Seismic properties of rocks with distributions of small cracks

S. Peacock¹ and J. A. Hudson²

¹ Postgraduate Research Institute for Sedimentology, University of Reading, Whiteknights, PO box 227, Reading RG6 2AB, UK
² Department of Applied Mathematics and Theoretical Physics, Silver Street, Cambridge CB3 9EW, UK

Accepted 1990 March 14. Received 1990 February 9; in original form 1989 October 2

SUMMARY

Hudson's theory of multiple scattering for the dynamic properties of cracked rock is applied to media containing cracks of varying orientation, radius and aspect ratio. Angular variation of velocity and attenuation due to parallel and nearly parallel cracks and cracks with normals at a fixed angle to a given axis are presented. Anisotropy gradually decreases as the distribution of orientations is widened from exactly parallel to random. Increasing the aspect ratio of thin saturated cracks generally causes the elastic properties to approach those due to dry (gas-filled) cracks. For fluid-filled cracks the line singularity where the birefringent shear waves have equal velocity moves closer to the symmetry axis as either the distribution of orientations is widened or the aspect ratio is increased. Calculated attenuations due to Rayleigh scattering and viscous dissipation in crack-filling fluids are many orders of magnitude less than reported attenuations in the Earth. Viscous dissipation varies linearly with frequency and aspect ratio of thin fluid-filled cracks. A power-law distribution of crack radii, intended to represent that found in the Earth's crust, has no effect on the angular variations of seismic velocities or viscous attenuation but scattering attenuation is sensitive to the power-law exponent. Lognormal distributions of aspect ratio, also corresponding to observations in real rock, affect velocity as well as attenuation. Velocities derived from the theory are compared with those measured in laboratory specimens with known geometries of cracks.

Key words: anisotropy, attenuation, cracks, seismic velocities, shear wave splitting.

1 INTRODUCTION

Hudson's (1980, 1981, 1986) formulae for the effective elastic properties of rocks containing arrays of cracks have been used by several authors to interpret seismic anisotropy (Crampin et al. 1986a; Sayers 1988). They yield the elastic properties of the homogeneous medium with identical bulk properties to the cracked medium. Because the formulae are for overall properties of the cracked medium, the crack radii are assumed to be much shorter than the seismic wavelength and the cracks separated by distances larger than the crack size but smaller than the wavelength. The total crack density \( v \) must be small (\( v \) is the number density, or number of cracks per unit volume, and \( a \) is an average radius). For mathematical convenience the cracks are assumed to be smooth-faced highly oblate spheroids, sparsely distributed through the matrix and isolated from each other. Even with this simple model the number of variables governing seismic behaviour is large. At present the most powerful method of interpreting seismograms for crack geometry is by synthetic modelling of every detail of 3-D particle motion of the shear wave arrivals (Crampin et al. 1986a). Material properties for the model are derived by trial-and-error substitution of hypothetical crack and matrix parameters into Hudson's or similar formulæ.

Cracks, faults and joints in the real Earth are in general far from being smooth-faced ellipsoids and presumably have as great a variety of sizes, shapes and orientations as those in thin sections (Simmons & Richter 1976). However, laboratory and field observations may control the range of values of the parameters in the model. Cracks grown at subcritical rates in stressed rock in the laboratory have an approximately power-law distribution of lengths (Meredith & Atkinson 1983). Aspect ratios of mineral-filled cracks in hand specimens of gneiss show an approximately lognormal distribution (Hay et al. 1988; Hay 1988), which presumably pertains to open cracks at depth in the crust. The normals to cracks grown under true triaxial stress are approximately parallel to the direction of minimum compression (Hadley 1975), but the exact alignment depends on existing weaknesses in the rock as well as the ambient stress direction (Main, Peacock & Meredith 1990). Under uniaxial...
compression crack normals are expected to be randomly oriented in the plane perpendicular to the direction of compression, but even in laboratory specimens this orientation is perturbed by pre-existing heterogeneities (Hadley 1975).

It is not possible to reconcile Hudson's model any further with real cracks without sacrificing its simplicity. The limits on crack size and spacing are fundamental to the whole concept of an effective medium, which treats cracks by means of a statistical average rather than as separate entities. Extension to higher crack densities is a hope for the future but is not yet available. Crack shapes more complicated than oblate spheroids are analytically intractable and would require numerical schemes. We can however create a more 'realistic' crack geometry without great loss of simplicity by allowing the oblate spheroidal cracks to have distributions of orientation, radius and aspect ratio. The distributions need to be simple so as to introduce as few extra parameters as possible. Material constants are easily determined by computer programs using the principle (Hudson 1986) of summing the first-order contributions to the effective elastic constants from several sets of cracks with different geometries, then calculating the second-order contributions from the summed first-order contributions. In this paper we present results from such a program. We determine the angular variations of velocity and attenuation expected from crack distributions measured in the laboratory and assess the effect of such distributions on modelling and interpretation of shear wave seismograms.

2 THEORY—EFFECTIVE ELASTIC CONSTANTS OF CRACKED MEDIA

Hudson's theory gives complex effective elastic constants of a cracked medium. Crampin (1981) describes converting such elastic constants to velocities and attenuations of seismic waves. Before looking at Hudson's theory of the effects of cracks on seismic velocity and attenuation we shall briefly review the effects of anisotropy in general on seismic wave propagation.

Two properties of seismic waves in anisotropic media concern us. One is that their velocities and attenuations vary with direction. As a consequence, the group velocity or energy transport direction does not coincide with the phase velocity direction (i.e. the normal to the surface of constant phase). The other is that three body waves exist in anisotropic media, with orthogonal polarizations fixed within the medium and not usually exactly aligned with the direction of propagation. The fastest wave (quasi-P or qP) has polarization closest to the propagation direction. The two slower waves (quasi-shear waves) generally have different velocities, which leads to shear wave splitting (birefringence or double refraction). The faster split shear wave is called qS1 and the slower, qS2. In planes of mirror symmetry in an anisotropic medium, the shear wave polarized parallel to the plane is sometimes called qSP and the shear wave polarized at right angles to the plane is called qSR (Crampin 1981).

Attenuation of the three body waves in anisotropic media also varies with direction. It is generally low in directions in which velocity is high (Crampin 1984).

