_i, M_i)$ and $(J_f, M_f)$ and pions with momenta $\hat{k}_f$ and $\hat{k}_i$ is written as

\[
\langle J_f, M_f; \hat{k}_f | T | J_i, M_i; \hat{k}_i \rangle \sum_{J, M} (J_f M_f | J M) (J_i M_i | J M) T^J, \tag{4.5}
\]
where

\[ T^J = \langle (J_f l_f) J_f | T | (J_i l_i) J_i \rangle = T^J_{bg} + T^J_{\Delta}. \]  \hfill (4.6)

The scattering amplitude \( F \) is related to the \( T \)-matrix as

\[ F = -4\pi^2 \sqrt{\frac{\omega_\pi(k_f)E_A(k_f)\omega_\pi(k_i)E_A(k_i)}{W_A}} T, \]  \hfill (4.7)

where \( E_A(k) = \sqrt{M_i^2 + k^2} \) and \( W_A = \omega_\pi + E_A \) are the total energy of the nucleus, and of the whole system, respectively. The corresponding scattering amplitude \( F \) is divided into two terms as

\[ F = F_{bg} + F_\Delta, \]  \hfill (4.8)

which is expanded into the partial waves as follows.

Case (i) for nuclear state with \( J_i = J_f = 0 \),

\[ F = \sum_J (2J + 1) F^J P_J(\hat{k}_f \cdot \hat{k}_i). \]  \hfill (4.9)

Case (ii) for nuclear state with \( J_i = J_f = 1/2 \),

\[ F = \delta_{M_f, M_i} f + i(\sigma)_{M_f, M_i} \cdot \hat{n} \ g \]  \hfill (4.10)

with

\[ f = \sum_L [(L + 1) F_L^{L+1/2} + LF_L^{-1/2}] P_L(\hat{k}_f \cdot \hat{k}_i), \]  \hfill (4.11)

\[ g = \sum_L [F_L^{L+1/2} - F_L^{-L-1/2}] P_L(\hat{k}_f \cdot \hat{k}_i), \]  \hfill (4.12)

\[ \hat{n} = \frac{\hat{k}_i \times \hat{k}_f}{|\hat{k}_i \times \hat{k}_f|}. \]  \hfill (4.13)

The differential cross section of elastic pion-nucleus scattering and the total cross section are given, respectively, by

\[ \frac{d\sigma_{el}}{d\Omega} = \frac{1}{2J_f + 1} \sum_{M_i, M_f} |F|^2, \]  \hfill (4.14)

\[ \sigma_{tot} = \frac{4\pi}{k} \text{Im} F(0). \]  \hfill (4.15)

In case (ii), the analyzing power \( A_y \) is another observable. It is defined as

\[ A_y = \frac{2\text{Im}(fg^*)}{|f|^2 + |g|^2}. \]  \hfill (4.16)
4.2. The resonant part of the $T$-matrix element

First, we drop the effect of pion distortion in the matrix element of $T_\Delta$ for simplicity. Inclusion of pion distortion is trivially accomplished by replacing the pion plane wave by a distorted wave. Then, we express the matrix element of $T_\Delta$ as

$$
\langle \gamma_i^A; k_f, \tau_f | T_\Delta | \gamma_i^A; k_i, \tau_i \rangle. \tag{4.17}
$$

We specify the nuclear state with angular momentum $J_{i,f}$ and isospin $T_{i,f}$ by $\gamma_i^A = (J_{i,f}, T_{i,f})$ and the charge state of a pion by $\tau_{i,f}$. We insert a complete set of states $\{D_{f,i}\}$ with a $\Delta$ and $(A - 1)$ nucleons into Eq. (3.29) (or Eq. (4.2)), as

$$
T_\Delta = \langle \gamma_f^A; k_f, \tau_f | T_\Delta | \gamma_i^A; k_i, \tau_i \rangle = \sum_{D_{f,i}} \langle \gamma_f^A; k_f, \tau_f | H_{\pi N \Delta} | D_f \rangle 
\times \langle D_f | G_\Delta(D_i) | D_i | H_{\pi N \Delta}^\dagger | \gamma_i^A; k_i, \tau_i \rangle, \tag{4.18}
$$

with

$$
\langle D_i | H_{\pi N \Delta}^\dagger | \gamma_i^A; k_i, \tau_i \rangle = (J_i M_i \ l_i \ m_i \ | \ J \ M) Y_i^* \left( \gamma_i \right) (T_i T_{i_z} \ 1 \ \tau_i \ | \ T \ T_z)
\times \langle \langle \gamma_{i-1}^A \otimes \gamma_i^A \rangle | H_{\pi N \Delta} \langle \langle \gamma_{i-1}^A \otimes \gamma_i^A \rangle | k_i \rangle \rangle. \tag{4.19}
$$

The initial and final nuclear states are described in the nuclear configuration space with $(0s)^4(0p)^{A-4}$ configurations. Then, in the $D$-space nuclear configuration states are spanned by $(0s)^n(0p)^{A-n-1}$ configuration with $n = 3$ or 4. To perform a partial wave expansion of the matrix element of $T_\Delta$, we have written the matrix element of the $\pi N \Delta$ vertex between the state in $D$-space and the pion-nucleus state in $P$-space with angular momentum and isospin $\gamma_i^A = (J_i, T_i)$. The symbols $\gamma_i^\Delta$ and $\gamma_i^{A-1}$ represent the angular momentum and isospin, $(J_{i-1}^{A-1}, T_i^{A-1})$ and $(J_{i}^{A-1}, T_i^{A-1})$, of a $\Delta$ and of $(A - 1)$ nucleon state, respectively. $\gamma_i^\pi$ represents the angular momentum $l$ and the isospin of unity of a pion.

Then, we obtain the expression of the matrix element in isospin space as

$$
T_\Delta^\prime = \sum_T (T_i T_{i_z} \ 1 \ \tau_i \ | \ T \ T_z \rangle \langle T_f T_{f_z} \ 1 \ \tau_f \ | \ T_T \rangle) \ T_\Delta^\prime, \tag{4.20}
$$

where $\gamma$ stands for quantum numbers $(J,T)$. The matrix element $T_\Delta^\prime$ is expressed as

$$
T_\Delta^\prime = \frac{1}{4\pi} \sum_{\gamma_i^{A-1}, \gamma_i^\Delta, \gamma_i^\pi} \langle \langle \gamma_i^A \otimes \gamma_f^\pi \rangle | k_f \rangle | H_{\pi N \Delta} \langle \langle \gamma_{i-1}^A \otimes \gamma_f^\pi \rangle \rangle
\times \langle \langle \gamma_i^A \otimes \gamma_f^\pi \rangle | G_\Delta(E) \langle \langle \gamma_{i-1}^A \otimes \gamma_f^\pi \rangle \rangle
\times \langle \langle \gamma_i^{A-1} \otimes \gamma_f^\pi \rangle | H_{\pi N \Delta}^\dagger \langle \langle \gamma_i^{A-1} \otimes \gamma_f^\pi \rangle \rangle. \tag{4.21}
$$

Therefore, the remaining task is to evaluate the matrix element of the $\pi N \Delta$ vertex and the matrix element of the $\Delta$-nucleus Green’s function with a given angular momentum and isospin $\gamma = (J,T)$.

(i) $\pi N \Delta$ Vertex
Here we apply standard shell model techniques. The matrix element of the $\pi N\Delta$ vertex is given as

$$
\langle \gamma^A - 1_i \otimes \gamma^\pi | H^\dagger_{\pi N\Delta} | \gamma^\pi \otimes \gamma^A | \rangle = \sum_\beta (-1)^{\beta + \gamma - \gamma^\pi - \gamma^A} W(\beta \gamma^\pi \gamma^A - 1_i \gamma^A) \langle \gamma^\pi || H^\dagger_{\pi N\Delta} || \beta \rangle \langle \gamma^A - 1_i || N_\beta || \gamma^A \rangle.
$$

(4.22)

The spin-isospin reduced matrix element $\langle \gamma^A - 1_i || N_\beta || \gamma^A \rangle$ on the right-hand side of Eq. (4.22) includes all the information of the nuclear structure, and it can be evaluated by using coefficients of fractional parentage.