2.1 Overall material properties—wavespeeds to first order

Hudson's theory is summarized as \( e = e_0 + \sum e'_I + e_2 \) where \( e \) is the fourth-order tensor of elastic constants of the cracked medium, \( e_0 \) are the constants of the uncracked matrix, \( e'_I \) are the first-order perturbations due to the \( I \)th crack set and \( e_2 \) are the second-order contributions due to the interaction of all the crack sets, determined by operating on the cumulative first-order constants \( \sum e'_I \).

Hudson (1981) gives the first-order contributions \( e'_I \) due to a set of parallel cracks with radius \( a \) and normals in the \( x_3 \) direction as

\[
\begin{align*}
    c_{iju}^1 &= (-v a^3/\mu)\epsilon_{k30}\epsilon_{l3q} \tilde{U}_{kl}(ka),
\end{align*}
\]

where \( v = \) number density of cracks (number per unit volume); \( k = \) wavenumber; and \( \mu = \) shear modulus of matrix. The summation convention for repeated suffixes is assumed throughout. The crack normals are taken to lie parallel to the \( x_3 \) direction following the convention in Hudson (1980, 1981, 1986), although in many geophysical publications the crack normals lie parallel to the \( x_1 \) direction (Crampin 1984).

\( \tilde{U}_{kl} \) is the \( k \)th component of the discontinuity in displacement across the crack due to unit imposed traction in the \( x_i \) direction, integrated over the crack and normalized by \( \mu/a \) (Hudson 1981). For cracks with circular symmetry \( \tilde{U}_{kl} \) vanishes unless \( k = l \); also \( \tilde{U}_{11} = \tilde{U}_{22} \). The formulae for \( \tilde{U}_{11} \) and \( \tilde{U}_{23} \) contain the crack aspect ratio \( d/a \), where \( d \) is the half-thickness of the cracks, and the bulk and shear moduli of the filling material, \( \kappa' \) and \( \mu' \):

\[
\begin{align*}
    \tilde{U}_{11} &= (16/9)(3\kappa + 4\mu)/(3\kappa + 2\mu)(1 + M),
    \\
    \tilde{U}_{23} &= (4/3)(3\kappa + 4\mu)/(3\kappa + \mu)(1 + K),
\end{align*}
\]

where

\[
M = (4\pi/3)(a/\mu')(d\kappa + \mu')(3\kappa + 4\mu)/(3\kappa + \mu)
\]

and

\[
K = (1/3\pi)(d/\mu')(3\kappa + 4\mu')(3\kappa + 4\mu)/(3\kappa + \mu)
\]

where \( \kappa \) and \( \mu \) are the bulk and shear moduli of the uncracked matrix. Hudson (1986, but with corrections as shown in Appendix A) gives the perturbation of elastic constants due to cracks with normals at a fixed angle \( \psi \) to the \( x_3 \)-axis (with \( \lambda = \kappa - 2\mu/3 \)):

\[
\begin{align*}
    c_{1111}^1 &= c_{1222}^1 = (-v a^3/2\mu)(\tilde{U}_{23}(2\theta^2 + 4\lambda\sin^2 \theta + 3\mu^2 \sin^4 \theta) &+ \tilde{U}_{11}\mu^2 \sin^2 \theta (4 - 3\sin^2 \theta)),
    \\
    c_{2333}^1 &= (-v a^3/\mu)(\tilde{U}_{33}(\lambda + 2\mu\cos^2 \theta)^2 &+ \tilde{U}_{11}\mu^2 \cos^2 \theta \sin^2 \theta),
    \\
    c_{1112}^1 &= (-v a^3/2\mu)(\tilde{U}_{33}(2\theta^2 + 4\lambda\sin^2 \theta + \mu^2 \sin^4 \theta) &- \tilde{U}_{11}\mu^2 \sin^2 \theta),
    \\
    c_{1133}^1 &= c_{2133}^1 = (-v a^3/\mu)(\tilde{U}_{33}(\lambda + \mu \sin^2 \theta) &\times (\lambda + 2\mu \cos^2 \theta) - \tilde{U}_{11}\mu^2 \sin^2 \theta \cos^2 \theta),
    \\
    c_{2323}^1 &= c_{1313}^1 = (-v a^3/2)(\tilde{U}_{23}\sin^2 \theta \cos^2 \theta &+ \tilde{U}_{11}(\sin^2 \theta + 2\cos^2 \theta - 4\sin^2 \theta \cos^2 \theta)),
    \\
    c_{1212}^1 &= (-v a^3/2)(\tilde{U}_{33}\sin^4 \theta + \tilde{U}_{11}\sin^2 \theta(2 - \sin^2 \theta)).
\end{align*}
\]
In all these expressions the following conditions are assumed:

1. the crack radius $a$ and the spacing between cracks are much less than the seismic wavelength ($ka \ll 1$ where $k$ is the wavenumber);
2. the distribution of positions of cracks is random within the scale of a seismic wavelength;
3. the cracks are sparsely distributed and disconnected, and their total volume is a small fraction of the volume of the rock;
4. the cracks are oblate spheroids of small aspect ratio; and
5. the crack contents are softer than the matrix.

These formulae agree with all others to first order in the crack density $v a^3$. Douma (1988) compares Hudson's formulae to those of Nishizawa (1982), which are valid for all aspect ratios, and shows that the two models agree up to aspect ratio $d/a$ approximately 0.3.

If more than one set of cracks exists in the material, the first-order corrections to the elastic parameters are given by summing the terms that would be obtained for each set of cracks separately. For instance if cracks of different radii but the same aspect ratio occur, the term $v a^3$ should be interpreted as the overall number density multiplied by the mean cubed radius. Cracks with different aspect ratios, different internal conditions and different orientations can be accommodated in a similar way.

The number density $v$ is included with the crack radius in the crack density $v a^3$. These quantities cannot be determined separately by any measurement of wavespeed or polarization alone. Similarly the aspect ratio $d/a$ appears in the factors $a u'/d u$ and $(a u'/d u)(3x' + 4u')/u$ and it is impossible to find the aspect ratio alone, only in conjunction with the ratio $u'/u$ unless the latter is known independently (which in practice it often is).

2.2. Attenuation—imaginary parts of first-order terms

Hudson (1981) deals with two distinct mechanisms of attenuation due to cracks: viscous dissipation within fluid-filled cracks, and scattering. Viscous dissipation is simulated by adding the imaginary part $\imath \omega \eta$ to the shear modulus $\mu'$ of the crack-filling material ($\eta$ is the dynamic viscosity of the fluid and $\omega$ the angular frequency). $\tilde{U}_{11}$ and $\tilde{U}_{33}$ are then complex and cause $c^1$ to have an imaginary part. To first order, for $1/Q \ll 1$, viscous dissipation varies linearly as frequency $\omega$ ($Q$ is independent of frequency).