The term $\langle \gamma^\pi || H^\dagger_{\pi N\Delta} || \beta \rangle$ is the single-particle matrix element of the $\pi N\Delta$ vertex, whose irreducible tensor operator is expressed as

$$
H^\dagger_{\pi N\Delta} = \frac{m_N}{2(2\pi)^3 2\omega \pi \epsilon_N} F(k_i) S^i \cdot k_i T^j_{\pi N} 4\pi \delta^i j_i (k_i r) Y_{l_i m_i} (\hat{r}).
$$

(4.23)

We take into account the center-of-mass correction in the shell model wave function. This can be approximately done by modifying the size parameter $b$ of the single-particle wave function of the harmonic oscillator. The correction is of order $1/A$, which is rather small in pion scattering from $^{12}$C and $^{16}$O. The details of the single-particle matrix element of the $\pi N\Delta$ vertex are given in Appendix A.

(ii) $\Delta$-nucleus Green’s Function

The $\Delta$-nucleus Green’s function is obtained by calculating the doorway-state matrix element of $G^{-1}_\Delta$. Here in the matrix element of $G^{-1}_\Delta$, the doorway state is not necessarily restricted to the nuclear configuration of $|D_{i,f}\rangle$ in principle. Since the doorway-state expansion gives quick convergence, it is a good approximation to restrict ourselves to dominant configurations $|D_{i,f}\rangle$ when we calculate the inverse of the $\Delta$-nucleus Green’s function. Nevertheless, one encounters a problem of a relatively large-scale matrix calculation. To obtain the doorway-state matrix element of the Green’s function between the doorway-states $|D_{i,f}\rangle$, we use the doorway-state expansion series of continued fractions, which was adopted by Hirata et al. $^{26}$

$G^{-1}_\Delta$ is given as

$$
G^{-1}_\Delta = E - H_\Delta - H_{A-1} - \Sigma(E) - W(E) - W_{SP},
$$

(4.24)

when $H_\Delta$ can be written down explicitly as

$$
H_\Delta = T_\Delta + V_\Delta + \Delta m_\Delta,
$$

(4.25)

with $\Delta m_\Delta = m_0^\Delta - m_N$. The kinetic energy of a $\Delta$ is expressed as the relative kinetic energy of the system with a $\Delta$ and a cluster of $(A - 1)$ nucleons as

$$
T_\Delta = \frac{p_\Delta^2}{2\mu_\Delta}, \quad \text{with } \frac{1}{\mu_\Delta} = \frac{1}{m_\Delta} + \frac{1}{(A - 1)m_N}.
$$

(4.26)
Here, the binding potential of a $\Delta$ is chosen as the single-particle potential of a nucleon as

$$V_{\Delta} = V_N.$$  \hfill (4.27)

The self-energy operator $\Sigma$ is given by Eq. (3.24), which should be calculated in the center-of-mass system of a pion and a nucleus, and the operator is still a many-body operator. The self-energy of the elementary process is given by Eq. (2.15), which was expressed in the pion-nucleon center of mass system. In the simplest approximation, $\Sigma$ may be replaced by that of an elementary process. Since a $\Delta$ is not at rest in the pion-nucleus reaction, we will take into account this recoil effect approximately. The energy denominator of $\Sigma$ can be approximated as

$$E - (H_{A-1} + T_{\Delta} + V_{\Delta} + H^{\text{rel}}_{\pi N} - m_N).$$  \hfill (4.28)

Here we separated out the Hamiltonian of a nucleon which is produced by the decay of a $\Delta$. Then we rewrite it in terms of the relative energy of a pion and a nucleon $H^{\text{rel}}_{\pi N}$, which is the kinetic energy in the elementary process, and also the residual nucleus as $T_{\Delta}$.

The energy of the system is given as a function of the initial momentum of a pion $k_0$:

$$E = \sqrt{m_{\pi}^2 + k_0^2} + \sqrt{(Am_N)^2 + k_0^2} - Am_N.$$  \hfill (4.29)

This can be written as a sum of the pion-nucleon center-of-mass energy $E_{\pi N}$ and the remainder $\epsilon_0^\Delta$ as

$$E = E_{\pi N} - m_N + \epsilon_0^\Delta.$$  \hfill (4.30)

Here we choose the pion-nucleon scattering energy $E_{\pi N}$ by neglecting the Fermi-motion of a nucleon as

$$E_{\pi N} = \sqrt{\left(\sqrt{m_{\pi}^2 + k_0^2} + \sqrt{m_N^2 + (k_0/A)^2}\right)^2 - \left(k_0 \left(1 - \frac{1}{A}\right)\right)^2}.$$  \hfill (4.31)

Then, Eq. (4.28) reads

$$(E_{\pi N} - H_{A-1} - T_{\Delta} - V_{\Delta} + \epsilon_0^\Delta) - H^{\text{rel}}_{\pi N},$$  \hfill (4.32)

which corresponds to the energy denominator, $E_{\pi N} - H^{\text{rel}}_{\pi N}$, of the self-energy $\Sigma_{\pi N}(E_{\pi N})$ in the elementary process. From this result, we find that the self-energy operator is simply expressed by

$$\Sigma(E) \sim \Sigma_{\pi N}(E_{\pi N} - H_{A-1} - T_{\Delta} - V_{\Delta} + \epsilon_0^\Delta).$$  \hfill (4.33)

The recoil energy correction of $\Sigma$ is approximately included as

$$\Sigma(E) \sim \Sigma_{\pi N}(E_{\pi N}) - \frac{\partial \Sigma_{\pi N}(E_{\pi N})}{\partial E_{\pi N}} (H_{A-1} + T_{\Delta} + V_{\Delta} - \epsilon_0^\Delta).$$  \hfill (4.34)
Finally, the $\Delta$-nucleus Green’s function is given as

$$G_{\Delta}^{-1} = E_{\pi N} - m_{\Delta}^0 - \Sigma_{\pi N}(E_{\pi N})$$

$$- (T_\Delta + V_\Delta + H_{A^{-1}} - \epsilon^0_\Delta) \left( 1 - \frac{\partial \Sigma_{\pi N}(E_{\pi N})}{\partial E_{\pi N}} \right) - \hat{W} - W_{SP}. \quad (4.35)$$

Here, all the operators except $\hat{W}$ are written in terms of single-particle operators of a $\Delta$, which can be evaluated trivially. The following is an explicit form of the matrix element of two-body operator $\hat{W}$:

$$\hat{W} = \sum_{\alpha\beta\alpha'\beta'} \sum_{L} \int k^2 dk \langle \alpha' | H_{\pi N\Delta}^{\pi\pi} | \beta' \rangle G_{\pi A}^{\beta\beta'}(k) \langle \beta | H_{\pi N\Delta}^{\pi\pi} | \alpha \rangle$$

$$\times (-)^{\alpha'-\alpha+\beta'-\beta} W(\alpha' \alpha \beta \beta' ; l L) \sqrt{[L]} \times \left[ \bar{\Delta}_\alpha \otimes \tilde{\Delta}_\alpha \right]_{(L)} \otimes \left[ N^\dagger_{\beta} \otimes \tilde{N}^\dagger_{\beta'} \right]_{(0)} \right), \quad (4.36)$$

with

$$G_{\pi A}^{\beta\beta'} = \frac{1}{E - (H_A + \epsilon_{\beta} + \epsilon_{\beta'}) - \omega_{\pi}(k) + i\epsilon}. \quad (4.37)$$

The matrix element of $\hat{W}$ between $|D_{i,f}\rangle$ is obtained as

$$\left\langle \left[ \gamma_{i}^{A-1} \otimes \gamma_{f}^{A} \right]_{(\gamma)} \right| \hat{W} \left| \left[ \gamma_{i}^{A-1} \otimes \gamma_{f}^{A} \right]_{(\gamma)} \right\rangle$$

$$= - \sum_{\gamma_f^{A-1}, \gamma_i^{A-1}} \sum_{\gamma_f^{A}, \gamma_i^{A}} \sum_{L} \sum_{l} \int k^2 dk \langle \gamma_f^{A} | H_{\pi N\Delta}^{\pi\pi} | \gamma_i^{A} \rangle G_{\pi A}^{\beta\beta'}(k) \langle \beta | H_{\pi N\Delta}^{\pi\pi} | \alpha \rangle$$

$$\times W(\gamma_f^{A} \beta' \gamma_i^{A} \beta ; l L) W(\gamma_f^{A} \gamma_i^{A-1} \gamma_i^{A-1} ; \gamma L)$$

$$\times \langle \gamma_f^{A-1} | N^\dagger_{\beta} \otimes \tilde{N}^\dagger_{\beta'} \rangle_{(L)} \langle \gamma_i^{A-1} \rangle. \quad (4.38)$$

Here, the spin-isospin reduced matrix element in the last line of Eq. (4.36) also includes all the information of the nuclear structure.