Attenuation due to scattering is found by evaluating $c^1$ to higher order in $ka$, where $k$ is the wavenumber, and taking the imaginary part. This turns out to be equivalent to evaluating the energy loss by summing over the scattering cross-sections of the cracks. The equations of plane wave attenuation versus incidence angle $\theta$ were derived by Hudson (1981) for parallel and randomly oriented cracks. They are subject to the same limits on crack size, shape and number density as the expressions for the real elastic constants given above. Similar expressions for cracks with normals at a fixed angle to one axis are given in Appendix B.

Hudson’s (1981) expressions are for scattering attenuation coefficients $\gamma_p$, $\gamma_s$, and $\gamma_p$ of the three different body waves that propagate in anisotropic media. Crampin (1984) derived elastic constants from these expressions using his (1981) approximate ‘reduced equations’ for angular variation of attenuation in planes of sagittal symmetry. For parallel cracks and cracks with normals at a fixed angle to one axis Hudson’s expressions contain the same angular variation as Crampin’s reduced equations: $\cos 2\theta$ and $\cos 4\theta$ terms for $q P$-waves, $\cos 4\theta$ terms only for $q S$-waves and $\cos 2\theta$ terms only for $q SR$-waves. Elastic constants are derived by equating the attenuations predicted by Hudson’s formulae at $\theta = 0^\circ$, $45^\circ$ and $90^\circ$ to the reduced equations and solving simultaneously for the imaginary elastic constants that appear in linear combinations in the factors.

All of these scattering attenuation expressions show the $(\omega a/\alpha)^2$ dependence of quality factor $Q$ for Rayleigh scattering, where $a$ is the crack radius and $\alpha$ the wavespeed.

2.3 Second-order contributions to elastic constants

Second-order terms, due to crack-crack interaction, are given by Hudson (1986) and are simple functions of the first-order terms:

$$ c^2_{ijpq} = (1/\mu)c^1_{ijpq}\chi_{rkn} $$

where

$$ \chi_{rkn} = [\delta_{rk} \delta_{sn}(4 + \beta^2/\alpha^2) $$

$$ - (\delta_{rm} \delta_{nk} + \delta_{rn} \delta_{km})(1 - \beta^2/\alpha^2)]/15; $$

$$ \beta^2 = (\lambda + 2\mu)/\rho $$

and $\alpha^2 = (\lambda + 2\mu)/\rho$ ($\rho$ is the density of uncracked material). $c^1_{ijpq}$ etc. are the overall first-order constants, including real and viscous imaginary parts but not the imaginary terms due to scattering; if these are included they generate spurious second-order terms.

2.4 Variations in crack properties

A program has been written to calculate effective elastic constants of media containing cracks with a range of radii, aspect ratios and orientations as follows:

1. the range of the varying parameter (radius, aspect ratio or orientation) is divided into variable small intervals;
2. the crack density within each interval is calculated according to a probability density function of the varying parameter;
3. the first-order contribution to the elastic constants due to cracks within each interval is calculated and scaled by the appropriate crack density;
4. the contributions from all the intervals within the range of the varying parameter are summed;
5. steps 1–4 are repeated as necessary for each different set of cracks in the material; and
6. the second-order constants are calculated from the summed first-order constants and added to the matrix and summed second-order constants to give the final effective elastic constants of the cracked medium.

The first-order constants are calculated using equation (1) or (3) for the real and viscous imaginary parts and the equations given in Appendix B for the scattering imaginary parts. The probability density function (PDF) is taken to be uniform, Gaussian, lognormal or power law for crack radius.
or aspect ratio; and spherical uniform or Fisher for orientations.

The PDFs for orientation, Fisher and spherical uniform (Mardia 1972), are symmetrical about the mean orientation of the crack normals, so the set of cracks with distributed orientations still has hexagonal symmetry. This eases computation and reduces the number of variables. More complex distributions, lacking hexagonal symmetry, can be obtained by superimposing several crack sets with hexagonal symmetry about different axes.

The PDFs must be modified to accommodate the axial (bidirectional) crack orientation vectors. The uniform distribution is simply cut off at $\pi/2$:

$$ p(\theta) = \frac{1}{2\pi}, \quad 0 < \theta \leq \pi/2, \quad (5) $$

where $p$ is defined such that $p(\theta) d\Omega$ is the probability that an orientation lies within solid angle $d\Omega = \sin \theta d\theta d\phi$ centred on the polar direction $(0,0)$. It is directly related to the number density of orientations on the unit sphere. We simulate axiality in the Fisher distribution by summing two antipodeal distributions with equal concentration parameters $s$, giving

$$ p(\theta) = \left( \frac{s}{4\pi \sinh s} \right) \left[ e^{s \cos \theta} + e^{s \cos(\pi - \theta)} \right], \quad 0 < \theta \leq \pi/2. \quad (6) $$

The concentration parameter $s$ determines the sharpness of the peak in the distribution: the larger $s$, the sharper the peak. For large $s$ the distribution is approximately Gaussian in $\theta$ with mean deviation $\sqrt{\pi/2s}$ from the axis: $p(\theta) = (se^{-s\theta^2})/\pi$. This is easier to program than distributions such as the Dimroth–Watson (Mardia 1972) that are specifically for axial data. Example PDFs of crack orientation are shown in Fig. 1 for Fisher distributions in which 50 per cent of the crack normals are inclined within $10^\circ$, $25^\circ$ and $45^\circ$ of the mean direction (corresponding to concentration parameters $s$ of 46.1, 7.5 and 2.3 respectively). If $s$ is set to zero the Fisher becomes a uniform distribution, in which case 50 per cent of crack normals are within $60^\circ$ of the axis.