4.3. Non-resonant part of the $T$-matrix element and pion-nucleus optical potential

We describe the non-resonant part of the $T$-matrix element using the optical potential. We adopt the first order optical potential following the formalism of Kerman, McManus and Thaler.\(^{23}\) The pion-nucleus optical potential can be written in terms of the pion-nucleon $T$-matrix $t_{\pi N}$ and the nuclear density $\rho$ as

$$U_{bg} = A t_{\pi N} \rho, \quad (4.39)$$

where the $P_{33}$ partial wave is subtracted from the $T$-matrix $t_{\pi N}$, and $A$ is the nuclear mass number. The non-resonant pion-nucleus $T$-matrix is given as

$$T_{bg} = \frac{A}{A - 1} T_{bg}^\ast, \quad (4.40)$$
with
\[ T_{bg} = U_{bg} + U_{bg}G_0^+(E)T_{bg}'. \tag{4.41} \]

The optical potential can be expanded into a multipole series as
\[
\langle \gamma_f^A ; k_f, \tau_f | U_{bg} | \gamma_i^A ; k_i, \tau_i \rangle = \sum_{J,M,T} \sum_{l_f,m_f,l_i,m_i} U_{bg}^\gamma \times Y_{l_f,m_f}(\hat{k_f}) (J_f M_f l_f m_f | J M) (T_f T_z f 1 \tau_f | T T_z), \tag{4.42} \]

where \( k_i \) and \( k_f \) are the off-shell momenta of a pion for the initial and final state in the pion-nucleus center-of-mass system, respectively. The optical potential with angular momentum \( \gamma = (J,T) \) is defined as
\[
U_{\gamma bg} = \frac{1}{4\pi} (|\gamma_f^A \otimes \gamma_i^A|_{\gamma}) ; k_f | U_{bg} | (|\gamma_i^A \otimes \gamma_f^A|_\gamma) ; k_i. \tag{4.43} \]

To calculate the optical potential by using the pion-nucleon \( T \)-matrix, we must take into account two kinds of effects. One is the off-shell effect of the \( T \)-matrix, since the pion in a nucleus is not necessarily on the energy shell. The other is the effect due to the transformation of the \( T \)-matrix from the pion-nucleon center-of-mass system to the pion-nucleus center-of-mass system. Here it is noted that the information regarding the pion-nucleon scattering obtained from experiments is only the pion-nucleon scattering phase shifts. Nevertheless, we manage to include the off-shell effect of the pion-nucleon scattering with a reasonable approximation.

(i) Off-shell pion-nucleon \( T \)-matrix

We write the off-shell pion-nucleon \( T \)-matrix required in the optical potential as
\[
\langle k_f, p_f | \ell_{\pi N} | k_i, p_i \rangle, \tag{4.44} \]

where \( p_i \) and \( p_f \) are the momenta of a nucleon in the initial and final states, respectively. The pion-nucleon phase shift is related to the on-shell \( T \)-matrix in the pion-nucleon center-of-mass system as
\[
\langle \kappa_0^I | \ell_{\pi N}(\epsilon^I) | \kappa_0^I \rangle = \sum_\alpha A^\alpha(\kappa_0^I, \kappa_0^I) t_0^\alpha, \tag{4.45} \]

where \( \alpha \) represents the total and orbital angular momenta \( j \) and \( l \) and isospin \( i \) of the pion-nucleon system, and \( A^\alpha \) is the relevant projection operator. \( \kappa_0^I \) is the relative momentum of a pion and a nucleon on the energy shell. The on-shell \( t_0^\alpha \) is given by the phase shift as
\[
t_0^\alpha = -\frac{1}{4\pi^2} \frac{\omega_\pi(\kappa^0) + \epsilon_N(\kappa^0) \eta^\alpha e^{2i\delta^\alpha} - 1}{\omega_\pi(\kappa^0)\epsilon_N(\kappa^0) - 2i\kappa^0}. \tag{4.46} \]

In order to relate the pion-nucleon \( T \)-matrix (4.44) with the on-shell \( T \)-matrix (4.45), we first transform the \( T \)-matrix from the pion-nucleus center-of-mass system to the
pion-nucleon center-of-mass system, assuming that the pion and nucleon are on-shell. By neglecting the Fermi motion of nucleons, we have \( p_i = -k_i/A \) and \( p_f = p_i + k_i - k_f \). Then the \( T \)-matrix in the pion-nucleon center-of-mass system can be expressed as

\[
\langle k_f, p_f | t_{\pi N} | k_i, p_i \rangle \sim \sqrt{\frac{\omega_\pi(k_f)\omega_\pi(k_i)\epsilon_N(k_f)\epsilon_N(k_i)}{\omega_\pi(k_f)\omega_\pi(k_i)\epsilon_N(p_f)\epsilon_N(p_i)}} \langle \kappa_f | t_{\pi N}(\epsilon^0) | \kappa_i \rangle, \tag{4.47}
\]

where the relative momentum \( \kappa \) is obtained by the Lorentz transformation as

\[
\kappa = \frac{1}{\sqrt{1-V^2}}[k - V\omega_\pi(k)]\hat{k} \tag{4.48}
\]

with the velocity of the pion-nucleon system \( V = (k + p)/(\omega_\pi(k) + \epsilon_N(p)) \). The total energy of the pion-nucleon system in the pion-nucleon center-of-mass system is determined by the incident momentum of a pion and a nucleon in the pion-nucleon center-of-mass system as

\[
\epsilon^0 = \sqrt{[\omega_\pi(k_i^0) + \epsilon_N(k_i^0/A)]^2 - (k_i^0 - k_i^0/A)^2}, \tag{4.49}
\]

in which the Fermi motion is neglected. The relative momentum is still off-shell, and we need assumptions concerning the off-shell behavior of the \( T \)-matrix. We take into account the threshold behavior of \( T \)-matrix as the \( l \)-th power of the pion momentum for each partial wave \( l \), and we assume a Gaussian form of the momentum dependence at the off-shell state. Then, the \( T \)-matrix can be expressed in terms of the pion-nucleon scattering phase shifts as

\[
\langle k_f, p_f | t_{\pi N} | k_i, p_i \rangle = \frac{\omega_\pi(k_f)\omega_\pi(k_i)\epsilon_N(k_f)\epsilon_N(k_i)}{\omega_\pi(k_f)\omega_\pi(k_i)\epsilon_N(p_f)\epsilon_N(p_i)} \times \sum_\alpha A^\alpha(\kappa_f, \kappa_i) g^l(\kappa_i) g^l(\kappa_i)|t^0_0| \tag{4.50}
\]

with

\[
g^l(\kappa) = \kappa^l e^{-\kappa^2/\kappa_c^2}, \tag{4.51}
\]

where \( \kappa_c \) is a cutoff parameter.

(ii) Optical potential and nuclear form factor

The pion-nucleon \( T \)-matrix in Eq. (4.50) is a complicated function of \( \kappa_i \) and \( \kappa_f \). For convenience in the nuclear structure calculation, we rewrite the \( T \)-matrix as

\[
\langle k_f, p_f | t_{\pi N} | k_i, p_i \rangle = t^{00}(k_f, k_i) + t^{10}(k_f, k_i)i\sigma \cdot \hat{n}
+ [t^{01}(k_f, k_i) + t^{11}(k_f, k_i)i\sigma \cdot \hat{n}]\tau \cdot I_\pi, \tag{4.52}
\]

where \( \sigma \) is the spin operator of a nucleon, and \( I_\pi \) and \( \tau \) are isospin operators of a pion and a nucleon, respectively. They are local single-particle operators of nucleons.
We derive the optical potential for elastic scattering from nuclei with $J_i = 0$, $T_i = 0$, and also with $J_i = \frac{1}{2}$, $T_i = \frac{1}{2}$. The optical potential can be expressed in terms of nuclear form factors $F^{ji}$ defined as

$$F^{ji}(k_f, k_i) = \sqrt{\frac{\pi}{(2j + 1)(2i + 1)}} \times (\gamma_i^A \parallel [Y_0(\hat{r}) \otimes \sigma^{(j)}]_{(j)j0}(qr) - \frac{1}{\sqrt{2}} [Y_2(\hat{r}) \otimes \sigma^{(j)}]_{(j)j2}(qr)) \tau^{(i)} || \gamma_i^A),$$

with $q = k_f - k_i$. This corresponds to the $T$-matrix $t^{ji}$.