We use equation (3) and the equations in Appendix B for cracks at a fixed angle to an axis, to calculate the elastic constants due to cracks with a distribution of orientations. Let $\psi$ be the angle between the crack normals and the axis of symmetry. When $\psi = 0$ the cracks are parallel. Elastic constants due to cracks with a uniform or Fisher distribution of orientations come from making $\psi$ the varying parameter in steps 1–6 above with a range between 0 and $\pi/2$.

$$ Figure 1. Fisher probability density function of crack density versus orientation of crack normals, measured from the mean direction. 50 per cent of crack normals are within $10^\circ$ (solid line), $25^\circ$ (long dash) or $45^\circ$ (short dash) of the mean direction. $ Figure 2. Angular variation of velocities and attenuation in a medium containing perfectly aligned cracks (solid line) and cracks with Fisher distribution of normals about the mean direction ($0^\circ$). 50 per cent of crack normals are within $10^\circ$ (long dash), $25^\circ$ (medium dash) or $45^\circ$ (short dash) of the mean direction. (a) Phase and group velocities of $qP$ and the two $qS$-waves in a medium containing dry (empty or gas-filled) cracks. (b) As for (a) but for water-saturated cracks. (c) Scattering attenuation $1/Q$ of $qP$ and the two $qS$-waves at 10 Hz in a medium containing dry cracks. (d) Viscous and scattering attenuation $1/Q$ of $qP$ and $qS$ at 10 Hz in a medium containing saturated cracks. Matrix and crack properties are given in Table 1. The $qSR$-wave is the one with higher velocity and lower attenuation at $90^\circ$. 
3.1 NUMERICAL RESULTS

3.1 Effects of a spread in crack orientation

When the cracks are no longer exactly parallel but have a distribution of orientations, the anisotropy of the cracked medium is reduced. Fig. 2 shows this for thin cracks (aspect ratio $10^{-4}$) with Fisher distributions of crack normals about a mean axis. The cracks are dry (Fig. 2a) and fluid-saturated (Fig. 2b). The phase and group velocity variations with direction are fundamentally different in the two cases. For dry cracks the main effect of the cracks is an increased compressibility for waves propagating parallel to the mean direction of the crack normals and a reduced shear modulus at 45°. The velocity variation with angle is monotonically increasing from 0° to 90°. When the cracks are saturated with fluid the effect on the compressibility is much reduced, especially for cracks such as those with small aspect ratios ($10^{-4}$), but the reduction in shear modulus is unchanged. It follows that the quasi-$P$- ($qP$-) wave velocity variation has a minimum around 45° while the velocity variation of the quasi-shear wave polarized in the plane containing the symmetry axis ($qSP$) has a maximum. The velocity variation of the other quasi-shear wave ($qSR$), polarized at right angles to the symmetry axis, is unaffected by the presence of the fluid.

The Fisher distributions in Fig. 2 have 50 per cent cumulative bounds of $10^6$, 25° and 45°. For ‘dry’ (gas-filled) cracks the reduced anisotropy and increased minimum shear wave velocity are the only effects of the widening distribution of orientations; these are equivalent to the effects of reducing the number density of perfectly aligned cracks. For saturated cracks a second, more subtle but possibly important effect is that the line singularity (or intersection singularity, Crampin & Ye Dien 1981), where the phase velocities of the two quasi-shear waves are equal, moves towards the mean axis as the distribution widens. This also happens to the singularity of the group velocities. The angle of incidence at which the singularity occurs is plotted against the 50 per cent cumulative limit of the Fisher distribution for saturated cracks in Fig. 3(a).

Attenuation shows the same reduction in anisotropy as velocity when the crack orientation distribution is widened (Fig. 2c, d). The dissipation, both by scattering and viscosity, predicted by this model is very much smaller than that measured in laboratory experiments or in the Earth. $1/Q_{\text{e}}$ due to viscosity has a maximum value of approximately $10^{-9}$ and scattering at this large wavelength/crack radius ratio ($3 \times 10^4$ for shear waves) is even less.

3.2 Cracks oriented at a fixed angle to a given axis

Once again, so long as the distribution of normals about the fixed direction is random, the overall symmetry of the cracked medium remains hexagonal. The fixed angle between the crack normals and the axis may lie between 0° (the degenerate case of parallel cracks) and 90° (cracks with coplanar normals, Crampin 1990a). Fig. 4 shows velocity and attenuation variations due to thin (aspect ratio $10^{-4}$) parallel cracks and cracks with coplanar normals. 0° in the figure is the axis of hexagonal symmetry, which for parallel cracks is the direction of the normal to the cracks and for coplanar-normal cracks is perpendicular to the plane containing the crack normals. Corresponding shear wave polarizations are shown in Fig. 5.

For saturated thin cracks $qP$-wave velocity variations with direction are similar for parallel and coplanar-normal cracks, making them difficult to distinguish. For dry cracks the velocities vary in antiphase to each other, so that coplanar-normal cracks could be recognized easily from equal-area projections of velocity. Crampin & Radovich (1982) calculate the first-order effects of dry vertical cracks with horizontal coplanar normals using the formulae of Garbin & Knopoff (1973, 1975) and Crampin (1978). The resulting velocities agree approximately with those in Fig. 4(a).

From Fig. 5 it is clear that shear wave polarizations could be used to distinguish parallel from coplanar-normal cracks if the orientation of the axis of symmetry is known. Both types of crack alignment have hexagonal symmetry so one shear wave is always polarized in a plane containing the symmetry axis. For cracks with coplanar normals this is
Figure 4. Angular variation of velocities for (a) dry and (b) saturated cracks that are parallel (solid line) and randomly oriented with coplanar normals (dashed line). Corresponding attenuations: (c) scattering attenuation in dry cracks and (d) viscous and scattering attenuation in saturated cracks. The 0° direction is normal to the parallel cracks and normal to the plane containing the normals of the cracks with coplanar normals.

always the faster shear wave, but for parallel cracks it is the slower shear wave except at low incidence angles in saturated cracks. If the cracks are aligned by stress, the rule that the faster shear wave is generally polarized parallel to the direction of maximum compressive stress (Crampin 1985) holds for coplanar-normal as well as parallel cracks. For saturated coplanar-normal cracks the lack of shear wave splitting in directions both in and perpendicular to the plane of crack normals may be a good diagnostic if shear waves propagating in both directions can be received.

Attenuation shows similar variation with incidence angle as velocity, but is high where velocity is low, for both parallel and coplanar-normal cracks (Fig. 4c, d); again the values are unrealistically low.

Figure 5. Equal-area projections of shear wave polarizations in a medium with (a) dry, (b) saturated parallel cracks and (c) dry and (d) saturated cracks with coplanar normals. The symmetry axis (0° in Fig. 4) runs from top to bottom of each figure. The unbroken bars are the polarizations of the faster shear wave and the broken bars the polarizations of the slower shear wave.