Case (a) $J_i = T_i = 0$. Here the optical potential is given by the angular momentum projection of a product of a nuclear form factor and pion-nucleon $T$-matrix defined as

$$U^{JT}(k_f, k_i) = \frac{1}{2} \int_{-1}^{1} dx \ F^{00}(k_f, k_i) t^{00}(k_f, k_i) P_J(x), \quad (4.53)$$

$$x = \hat{k}_f \cdot \hat{k}_i, \quad l_i = l_f = J \quad \text{and} \quad T = 1.$$  

Case (b) $J_i = T_i = \frac{1}{2}$. The optical potential is given as

$$U^{J=L+1/2, T}(k_f, k_i) = \frac{1}{2} \int_{-1}^{1} dx \ [H^{1T} P_L(x) + H^{2T} P_{L+1}(x)], \quad (4.54)$$

and

$$U^{J=L-1/2, T}(k_f, k_i) = \frac{1}{2} \int_{-1}^{1} dx \ [H^{1T} P_L(x) + H^{2T} P_{L-1}(x)], \quad (4.55)$$

$$l_i = l_f = L, \quad \text{and} \quad T = \frac{1}{2} \text{ or } \frac{3}{2},$$

with

$$H^{1/2} = \left\{ F^{00}(k_f, k_i) t^{00}(k_f, k_i) - x F^{10}(k_f, k_i) t^{10}(k_f, k_i) - F^{01}(k_f, k_i) t^{01}(k_f, k_i) - x F^{11}(k_f, k_i) t^{11}(k_f, k_i) \right\}, \quad (4.56)$$

$$H^{3/2} = \left\{ F^{00}(k_f, k_i) t^{00}(k_f, k_i) - x F^{10}(k_f, k_i) t^{10}(k_f, k_i) + 2 F^{01}(k_f, k_i) t^{01}(k_f, k_i) - x F^{11}(k_f, k_i) t^{11}(k_f, k_i) \right\}, \quad (4.57)$$

$$H^{2/2} = F^{10}(k_f, k_i) t^{10}(k_f, k_i) - F^{11}(k_f, k_i) t^{11}(k_f, k_i), \quad (4.58)$$

$$H^{2/2} = F^{10}(k_f, k_i) t^{10}(k_f, k_i) + 2 F^{11}(k_f, k_i) t^{11}(k_f, k_i). \quad (4.59)$$

The scattering amplitude is dominated by the spin-isospin nonflip term, $F^{00, 00}$, and the analyzing power is sensitive to the difference between amplitudes with $J = L \pm \frac{1}{2}$, which is determined by the spin-flip term $F^{11, 11}$. Finally the charge exchange amplitude is determined by the isospin-flip term $F^{j1, j1}$. 

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§5. Pion scattering from $p$-shell nuclei

We apply the isobar-doorway state model to the elastic and charge-exchange scattering of pions from $^{12}\text{C}$, $^{13}\text{C}$ and $^{15}\text{N}$. The spreading potential of a $\Delta$ is chosen to reproduce the differential cross section of the elastic pion scattering, the total cross section of pion scattering from $^{12}\text{C}$. Then we predict the elastic scattering of pions from $^{13}\text{C}$ and the analyzing power. Here we investigate the role of the spreading potential of a $\Delta$ and the effects of configuration mixing of nuclear wave functions on the spin-flip amplitude in detail. We then study a long-standing problem involving the charge-exchange reaction $^{13}\text{C}_{gr}(\pi^{+}, \pi^{0})^{13}\text{N}_{gr}$. We also study pion reactions with $^{15}\text{N}$ in a similar way to those with $^{13}\text{C}$. We propose an isospin-dependent spreading potential of a $\Delta$ so as to give a possible solution of the problem in the charge-exchange reaction.

The pion-nucleon interaction except for the $P_{33}$ channel is treated in terms of the pion-nucleus optical potential discussed in §4.

5.1. Nuclear model

5.1.1. Configuration mixing of the nuclear wave functions

In the conventional $\Delta$-hole model, $^{12}\text{C}$ and $^{13}\text{C}$ are treated as a $p_{3/2}$ closed shell and one $p_{1/2}$ neutron particle state, respectively. However, it is well known that the nuclear structures of these nuclei are far from those described by this simple jj-coupling scheme, and the configuration mixing within the $p$-shell configuration space plays an important role. As an effective interaction of nucleons in the $p$-shell, we adopt the POT model of Cohen-Kurath. The effective interaction was determined so as to reproduce the low-lying energy levels of $p$-shell nuclei systematically, and as a result, the magnetic moments and Gamow-Teller strengths were described satisfactorily. The ground states of $^{12}\text{C}$ and $^{13}\text{C}$ are expressed as

\[
|^{12}\text{C}_{gr}; 0^{+}0 \rangle = -0.613|p_{3/2}^{8}\rangle \\
+0.625|p_{3/2}^{6}\rangle_{01}(p_{1/2}^{2})_{01} - 0.261|p_{3/2}^{6}\rangle_{10}(p_{1/2}^{2})_{10} \\
-0.255|p_{3/2}^{5}\rangle_{1/2}(p_{1/2}^{2})_{1/2} - 0.319|p_{3/2}^{4}\rangle_{00}(p_{1/2}^{4})_{00}
\]  

(5.1)

and

\[
|^{13}\text{C}_{gr}; 1^{-}1/2 \rangle = +0.837|p_{3/2}^{8}\rangle_{p_{1/2}} \\
-0.012|p_{3/2}^{7}\rangle_{3/2}(p_{1/2}^{1/2})_{10} - 0.501|p_{3/2}^{5}\rangle_{01}(p_{1/2}^{3})_{01} \\
+0.187|p_{3/2}^{6}\rangle_{01}(p_{1/2}^{3})_{1/2} + 0.115|p_{3/2}^{5}\rangle_{1/2}(p_{1/2}^{4})_{00},
\]  

(5.2)

where the state with $n_{j}$ nucleons in the $p_{j}$ orbit with angular momentum $J_{j}$ and isospin $T_{j}$ is denoted by $|p_{j}^{n}\rangle_{J_{j}, T_{j}}$. The component $|p_{3/2}^{8}\rangle$ is just the state given by the simple jj-coupling model. It does not dominate other components. This shows that the configuration mixing is apparently large. In fact, it is known that the configuration mixing has a large effect on the electromagnetic form factor of elastic and inelastic electron scattering, beta-decay and pion photoproduction reactions.
The spin-nonflip amplitude in the elastic scattering is mainly determined by the monopole form factor,

\[ F_C^t(q) = \langle F \| \sum_{i=1}^{A} Y_0(\hat{r}_i) j_0(qr_i) \| I \rangle, \tag{5.3} \]

where |I⟩ and |F⟩ are the nuclear wave functions in the initial and final states, respectively. Since \( F_C^t \) is given by the spin-independent operator, it is determined essentially by the density of nucleons in s- and p-orbits. Therefore, the effect of the configuration mixing is expected to be small for \( F_C^t \).