The shear wave polarized in the plane of the symmetry axis, qSP, (Fig. 6c) is the faster (unbroken bar) for crack normals inclined at 70° to the axis, but is slower (broken bar) for crack normals inclined at 30° and 45° to the axis. For saturated cracks with normals at 30° and 70° to the axis the velocity variations (Fig. 6b, solid line and short dash) are similar in form to those for the corresponding dry cracks, although the range of values is smaller. Note that the shear wave singularity for saturated cracks with normals at 0° to the axis (parallel cracks, Fig. 4) disappears when the angle is increased to 30°. The shear wave polarization patterns [Fig. 6d, (i) and (iii)] are identical to those for dry cracks. The velocities due to saturated cracks with normals at 45° to the axis (Fig. 6b, long dash) have a unique pattern, with a line singularity for the quasi-shear waves near to but not in the plane of the crack normals. There is thus a narrow band of directions nearly normal to the symmetry axis in which the qSP-wave is the faster; in other directions it is the slower shear wave [Fig. 6d, (ii)].
Seismic properties of cracked rocks

477

Figure 6. Angular variation of velocities for (a) dry and (b) saturated randomly oriented cracks with normals at 30° (solid line), 45° (long dash) and 70° (short dash) to the principal axis (0°). (c) Equal-area projections of shear wave polarizations corresponding to the velocity variations in (a) for dry cracks, with the same notation as in Fig. 5. The angle between the principal axis and the crack normals is shown beside each projection. (d) Same as (c) for saturated cracks.

3.3 Crack radius and attenuation

When the aspect ratio and crack density are constant the radii of the cracks have no effect on velocity or viscous attenuation. Scattering attenuation depends on the cube of the radius and so is dominated by the largest cracks present.

This is illustrated in Fig. 7 in which velocity and scattering attenuation at 10 Hz due to parallel cracks with a uniform distribution of radii between 0.001 and 1 m are compared with velocity and attenuation due only to those cracks from the same distribution that have radii between 0.9 and 1 m. The large cracks cause about half the total scattering attenuation (Fig. 7c, d) but make a relatively small contribution to the velocity anisotropy in general (Fig. 7a, b).

Figure 7. Angular variation of velocities and scattering attenuation 1/Ω at 10 Hz in (a), (c) dry and (b), (d) saturated parallel cracks with a uniform distribution of radii between 0.001 and 1 m, crack density 0.1 (solid line) and between 0.9 and 1 m, crack density 0.01 (dashed line).
3.4 Aspect ratios

The aspect ratio (ratio of width to length of a crack) appears in Hudson's formulae only in combination with the elastic properties of the crack-filling substance, so the elastic properties of a medium containing empty cracks are independent of the value assigned to it. It can be seen from Crampin, McGonigle & Ando (1986b) that, in Hudson's formulae, increasing the aspect ratio of thin fluid-filled cracks (aspect ratio less than 0.1) reduces the effect of the fluid, so that the velocities (real parts of the effective constants) tend towards those for a medium with the same distribution of dry cracks as the aspect ratio increases. The same is true for both scattering and viscous attenuation of \( qP \)- and \( qS \)-waves. In particular, the increase in compressibility, which lowers the velocity of \( qP \)-waves travelling parallel to the crack normals, becomes more and more marked as the aspect ratio increases. This is shown in Fig. 8, in which velocities and attenuations from parallel dry cracks are compared with those from water-filled cracks with aspect ratio \( 10^{-3}, 10^{-2} \) and \( 10^{-1} \). The velocity and attenuation of \( qSR \)-waves (polarized in the plane of the cracks) are independent of aspect ratio.

Velocities calculated from Nishizawa's theory (Nishizawa 1982; Douma 1988; Crampin 1990b) for both dry and saturated cracks show the same pattern as Hudson's for aspect ratios up to approximately 0.3. Above 0.3, where the cracks are no longer 'thin' as required by Hudson's theory, Nishizawa's continues to be valid and shows an overall decrease in velocity of both dry and saturated cracks as aspect ratio is increased.

Increasing the aspect ratio of saturated cracks also causes the line singularity of the shear wavespeeds to move towards the crack normal. The angle at which the singularity occurs is plotted against aspect ratio in Fig. 3(b) so that the effect can be compared with that of widening the distribution of crack orientations.

4 VELOCITY VARIATIONS DUE TO OBSERVED CRACK DISTRIBUTIONS

Measuring distributions of crack orientation, radius or aspect ratio in laboratory thin sections of rock is tedious and therefore infrequently done. The few existing measurements of radius and aspect ratio distributions tend to support the hypothesis that cracks grown under stress are self-similar (Main 1988). This means that the cracks should have a power-law distribution of radii and a single value of aspect ratio. Cracks grown at subcritical rates (Atkinson 1982) in laboratory specimens show a power-law distribution of radii (S. Clifford, unpublished manuscript, 1988). This distribution has also been deduced from acoustic emission magnitudes and rates in rocks under stress (Meredith & Atkinson 1983). Aspect ratios of sealed cracks in Lewisian gneiss have a narrow lognormal distribution (Hay et al. 1988; Hay 1988). As cracks in the Earth's crust are thought to grow subcritically, a power-law distribution of radii and lognormal distribution of aspect ratios may be the most appropriate distributions for models of crustal rocks.

According to Hudson's formulae, aspect ratio distributions affect seismic velocities and viscous attenuation but crack radii as such affect only scattering attenuation. We investigate the scattering attenuation due to cracks with a power-law distribution of radii and the velocity variations due to a lognormal distribution of aspect ratios. We also simulate published crack orientation and P-wave velocity data from laboratory samples.

4.1 Power-law distribution of crack radii

The power law takes the form

\[
p(a > x) \propto x^{-D}; \quad x_{\text{min}} < x < x_{\text{max}},
\]

where \( D \) is a parameter that theory and experiment show is constrained to lie between 1 and 3 (Main et al. 1990). It is necessary to limit the distribution to a finite length scale.
Seismic properties of cracked rocks

4.2 Lognormal distribution of aspect ratios

The PDF of a lognormal distribution about modal aspect ratio 0.0036 with standard deviation 0.799 natural-log units is shown in Fig. 11 superimposed on a graph of Hay's (1988) results from sealed cracks in Lewisian gneiss. Angular variations of velocity due to this distribution of aspect ratio are shown in Fig. 12. They are not appreciably different from those due to cracks with a single aspect ratio 0.01 (Fig. 8, medium dash), so it seems unlikely that velocity measurements can show whether lognormal distributions of aspect ratio are the rule for open cracks in the Earth.