In contrast to this, we expect that the spin-flip amplitude is affected strongly by configuration mixing, and hence that the analyzing power should be sensitive to configuration mixing. The spin-flip amplitude is classified into two types of form factors, longitudinal and transverse, defined by

\[ F_{Tt}^\sigma = \langle F \| \sum_{i=1}^{A} \left[ \sqrt{\frac{1}{3}} Y_0(\hat{r}_i) \otimes \sigma_i \right]_{(1)} j_0(qr_i) \sigma_t^i \| I \rangle, \]

\[ F_{Lt}^\sigma = \langle F \| \sum_{i=1}^{A} \left[ \sqrt{\frac{2}{3}} Y_2(\hat{r}_i) \otimes \sigma_i \right]_{(1)} j_0(qr_i) \tau_t^i \| I \rangle. \]

Of these, only the transverse form factor \( F_{Tt}^\sigma \) is dominant in pion-nucleus elastic scattering. In the present work we consider the POT model of Cohen-Kurath and use the harmonic oscillator model as a single-particle wave function of a nucleon. The size parameter of the single-particle wave function is determined so as to reproduce the charge form factor as

\[ b = 1.635 \text{ fm for } ^{12}\text{C and } ^{13}\text{C}. \] 

5.1.2. Configuration of isobar-doorway state

The doorway state of the first step is written as

\[ |D_0\rangle = H_{\pi N \Delta} |\pi I\rangle. \]

In the conventional isobar-doorway state model, this is simply a \( \Delta \)-hole state, \( |\Delta h\rangle \), for \(^{12}\text{C}, \) and \( |\Delta \rangle + |\Delta ph\rangle + \cdots \) for \(^{13}\text{C} \). In our model, the nuclear state is a many-particle and many-hole state due to the configuration mixing. The doorway state with a \( \Delta \) and \((A - 1)\) nucleons is expressed as

\[ |D_0\rangle = |\Delta, (A - 1)N\rangle, \]

where the nuclear state \((A - 1)\) is a linear combination of the configurations \( |(A - 1)N\rangle = |s_{1/2}^n_p^3/n_p^1\rangle \) with \( n_s = 3 \) or 4 and \( n_s + n_p^3 + n_p^1 = A - 1 \). In general, the
doorway states in the $n$-th step can be constructed from all possible configurations of $(A-1)$ nucleons states. The number of possible configurations becomes too large to be handled numerically. In this work we limit the configuration space of $(A-1)$ nucleons to the states which appear in $|D_0\rangle$.

The single-particle wave functions of $\Delta$ are taken as harmonic oscillator wave functions with the same size parameter as the nucleons. The angular momentum of the $\Delta$ state is taken to be large enough so as to give convergent results. Typically, we include the $\Delta$ states up to excitation energy of $14\hbar\omega$.

5.2. Elastic scattering of pions from $^{12}$C and spreading potential of $\Delta$

First, we study the effect of configuration mixing on the elastic scattering of pions from $^{12}$C. We use the standard form of the spreading potential.\(^4\)

$$W_{SP} = V_C \rho(r)/\rho(0) + V_{LS} 2L_\Delta \cdot \Sigma_\Delta \mu r^2 e^{-\mu r^2} \quad (5.9)$$

with

$$V_C = 12.3 - 43i, \quad \nu = 0.348, \quad (5.10)$$
$$V_{LS} = -5.4 - 4.9i, \quad \mu = 0.348. \quad (5.11)$$

We calculated the differential cross section of $\pi^-$ elastic scattering from $^{12}$C for incident energies of 150, 180, 200 and 230 MeV by adopting the Cohen-Kurath model and the simple jj-coupling model, and compared the results with experimental data.\(^27\) The differences between the Cohen-Kurath and jj-coupling models are seen to be small. This result was already expected for spin-nonflip amplitude. Slight differences appear only in the region of large scattering angle.

The calculated results agree well with the data over all energies. Nevertheless, at the first dip and at the second peak of the differential cross sections, the calculated values are systematically larger than the experimental values at higher incident pion energy. Then, we modified the spreading potential so as to obtain a better fit to the experimental data of the differential cross section. In Cases 2 and 3, we take the smaller value of the imaginary part of the central potential, and also the smaller magnitude of the spin-orbit potential, than those of the standard potential, as shown in Table I.

In Case 4 we change the sign of the spin-orbit potential to investigate its effect. The energy-dependences of the differential cross section are shown in Fig. 3. If Cases 2 and 3 are used, the forward peak, the first dip and the second peak of the cross section are improved. Figures 4(a) and (b) show the dependences of the total and integrated cross sections on the spreading potential. The weaker spin-orbit potential

<table>
<thead>
<tr>
<th>Case</th>
<th>$V_C$</th>
<th>$\nu$</th>
<th>$V_{LS}$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>$12.3 - 43i$</td>
<td>0.348</td>
<td>$-5.4 - 4.9i$</td>
<td>0.348</td>
</tr>
<tr>
<td>Case 2</td>
<td>$0.8 - 38i$</td>
<td>0.308</td>
<td>$-0.7 - 3.1i$</td>
<td>0.388</td>
</tr>
<tr>
<td>Case 3</td>
<td>$-1.7 - 31i$</td>
<td>0.348</td>
<td>$-0.7 - 3.1i$</td>
<td>0.488</td>
</tr>
<tr>
<td>Case 4</td>
<td>$4.0 - 46i$</td>
<td>0.308</td>
<td>$+3.3 + 4.1i$</td>
<td>0.428</td>
</tr>
</tbody>
</table>
improves greatly the energy dependence of the total and elastic cross sections and also the total cross section, while it gives a rather small absorption cross section. The role of the spin-orbit potential in the energy dependence of the cross section, which is pointed out in Ref. 4), is clearly seen by comparing the results of Cases 1, 2 and 4. Finally, the true absorption cross section is compared with data in Fig. 5. We conclude that Case 3 gives a satisfactory description of the elastic cross section and also the gross features of the pion-\(^{12}\)C reaction, i.e., the total and absorption cross section above the pion kinetic energy of 100 MeV. Below 100 MeV, the P\(_{33}\) channel
Fig. 4. Effects of the spreading potential on (a) the total cross section and (b) the integrated elastic cross section of $\pi^{-}{^{12}}C$ scattering. The data are taken from Ref. 27). (See the caption of Fig. 3.)

is no longer dominant in the pion-nucleus scattering. We must include effects of the nuclear medium on the pion-nucleon scattering with the $s$-and $p$-waves except for the $P_{33}$.

5.3. Elastic scattering of pions from $^{13}C$ and analyzing power

We now study the differential cross section and the analyzing power of the pion elastic scattering from $^{13}C$. The relevant scattering amplitudes are described by the amplitudes with isospin eigenvalues:

$$F_{\pi^{+}{^{13}}C} = \frac{1}{3} F_{3/2} + \frac{2}{3} F_{1/2}, \quad (5.12)$$

and

$$F_{\pi^{-}{^{13}}C} = F_{3/2}. \quad (5.13)$$

We have already determined the spreading potential of a $\Delta$ with $^{12}C$. We assume that the spreading potential with $^{13}C$ is not much different from that with $^{12}C$.
Fig. 6. The scattering amplitudes \( f(\theta) \) and \( g(\theta) \) for \( \pi^{-}\text{C} \) elastic scattering at \( T_\pi = 223 \text{ MeV} \) with isospin \( T = \frac{1}{2}, \frac{3}{2} \) in units of fm. The solid and dashed lines were obtained by using the Cohen-Kurath and the jj-coupling models, respectively.

Fig. 7. The differential cross section and analyzing power for \( \pi^{+}\text{C} \) elastic scattering at \( T_\pi = 223 \text{ MeV} \) calculated with the spreading potential in Case 3. (See the caption of Fig. 3.) The data are taken from Ref. 17.

We examine the effect of configuration mixing on pion scattering from \( ^{13}\text{C} \). The total isospin of the \( \pi^{-}\text{C} \) system is either \( \frac{1}{2}, \frac{3}{2} \). The spin-nonflip and spin-flip amplitudes, \( f \) and \( g \), with isospins of \( \frac{1}{2} \) and \( \frac{3}{2} \) are shown in Fig. 6 for the Cohen-Kurath and the jj-coupling models. We expected that the spin-flip amplitude might be more sensitive to the configuration mixing than the spin-nonflip amplitude. In the case of elastic scattering from \( ^{13}\text{C} \), the main part of the spin-flip amplitude is of the transverse type which, however, is modified little by the configuration mixing. The longitudinal part, which is appreciably modified, gives a small contribution to the cross section. The differential cross section and analyzing power of \( \pi^{+}\text{C} \) at \( T_\pi = 223 \text{ MeV} \) calculated for Case 3 are shown in Fig. 7. The effect of configuration mixing, indeed, does seem very small for both the cross section and the analyzing...
power. Almost the same results as above are obtained for $\pi^{-}\text{^{13}C}$ elastic scattering.

Now we examine how the differential cross section and the analyzing power depend on the spreading potential. As seen in Fig. 8, we obtained good results for all the cases of the spreading potential we considered. The analyzing power depends rather strongly on the choice of the spreading potential. The calculated differential cross sections at $T_\pi = 162$ and 180 MeV are compared with the experimental data in Fig. 9. Again, Case 3 gives good agreement with the data.

Fig. 8. $\pi^+\text{^{13}C}$ elastic scattering and analyzing power. The solid, dashed and dotted curves correspond to the spreading potential in Cases 1, 2 and 3, respectively.

Fig. 9. Effects of nuclear models on the differential cross section for $\pi^+\text{^{13}C}$ at $T_\pi = 162$ and 180 MeV. The spreading potential in Case 3 is used. The solid and dashed curves are the calculated results using the Cohen-Kurath and jj-coupling models, respectively. The data are taken from Refs. 28) and 29).
Finally, the analyzing power of $\pi^{-}{^{13}\text{C}}$ scattering from 114 to 223 MeV are compared with experimental data as a function of momentum transfer in Fig. 10. The calculated results describe the energy dependence and the structure of analyzing power satisfactorily. Here it is noted that results of the various calculations based on the DWIA\cite{31,32,33} exhibit very strong angular dependence, in particular, the deep second dip in the differential cross section, which is in contrast to the smooth angular dependence in our present results.
5.4. Pion charge-exchange reaction on $^{13}$C

The charge-exchange reaction with $^{13}$C, i.e., $^{13}$C$(\pi^+, \pi^0)^{13}$N, is described by the difference between the amplitudes of the isospin $\frac{3}{2}$ and $\frac{1}{2}$ channels as

$$F_{\text{CEX}} = \frac{\sqrt{2}}{3}(F_{3/2} - F_{1/2}). \quad (5.14)$$

The present isobar-doorway state model gives a good description of the elastic scattering of pions from $^{12}$C and $^{13}$C reactions with the $\Delta$-spreading potential of Case 3. We applied the same model to the charge-exchange reaction. The differential cross section and the analyzing power of the charge exchange reactions are shown in Fig. 11 for three cases of the spreading potential. Although the results

![Graphs showing differential cross section and analyzing power](image-url)
depend slightly on the choice of the spreading potential, the calculated cross section at forward scattering is almost by one order of magnitude smaller than the experimental data, and the angular dependence of the analyzing power is completely in disagreement with the experimental data. It is well-known from previous works that the standard \( \Delta \)-hole model predicts values of the charge-exchange cross section that are too small. Although we have improved the \( \Delta \) spreading potential and nuclear model, this problem remains unsolved.

Hirata introduced the Lane term into the spreading potential, a term which depends on the total isospin of the pion-nucleus system.\(^{13}\) It was found that this term is strongly energy dependent in order to describe the charge-exchange reaction. This is because Hirata’s spreading potential did not include the important spin-orbit term. In fact, he showed later that this difficulty can be removed by introducing the spin-orbit interaction into the spreading potential.\(^{15}\) Here it is not clear how this isospin-dependence affects the elastic scattering of pion from \( ^{12}\text{C} \) and \( ^{13}\text{C} \). Since the pion absorption effect is known to depend on the isospin of the system with a \( \Delta \) and a nucleon, it is reasonable to assume that the spreading potential depends on isospins of a \( \Delta \) and a residual nucleus, instead of just the total isospin of the system. In this work, we introduce into the \( \Delta \) spreading potential (5.9), the Lane term, which depends on the isospin of a \( \Delta \) and a residual nucleus, \( T_{\Delta} \) and \( T_{A-1} \), as

\[
V_{C+\tau} = V_C(1 + \alpha_{\tau} T_{\Delta} \cdot T_{A-1}) \rho(r)/\rho(0). \tag{5.15}
\]

An advantage of this form is that we can study the role of the isospin-dependent spreading potential on \( ^{13}\text{C} \) and \( ^{12}\text{C} \) in a unified way.

In order to determine the strength of the isospin-dependent spreading potential, we take the following approach. In the elastic scattering of pions from \( ^{12}\text{C} \), the isospin of the initial nucleus is zero, and the isospin of the system with \( (A-1) \) nucleons in the doorway state is uniquely given as \( 1/2 \). The spreading potential is given as

\[
V_{C+\tau}^{T=1} = V_C \left( 1 - \frac{5}{4} \alpha_{\tau} \right) \rho(r)/\rho(0). \tag{5.16}
\]

Therefore, the spreading potential is related to the spreading potential of \( ^{12}\text{C} \) already determined as

\[
V_{C+\tau} = \frac{1 + \alpha_{\tau} T_{\Delta} \cdot T_{A-1}}{1 - \frac{5}{4} \alpha_{\tau}} V_{C+\tau}^{T=1}. \tag{5.17}
\]

We apply the above form of the isospin-dependent spreading potential and find the parameter \( \alpha_{\tau} \) so as to explain both the elastic scattering and charge-exchange reactions of pions with \( ^{13}\text{C} \).

We found that the parameter \( \alpha_{\tau} \), whose real part is negative, gives a good description of the differential cross section and also the qualitative behavior of the analyzing power. The best value we found is \( \alpha_{\tau} = -0.2 \). The calculated results are compared with the experimental data in Fig. 12. The magnitudes of the differential cross section are improved. We could also explain the qualitative feature of the analyzing power with this isospin-dependent spreading potential, while the elastic scattering cross section is only slightly modified and is still in good agreement with
Fig. 12. Effects of the isospin dependence of the spreading potential on the differential cross section and analyzing power of $\pi^{\pm}$-$^{13}$Ca at $T_\pi = 163$ MeV. The solid lines represent the results without the isospin-dependent term in the spreading potential ($\alpha_\tau = 0.0$, Case 3). The dashed lines represent the results with the isospin-dependent term in the spreading potential ($\alpha_\tau = -0.2$).

the data. The differential cross section and analyzing power of the elastic scattering at $T_\pi = 223$ MeV are also affected little, as shown in Fig. 13. Since the charge-exchange reaction is given by the difference between the amplitudes with isospin $\frac{3}{2}$ and $\frac{1}{2}$, which almost cancel with each other, the rather weak isospin-dependent spreading potential affects the cross sections of charge-exchange scattering appreciably, but not those of the elastic scattering.

In Fig. 14(a), the energy dependence of the integrated cross sections of the charge-exchange reaction is compared with experimental data. The strong energy dependence of the isospin-independent spreading potential is removed by introducing the isospin-dependent spreading potential. It is noted that our spreading potential
is energy independent. The differential cross sections for charge-exchange reactions are shown in Fig. 14(b). This also supports our spreading potential.

In summary, we have shown that the longstanding puzzle of the charge-exchange reaction with $^{13}$C is partly solved by introducing an isospin-dependent spreading potential of a $\Delta$. This helps us to understand the differential cross section and analyzing power at $T_\pi = 163$ MeV and also the qualitative features of the energy dependence of the integrated cross section and the differential cross section at forward scattering of charge-exchange scattering without introducing any strong energy-dependent potential, and also without affecting the pion-$^{12}$C elastic scattering.
5.5. Pion charge-exchange reaction with $^{15}\text{N}$

To this time, the theoretical studies of pion scattering from $^{15}\text{N}$ have been carried out using DWIA\(^{31},37\) and the $\Delta$-hole model\(^{14}\) in a similar way to those involving $^{13}\text{C}$: Both nuclei are negative parity states with a spin of $\frac{1}{2}$ and an isospin of $\frac{1}{2}$. From the viewpoint of nuclear structure, the pion reaction with $^{15}\text{N}$ is closely related to the reaction with $^{13}\text{C}$, since in the jj-coupling model, $^{13}\text{C}$ is one $p_{1/2}$-particle state with a closed $p$-shell, while $^{15}\text{N}$ is one $p_{1/2}$-hole state with a closed $p$-shell. Furthermore, since description of $^{15}\text{N}$ as a hole state is justified, study of the pion scattering provides us with a good test of our choice of the spreading potential discussed in the previous subsection.

We determined the strength of the spreading potential in Eq. (5.18) so as to reproduce the data of the elastic scattering from $^{16}\text{O}$, as shown in Fig. 15:

$$V_C = 12.7 - 34.9i, \quad \nu = 0.30,$$
$$V_{LS} = -0.957 - 1.37i, \quad \mu = 0.30. \quad (5.18)$$

These parameters reproduce the experimental data at all energies.