4.3 Comparison with observations of velocity anisotropy and crack orientations

Thill, Willard & Bur (1969) report P-wave velocity measurements on Salisbury granite and correlate them with crack alignment seen in thin section. They measured velocities of 5.22–5.92 km s⁻¹ with a 750-kHz ultrasonic transducer, which gives wavelengths of 6.98–7.89 mm. The cracks that they measured were confined to quartz crystals, and 'frequently extended completely through many of the quartz grains'. They state the average grain size of Salisbury granite as 1 mm, so we may assume that the condition of Hudson's theory that the crack size be smaller than the wavelength is barely satisfied. Sayers (1988) makes this assumption to predict crack orientation from the velocities of Thill et al. with Hudson's theory.

We ignore departure of the observed crack distribution from hexagonal symmetry (seen in the petrofabric diagram of Fig. 13a) and approximate the principal alignment by a bipolar Fisher distribution with 50 per cent confidence limit 12° (s = 32.1, Fig. 13b). Our aim is not to make a perfect match to the observed velocities but to determine whether velocities due to a one-parameter Fisher distribution of cracks are a useful first approximation to the observed velocities.

To convert the crack orientations into velocities we must give values for several parameters not given by Thill et al. (1969). For the uncracked matrix properties, Sayers (1988) used the properties of Barre granite at 10 kbar: density...
2.65 g cm$^{-3}$, $v_p = 6.4$ km s$^{-1}$ and $v_s = 3.7$ km s$^{-1}$ ($\lambda = 36.28$, $\mu = 35.99 \times 10^9$ Pa, Nur & Simmons 1969). We found these velocities too high and used elastic constants $\lambda = 31.84$ and $\mu = 32.1 \times 10^9$ Pa. We found that a crack density of 0.05 gave the best fit to the observed velocities. This is not far from the crack density of 0.04 that may be derived by assuming that 35 per cent (the figure quoted by Thill et al.) by volume of the rock is made up of 1-mm quartz grains, each containing a single throughgoing crack.

The resulting P-wave velocities are contoured in equal-area projection in Fig. 13(c) and (d) alongside the observed velocities. Observed and theoretical velocities agree as well as can be expected given that the observed distribution of crack orientations is imperfectly approximated by a Fisher function.

5 DISCUSSION

5.1 Effect of crack geometry on shear wave splitting and singularities

Crampin et al. (1986a) suggest that as shear waves are sensitive to small and subtle changes in anisotropic properties, these properties can be determined precisely by forward modelling to fit observed shear wave motions in 3-D. They demonstrate this by modelling records from a three-component vertical seismic profile (VSP) from the Paris Basin with synthetic seismograms from a cracked medium. Crucial to their argument is locating the line singularity or intersection of the two shear wave velocities (Crampin 1981). Polarization diagrams (hodograms) become linear at these points where there is zero time (or phase) delay between the two shear waves.

We have seen that the angle of incidence at which this singularity occurs depends on both the aspect ratio and the distribution of orientations of the cracks. Its dependence on aspect ratio was recognized by Crampin et al. (1986b) and Crampin (1990b). Bush (personal communication) varied the aspect ratio of cracks in his model of the Paris Basin VSP to improve the fit to the observed shear wave polarization diagrams. Varying the orientation distribution rather than the aspect ratio might also allow the observed polarization diagrams to be reproduced.

5.2 Attenuation

Attenuation is apparently more sensitive than velocity to variation in crack parameters, especially crack length (radius). Unfortunately both scattering and viscous attenuation predicted by Hudson’s theories are orders of magnitude
Smaller than those observed in laboratory or field. Stretching Hudson's theory to the maximum with a crack density 0.1 and crack radius/wavelength ratio approximately 0.6 for shear waves, gives scattering attenuations that are almost plausible (Fig. 14). They still have the unrealistic dependence of $Q$; worse, the theory predicts that scattering attenuation in saturated rocks is smaller than in dry rocks, owing to the reduced contrast in acoustic impedance between the rock and fluid-filled cracks. This is contrary to laboratory observations of $Q$ (Johnston & Toksőz 1980), which indicate that fluids in cracks greatly increase attenuation. Rayleigh scattering does not appear to be the appropriate mechanism of attenuation for realistic models of the Earth.

Viscous attenuation by fluid-filled cracks, which has a frequency dependence closer to that observed, is negligible compared to scattering for the large cracks in Fig. 14. High viscous attenuations arise in very thin cracks, but even an aspect ratio of $10^{-5}$ gives unrealistically low attenuations (Fig. 15). Hudson's theory brings us no nearer an understanding of attenuation processes in the Earth other than to rule out Rayleigh scattering and simple viscous damping in water-filled cracks as the dominant mechanism. Crack-filling fluids with high viscosities may account for some high observed attenuations. Also, greater dissipation may occur in partially saturated cracks, through more complex viscous processes. Hudson (1988) and Miksis (1988) present formulae for attenuation due to partially saturated cracks, where dissipation is due to fluid flow under tension and (for Miksis 1988) movement of the contact line of the crack wall with the liquid–gas interface. Converting these formulae into computer code is the next stage of this work, along with finding more examples of distributions of crack parameters in real rock on which models can be based.

5.3 Introducing more variables to the crack model

It is always difficult to know when to introduce further complexity and additional parameters into a model in order to account for more and more detail, and when to stop. In general, enough parameters will enable a fit to be made to almost any anisotropic variation of velocity with angle, but the fit might have very little significance. In this paper we have considered axisymmetric distributions only. A better fit to the data of Thill et al. (1969) might have been obtained by using an asymmetric distribution of crack density about the axis (see Sayers 1988), by adding a small density of cracks symmetrically arranged about a mean direction inclined to the axis of the first set, or by using elliptical crack shapes. There is considerable ambiguity in the inversion of velocity data to give crack orientations and densities, especially in field situations where the data coverage may be poor and the effects of anisotropy are combined with those of heterogeneity. At present it is valuable simply to know that the model of embedded cracks can account for the observations. However, this ambiguity makes it vital to find an adequate model for attenuation due to cracks.

6 CONCLUSIONS

We have presented the theoretical effects of cracks with distributions of radii, aspect ratios or orientations on seismic velocity and attenuation in cracked media. These were determined using Hudson's formulae for sparse uniform arrays of small, thin cracks. The effects on velocity are small and will be significant only where the position of the shear wave singularity is the deciding parameter. In this case there is ambiguity between the aspect ratio of the cracks and the distribution of their orientations. Viscous and scattering attenuations are more strongly affected, especially by crack radius (to which velocity is insensitive), but because observed attenuations are so much higher than predictions from these formulae, these results have little use in interpreting observations. A more appropriate model of attenuation in cracked media is required.