Now, we calculate the differential cross section and the analyzing power of $\pi^+,-^{15}\text{N}$ elastic scattering, and also the differential cross section of the charge-exchange reaction, $^{15}\text{N}(\pi^+, \pi^0)^{15}\text{O}$, by applying this spreading potential and also the parameter $\alpha_\tau$ of the isospin dependence determined in the previous section. The results with and without the isospin-dependent parameter $\alpha_\tau$ are shown in Fig. 16. We also examined the effect of the imaginary part of the parameter, by simply adding a small imaginary part to the parameter $\alpha_\tau$. As in $^{13}\text{C}$, the isospin dependence of the spreading potential does not affect the differential cross section and the analyzing power of the elastic scattering, while it does affect both observables of charge-exchange scattering, and succeeds in explaining the experimental data of the differential cross sections.

It is noted that the signs of the analyzing power for charge-exchange scattering
Fig. 15. Differential cross sections of $\pi^\pm$-$^{16}$O elastic scattering at $T_\pi = 114, 163$ and 240 MeV. The solid lines are the theoretical results with the best choice of the strength parameter of the spreading potential. The data are taken from Ref. 38).

It is interesting to compare our results with those of the DWIA calculation based on the computer code PIPIT.\(^{25}\) They are shown in Fig. 17. Since the DWIA calculation with PIPIT does not include a nuclear medium correction, such as Pauli forbidden effects, this is naturally expected to be the cause of any difference between our results and those of Mach and Kamalov.\(^ {31}\) The differential cross sections of the elastic pion scattering are found to be described by both calculations. In spite of this, the analyzing power near forward scattering takes different signs. Furthermore, in the charge-exchange reaction, the DWIA calculation does not explain the magnitude
Fig. 16. Isospin dependence of the spreading potential for the differential cross section and analyzing power of $\pi^\pm$-$^{15}$Na at $T_\pi = 165$ MeV. The solid, dotted and dashed curves are the results with $\alpha_T = 0$, $-0.2$ and $-0.2 + 0.1i$, respectively. The data are taken from Refs. 39), 19) and 20) for elastic scattering, and from Ref. 30) for charge-exchange scattering.

of the cross section; i.e., it gives smaller cross sections by one order of magnitude, and the sign of the analyzing power is also different from ours. Unfortunately, since no experimental data for the analyzing power are available at present, it is difficult to draw a definite conclusion in respect to the sign problem. We strongly recommend that an experiment on the analyzing power of the charge-exchange reaction with this nucleus be performed.
Fig. 17. Comparison of the present calculations with those of PIPIT for the differential cross sections and analyzing power of $\pi^{\pm}^{15}\text{Na}$ at $T_\pi = 165 \text{ MeV}$. The solid and dashed curves are the present results with the isospin-dependent spreading potential $\alpha_\tau = -0.2$, and PIPIT calculation, respectively.

§6. Summary and conclusion

We have studied pion-nucleus reactions around the $\Delta$ resonance region to investigate the interactions of $\Delta$ in nuclei.

First, we devised a formalism of the isobar-doorway state model for open-shell nuclei by using the projection operator formalism. The $\Delta$ propagation is written as a sum of a one-body operator of a $\Delta$ and a $\Delta$-nucleon two-body interaction. The former part is the self-energy of $\Delta$ corresponding to the elementary process, and the latter interaction is the rescattering process of pions in nuclei. Pauli forbidden effects in the conventional $\Delta$-hole model are naturally taken into account in $\Delta$-nucleon two-
body interactions without introducing explicit Pauli forbidden states. Therefore, the present isobar-doorway state model is applicable to any complex nuclei. Additional non-trivial interactions of $\Delta$s in nuclei were described as the binding potential and spreading potential phenomenologically.

We started from pion elastic scattering from $^{12}\text{C}$. Here, the spreading potential of a $\Delta$ was fitted to the differential cross section. It was found that the model describes the global feature of the pion-$^{12}\text{C}$ reactions. We obtained satisfactory agreement with the total and integrated elastic cross section; that is, the strength of the reaction cross section was correctly described. Furthermore, the strength of the spreading potential was examined by comparing the absorption cross section. We approximately reproduced the experimental data, except at low energy, where the isobar-doorway state model may not be verified. It is noted that in the literature, the strength of the $p$-shell pion-nucleon interaction was adjusted to fit the differential cross section using the DWIA, which might not be a correct description of the total and absorption cross section and is not acceptable.

The elastic cross section of pions with $^{13}\text{C}$ was predicted by using the same spreading potential as with $^{12}\text{C}$. The results for the differential cross section are in good agreement with the experimental data. The predicted angular and energy dependence of the analyzing power agree semi-qualitatively with the data. This contrasts with the DWIA analysis, which gives a rather strong angular dependence. The effect of configuration mixing on the differential cross section and analyzing power was found to be almost negligible. This is partly because the relevant spin-flip amplitude is of the transverse type, and partly because the transverse spin-flip nuclear form factor in this reaction is affected little by the configuration mixing.

The ratio of spin-nonflip and spin-flip pion-nucleus amplitudes is mainly determined by the excitation of $\Delta$. It is this excitation, not the spin-orbit term, that governs the analyzing power in contradiction to our naive expectation.

Understanding the charge-exchange reaction with $^{13}\text{C}$ is a long-standing problem. Although we succeeded in explaining all the observables described above, the conventional model also failed in the charge-exchange reaction, as shown in many previous works. Here, we introduced an additional term in the $\Delta$ spreading potential implied by Hirata, which depends on the isospin of $\Delta$ and the residual nucleus. The strength of isospin-dependent potential was determined so as to reproduce the differential cross section of the charge-exchange reaction at pion energy 163 MeV. Then, the predicted energy dependence of the integrated cross section and forward scattering cross section of charge-exchange reaction were greatly improved, and were in good agreement with the data. The analyzing power of the charge-exchange reaction at forward angles changed signs by including the isospin-dependent spreading potential, and our results agree with the experimental data qualitatively. This phenomenological potential was examined in the charge exchange reaction with $^{15}\text{N}$. We were able to consistently explain the charge-exchange reaction with the same strength of the isospin-dependent spreading potential. In this work we have found the important role of the isospin-dependent potential of $\Delta$. The origin of this isospin dependence could be the $\Delta-N$ interaction. If a pion is absorbed in a two-body process through a $\Delta$ in a nucleus, only the $\Delta-N$ channel with isospin $T = 1$ can decay.
into the two-nucleon states, while the channel with isospin \( T = 2 \) cannot. Therefore, we expect an isospin dependence of the spreading potential.

To further explore the idea of the isospin-dependent spreading potential, we plan to apply this phenomenological approach to the other reactions. The best place is the inelastic scattering of \( \pi^+ \) and \( \pi^- \) from \(^{12}\text{C}\) leading to the state with \((J^\pi, T) = (1^+1)\) and \((1^+0)\). Describing the energy dependence of the ratio of the cross sections with excitation of \((1^+1)\) and \((1^+0)\), and also the \( \pi^+ / \pi^- \) ratio are other long-standing problems. Since the inelastic form factor for \((1^+0)\) is very much affected by the configuration mixing, we must use the open-shell isobar-doorway state model.

Another approach is to understand the spreading potential in a microscopic model. Although such an approach was tried, for example, by Lee and Ohta \(^40\) using the phenomenological \( \Delta-N \) interaction, it is necessary to extend this approach to exploring the relation between the phenomenological isospin-dependent spreading potential and the isospin dependence of the two-body \( \Delta-N \) interaction. A more ambitious project is to study the spreading potential from a microscopic interaction model of pions, nucleons and \( \Delta s \). \(^41\)

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Appendix A

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\( \pi N \Delta \) Vertex and Center of Mass Correction

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When we move from the 2-body center-of-mass system to the \( A \)-body center-of-mass system, we must take into account center-of-mass corrections, including the elimination of spurious states in the harmonic oscillator model and the pion-nucleon relative momentum correction participated in vertex function.