ACKNOWLEDGMENTS

We thank Bob McGonigle and David Taylor for useful discussions on the computer code and Stuart Crampin and Ian Main for general advice. We are grateful for the use of computers and assistance from computer operators at the British Geological Survey, Edinburgh, and the University of Reading. R. E. Thill kindly let us reproduce figures from his paper and Stephen Hay sent copies of data on crack dimensions from his thesis. We acknowledge Macroc Ltd and Applied Geophysical Software Inc., for permission to use the Aniseis package. S.P. acknowledges financial support from the University of Reading Research Board. Reading University PRIS Contribution No. 062.

REFERENCES


Crampin, S., 1978. Seismic wave propagation through a cracked
APPENDIX A

Corrections to formulae for media containing cracks at a fixed angle to an axis

Equations (39) and (40) of Hudson (1986), giving the real first-order contributions to effective elastic constants of a medium with cracks at a fixed angle ψ to one axis, should read

\[ c_{ijpq}^{(a)} = (-\nu a^3 / \mu) \left[ \tilde{F}_{ijpq} + \tilde{F}_{ijpq}' + 2a \tilde{F}_{ijpq}'' + 4\nu^2 \tilde{F}_{ijpq}''' + 8\nu^3 \tilde{F}_{ijpq}'''' + 16\nu^4 \tilde{F}_{ijpq}'''''ight] \]

where there is no summation over bracketed suffixes and where

\[ a^{(1)} = (1/2\pi) \int_0^{2\pi} \sin^2 \psi \cos^2 \phi \, d\phi = \frac{1}{2} \sin^2 \psi = a^{(2)}, \quad a^{(3)} = (1/2\pi) \int_0^{2\pi} \cos^2 \psi \, d\phi = \cos^2 \psi, \]

\[ a^{(11)} = (1/2\pi) \int_0^{2\pi} \sin^4 \psi \cos^4 \phi \, d\phi = \frac{3}{8} \sin^4 \psi = a^{(22)}, \quad a^{(33)} = (1/2\pi) \int_0^{2\pi} \cos^4 \psi \, d\phi = \cos^4 \psi, \]

\[ a^{(23)} = (1/2\pi) \int_0^{2\pi} \cos^2 \psi \sin^2 \phi \, d\phi = \frac{1}{8} \cos^2 \psi \sin^2 \psi = a^{(32)} = a^{(31)} = a^{(13)}, \]

\[ a^{(12)} = (1/2\pi) \int_0^{2\pi} \sin^4 \psi \, d\phi = \frac{3}{8} \sin^4 \psi = a^{(21)}. \]
Equations (41) become

\[ c_{\text{1111}} = (-va^3/2\mu)(\bar{U}_{33}(2a^2 + 4\lambda\mu\sin^2\psi + 3\mu^2\sin^4\psi) + \bar{U}_{11}\mu^2\sin^2\psi(4 - 3\sin^2\psi)) = c_{\text{1222}}, \]

\[ c_{\text{1233}} = (-va^3/\mu)(\bar{U}_{33}(\lambda + 2\mu\cos^2\psi) + \bar{U}_{11}\mu^2\cos^2\psi\sin^2\psi), \]

\[ c_{\text{1122}} = (-va^3/2\mu)(\bar{U}_{33}(2a^2 + 4\lambda\mu\sin^2\psi + \mu^2\sin^4\psi) - \bar{U}_{11}\mu^2\sin^4\psi), \]

\[ c_{\text{1233}} = (-va^3/\mu)(\bar{U}_{33}(\lambda + \mu\sin^2\psi)(\lambda + 2\mu\cos^2\psi) - \bar{U}_{11}\mu^2\cos^2\psi\sin^2\psi) = c_{\text{1133}}, \]

\[ c_{\text{1212}} = (-va^3/2\mu)(\bar{U}_{33}\sin^4\psi + \bar{U}_{11}\sin^2\psi(2 - \sin^2\psi)) \]

and, as required by hexagonal symmetry,

\[ c_{\text{1111}} - c_{\text{1122}} = 2c_{\text{1212}} = -(va^3/2\mu)(\bar{U}_{33}\mu^2\sin^4\psi + \bar{U}_{11}\mu^2\sin^2\psi(2 - \sin^2\psi)). \]

**APPENDIX B**

**Scattering attenuation due to cracks oriented at a fixed angle to a given axis**

The attenuation coefficients for P- and shear waves for such a system are calculated by summing the attenuation due to a single orientation over the set of orientations of the crack normals.

Hudson (1981) gives the scattering attenuation coefficients for qP-waves of frequency \( \omega \) due to a number density \( v \) of cracks of radius \( a \) in material with wavespeeds \( \alpha \) and \( \beta \) to be

\[ \gamma_P = \frac{\omega}{\beta}(va^3)(\frac{\omega\alpha}{\alpha})^3 \frac{1}{30\pi \ell}[\frac{3}{2} + \frac{\beta^5}{\alpha^5}](\bar{U}_{11})^2(1 - c^2)c^2 + \left(2 + \frac{15\beta}{4\alpha} - \frac{10\beta^3}{3\alpha} - 8\frac{\beta^5}{5\alpha}\right)(\bar{U}_{33})^2\left(\frac{\alpha^2}{\beta^2} - 2 + 2c^2\right)^2, \]

where \( \bar{U}_{11} \) and \( \bar{U}_{33} \) depend on the conditions of the crack faces, as defined in equation (2) of the main text: \( c = n \cdot m \) where \( m \) is the unit normal to the crack and \( n \) the direction of propagation of the wave. The quality factor \( Q_P \) is related to \( \gamma_P \) by \( Q_P^{-1} = 2\gamma_P/\omega \).

If we put \( m = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \) then \( c = n_1 \sin \theta \cos \phi + n_2 \sin \theta \sin \phi + n_3 \cos \theta \). Summing over all orientations of the cracks we get

\[ \gamma_P = \frac{\omega}{\beta}(va^3)(\frac{\omega\alpha}{\alpha})^3 \frac{1}{30\pi \ell}[\frac{3}{2} + \frac{\beta^5}{\alpha^5}](\bar{U}_{11})^2(1 - c^2)c^2 + \left(2 + \frac{15\beta}{4\alpha} - \frac{10\beta^3}{3\alpha} - 8\frac{\beta^5}{5\alpha}\right)(\bar{U}_{33})^2\left(\frac{\alpha^2}{\beta^2} - 2 + 2c^2\right)^2, \]

where \( v \) remains the overall number density of cracks,

\[ C_2 = \int_{0}^{2\pi} p(\theta, \phi)c^2 \sin \theta d\theta d\phi \]

and

\[ C_4 = \int_{0}^{2\pi} p(\theta, \phi)c^4 \sin \theta d\theta d\phi, \]

where \( p(\theta, \phi) \) is the probability density for orientations of crack normals along the direction \( (\theta, \phi) \).

The attenuation coefficient for the quasi-shear waves is (Hudson 1981)

\[ \gamma_S = \frac{\omega}{\beta}(va^3)(\frac{\omega\alpha}{\alpha})^3 \frac{1}{30\pi \ell}[\frac{3}{2} + \frac{\beta^5}{\alpha^5}](\bar{U}_{11})^2(4c^4 - 4c^2 + 1 - d^2 + 4c^2d^2) + 4\left(2 + \frac{15\beta}{4\alpha} - \frac{10\beta^3}{3\alpha} - 8\frac{\beta^5}{5\alpha}\right)(\bar{U}_{33})^2(c^2 - c^2d^2), \]

where \( c = n \cdot m \) as before and \( d = (n \times b) \cdot m \), where \( b \) is the polarization of the wave. The quality factor for shear waves, \( Q_S \), is given by

\[ Q_S^{-1} = 2\beta\gamma_S/\omega. \]

Summing the cracks again we get

\[ \gamma_S = \frac{\omega}{\beta}(va^3)(\frac{\omega\alpha}{\alpha})^3 \frac{1}{30\pi \ell}[\frac{3}{2} + \frac{\beta^5}{\alpha^5}](\bar{U}_{11})^2(4C_4 - 4C_2 + 1 - D_2 + 4D_4) + 4\left(2 + \frac{15\beta}{4\alpha} - \frac{10\beta^3}{3\alpha} - 8\frac{\beta^5}{5\alpha}\right)(\bar{U}_{33})^2(C_2 - C_4 - D_4), \]

where

\[ D_2 = \int_{0}^{2\pi} p(\theta, \phi)d^2 \sin \theta d\theta d\phi \]
Cracks oriented randomly at a fixed angle to a given direction

Let the given direction be the \( x_3 \) axis and the fixed angle be \( \psi \). Then \( \mathbf{m} = (\sin \psi \cos \phi, \sin \psi \sin \phi, \cos \psi) \) with \( \phi \) taking any value between 0 and \( 2\pi \). The attenuation coefficients are given by equations (B2) and (B5) with

\[
C_2 = (1/2\pi) \int_0^{2\pi} \sin^2 \phi \, d\phi = (1/2\pi) \int_0^{2\pi} (n_1 \sin \psi \cos \phi + n_2 \sin \psi \sin \phi + n_3 \cos \psi)^2 \, d\phi,
\]

and similar equations for \( C_4 \), \( D_2 \) and \( D_4 \). After integration we have

\[
C_2 = \frac{(n_1^2 + n_2^2) \sin^2 \psi}{2} + n_3^2 \cos^2 \psi
\]

(B8a)

and

\[
D_2 = \frac{(q_1^2 + q_2^2) \sin^2 \psi}{2} + q_3^2 \cos^2 \psi,
\]

(B8b)

where \( \mathbf{q} = \mathbf{n} \times \mathbf{b} \).

\[
C_4 = \frac{1}{4} (n_1^2 + n_2^2)^2 \sin^4 \psi + 3(n_1^2 + n_2^2)n_3^2 \sin^2 \psi \cos^2 \psi + n_3^4 \cos^4 \psi,
\]

(B9)

\[
D_4 = \frac{1}{4} \sin^4 \psi (3(n_1^4 + n_2^4) + (n_1^2 - n_2^2)^2) + \frac{1}{2} \sin^2 \psi \cos^2 \psi \left[ 6(n_1n_2 + n_2n_3)n_3 \right]
\]

\[
+ (n_1n_3 - n_2n_2)^2 + (n_2n_3 - n_3n_3)^2 + n_3^2 \sin^2 \psi \cos^2 \psi.
\]

As the \( z \) axis is an axis of symmetry for the overall material, we may choose \( \mathbf{n} = (n_1, 0, n_3) \) without loss of generality. The level of approximation used in this first-order term allows us to substitute for \( \mathbf{b} \) and \( \mathbf{q} \) the corresponding quantities for an isotropic material:

<table>
<thead>
<tr>
<th>Polarization</th>
<th>( \mathbf{q} = \mathbf{n} \times \mathbf{b} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( qP )</td>
<td>( \mathbf{b} = (n_1, 0, n_3) )</td>
</tr>
<tr>
<td>( qSR )</td>
<td>( \mathbf{b} = (0, 1, 0) )</td>
</tr>
<tr>
<td>( qSP )</td>
<td>( \mathbf{b} = (n_3, 0, -n_1) )</td>
</tr>
</tbody>
</table>

The attenuation coefficients are now given by equations (B2) and (B5) with

\[
C_2 = \left( \frac{n_1^2}{2} \right) \sin^2 \psi + n_3^2 \cos^2 \psi
\]

(B10a)

and

\[
C_4 = \frac{3}{4} n_1^4 \sin^4 \psi + 3n_1^2 n_3^2 \sin^2 \psi \cos^2 \psi + n_3^4 \cos^4 \psi,
\]

(B10b)

for all three waves. For \( qSR \),

\[
D_2 = \left( \frac{n_1^2}{2} \right) \sin^2 \psi + n_3^2 \cos^2 \psi
\]

(B11a)

and

\[
D_4 = \frac{3}{4} n_1^2 n_3^2 \sin^4 \psi + \frac{1}{2} (n_1 - 4n_1^2 n_3^2 + n_3^4) \sin^2 \psi \cos^2 \psi + n_3^2 n_3^4 \cos^4 \psi,
\]

(B11b)

and for \( qSP \),

\[
D_2 = \frac{1}{4} \sin^2 \psi
\]

(B12a)

and

\[
D_4 = \left( \frac{n_1^2}{8} \right) \sin^4 \psi + (n_3^2/2) \sin^2 \psi \cos^2 \psi.
\]

(B12b)

These angular variations of attenuation are converted into elastic constants by the method of Crampin (1984) as described in the main text, Section 2.2.