A.1. Center-of-mass correction of the shell model wave function

The matrix elements of the \( \pi N \Delta \) vertex function \( H_{\pi N \Delta} \) are given by the matrix element

\[
M = \langle \phi_\Delta(r_{rel}) \psi_{A-1}^{\text{int},f} | H_{\pi N \Delta}^\dagger | \psi_{A}^{\text{int},i}, k\tau_\alpha \rangle, \tag{A.1}
\]

where \( \psi_{\text{int},i}, \psi_{\text{int},f} \) are the internal wave functions for the initial and final nuclear states, respectively, and \( \phi_\Delta(r_{rel}) \) is a single-particle wave function of the \( \Delta \) in the relative coordinate \( r_{rel} = r_A - R_{A-1} \). The center-of-mass coordinate of the nucleus is given as

\[
R_A = \sum_{i=1}^{A} \frac{r_i}{A} \quad \text{and} \quad R_{A-1} = \sum_{i=1}^{A-1} \frac{r_i}{A-1}. \tag{A.2}
\]
The matrix element $M$ is then expressed as
\[ M = \langle \phi_\Delta(r_{\text{rel}})\psi_{A-1}^{\text{int},j} | i f_{\pi N \Delta} \sqrt{\frac{m_N}{m_\pi}} \frac{m_N}{(2\pi)^3 2\omega_N \epsilon_N(\kappa)} F(\kappa) S^\dagger \cdot \kappa T^\dagger \alpha e^{-i \kappa \cdot (r_A - R_A)} | \psi_A^{\text{int},i} \rangle, \]
(A.3)
where $\kappa$ is the relative momentum of the pion, and the $A$-th nucleon is defined in terms of pion momentum $k$ and the $A$-th nucleon momentum $p_A$ as
\[ \kappa = \frac{\epsilon_N(p_A)k - \omega_\pi(p_A)p_A}{\sqrt{S}}. \]
(A.4)

It is noted that the pion wave function is given as
\[ e^{-i \kappa \cdot r_{\text{rel}}} = e^{i k_{\text{rel}} \cdot r_{\text{rel}}}, \]
(A.5)
where $k_{\text{rel}} = \frac{A-1}{A} k$.

The shell model wave function includes the center-of-mass coordinate. However, we can separate the center-of-mass coordinate using a harmonic oscillator wave function. The potential of the harmonic oscillator is given by
\[ \frac{m_\omega^2}{2} \sum_{i=1}^A r_i^2 = \frac{m_\omega^2}{2} \sum_{i=1}^A (r_i - R_A)^2 + \frac{Am_\omega^2}{2} R_A^2, \]
(A.6)
where the center-of-mass motion is separated out. We can also rewrite this as
\[ \frac{m_\omega^2}{2} \sum_{i=1}^A r_i^2 = \frac{m_\omega^2}{2} \sum_{i=1}^{A-1} (r_i - R_{A-1})^2 + \frac{m_\omega^2}{2} A^{-1} r_{\text{rel}}^2 + \frac{Am_\omega^2}{2} R_A^2. \]
(A.7)
Therefore we use the effective single particle $b_{\text{rel}} = \sqrt{\frac{A-1}{A}} b$ for the $\pi N \Delta$ vertex to take into account center-of-mass correction.

A.2. Center-of-mass correction for the $\pi N \Delta$ vertex

The momentum conjugate to $r_{\text{rel}}$ is
\[ p_{\text{rel}} = \frac{1}{A} [(A-1)p_N - P_{A-1}] = p_N - \frac{1}{A} P_A = p_N + \frac{1}{A} k, \]
(A.8)
where $P_A$ and $P_{A-1}$ are the center-of-mass momentum of the $A$ and the $(A-1)$ nucleon systems, respectively. Then, the relative momentum $\kappa$ is given by
\[ \kappa = \frac{(\epsilon_N + \omega_\pi)}{\sqrt{S}} k - \omega_\pi p_{\text{rel}}. \]
(A.9)

For the other kinematical factors and vertex function $F(\kappa)$, we approximate $p_N \cong -\frac{k}{A}$. Then, the $\pi N \Delta$ vertex is given as
\[ H_{\pi N \Delta}^\dagger = i f_{\pi N \Delta} \sqrt{\frac{m_N}{(2\pi)^3 2\omega_N \epsilon_N(\kappa)}} \cdot \left[ \alpha p_{\text{rel}} + \beta p_{\text{rel}}^N \right] T^\dagger_l. \]
(A.10)
where
\[
\bar{\alpha} = \frac{1}{\sqrt{S}} \left( \epsilon_N + \frac{\omega_\pi}{A} \right) \frac{A}{A-1},
\]
(A.11)
\[
\bar{\beta} = -\frac{1}{\sqrt{S}} \left[ \omega_\pi + \left( \epsilon_N + \frac{\omega_\pi}{A} \right) \frac{A}{A-1} \right],
\]
(A.12)
\[
\tilde{F}(k) = \frac{1}{S} \frac{\epsilon_N + \omega_\pi}{1 + k^2/\bar{\alpha}^2},
\]
(A.13)
with
\[
\bar{\alpha} = \frac{\epsilon_N + \omega_\pi}{\sqrt{(\epsilon_N + \omega_\pi)^2 - (A-1)^2 A^2}}.
\]
(A.14)

A.3. Reduced matrix element for the $\pi N\Delta$ vertex

The multipole operator for the $\pi N\Delta$ vertex is
\[
H^{\gamma \dagger}_{\pi N\Delta}(k) = \frac{f_{\pi N\Delta}}{m_\pi} \sqrt{\frac{m_N}{(2\pi)^3 2\omega_\pi \epsilon_N}} \tilde{F}(k) S^l \left[ \tilde{\alpha} p_{\text{rel}}^N + \tilde{\beta} p_{\text{rel}}^3 \right] T^l_\dagger
\]
\[
\times 4\pi i \ell_\pi^l j_\pi(k_{\text{rel}}r_{\text{rel}})Y_{l_\pi}(\hat{r}_{\text{rel}}).
\]
(A.15)
The single-particle matrix element of $H^{\gamma \dagger}_{\pi N\Delta}(k)$ is
\[
\langle n_\Delta l_\Delta j_\Delta | H^{\gamma \dagger}_{\pi N\Delta} | n_N l_N j_N \rangle = -4 \frac{f_{\pi N\Delta}}{m_\pi} \sqrt{\frac{m_N}{(2\pi)^3 2\omega_\pi \epsilon_N}} \tilde{F}(k) \sqrt{4\pi i \ell_\pi^l} \sqrt{[j_N j_\Delta l_\pi]}
\]
\[
\times \left\{ \begin{array}{ll} l_\Delta & 3/2 \\ l_N & 1/2 \\ L & 1 \end{array} \right\} \int r_{\text{rel}}^2 dr_{\text{rel}} R_{n_\Delta l_\Delta j_\Delta}(r_{\text{rel}}) h(r_{\text{rel}}) R_{n_N l_N j_N}(r_{\text{rel}}),
\]
(A.16)
where
\[
h(r_{\text{rel}}) = \bar{\alpha} \left[ \alpha_+ j_{\pi}(k_{\text{rel}}r_{\text{rel}}) \overrightarrow{D}(l_N) + \alpha_- j_{\pi}(k_{\text{rel}}r_{\text{rel}}) \overleftarrow{D}(l_N) \right]
\]
\[
+ \bar{\beta} \left[ \beta_+ \overrightarrow{D}(l_\Delta) j_{\pi}(k_{\text{rel}}r_{\text{rel}}) + \beta_- \overleftarrow{D}(l_\Delta) j_{\pi}(k_{\text{rel}}r_{\text{rel}}) \right],
\]
(A.17)
with
\[
\alpha_\pm = \pm \sqrt{[l_\Delta] \left( l_N + \frac{1}{2} \right) (l_\Delta 0 l_\pi 0 | l_N \pm 1 0) [L] W(l_N 1 l_\Delta l_\pi ; l_N \pm 1 L),
\]
(A.18)
and
\[
\beta_\pm = \mp \sqrt{[l_N] \left( l_\Delta + \frac{1}{2} \right) (l_N 0 l_\pi 0 | l_\Delta \pm 1 0) [L](-)^L W(l_N l_\pi l_\Delta 1 ; l_\pm 1 L).
\]
(A.19)
Here, we have defined the following differential operators:

\[
\overrightarrow{D^+}(l) = \frac{d}{dr} - \frac{l}{r}, \quad \overrightarrow{D^+}(l) = \frac{d}{dr} - \frac{l}{r},
\]

and

\[
\overrightarrow{D^-}(l) = \frac{d}{dr} + \frac{l+1}{r}, \quad \overrightarrow{D^-}(l) = \frac{d}{dr} + \frac{l+1}{r}.
\]

(A.20)

(A.21)

References

26) M. Hirata et al., Ann. of Phys. 120 (1979), 205.