Photon Structure Function at Small $x$

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The parton distributions of real photons at small $x$ are presented by taking account of the following two points. One is that the Pomeron-cut exchange rather than the Pomeron exchange is taken into the hadron-like parts of the gluon and the sea quark distributions. The other is that the $O(\alpha_s(Q^2))$ term is introduced into the point-like parts of the parton distributions. Using these distributions we can show that predictions for the photon structure function $F_2^\gamma(x, Q^2)$ are consistent with new data at small $x$.

§1. Introduction

The structure function of photons has been investigated through deep inelastic electron-photon ($e\gamma$) scattering. Experimentally, information regarding it is extracted from the process ($e^+e^-\rightarrow e^+e^-\text{hadrons}$), whose cross section is related to the $e\gamma$ cross section by the Weizsacker-Williams approximation.\(^1\) In particular, the structure function of a real photon in the single tagging experiment is studied by using a kinematical analogy to the usual deep inelastic lepton-nucleon scattering and in the kinematic region, where the transverse structure function $F_2^\gamma(x, Q^2)$ is only accessible. The function $F_2^\gamma(x, Q^2)$ can be factorized\(^2\) into parts of short distance-dependences (the coefficient functions) which are calculable in perturbative QCD, and parts of long distance-dependences (the parton distribution functions), which are incalculable. The parton distributions evolve according to the inhomogeneous Dokshitzer, Gribov, Lipatov, Altarelli and Parisi (DGLAP) Equations.\(^3\)-\(^5\) The solutions of the inhomogeneous DGLAP Equations are constructed from a superposition of a general solution of the homogeneous DGLAP Equation and a particular solution of the inhomogeneous equation. The general solution corresponds to a hadron-like part of the photon, which turns into a bound $q\bar{q}$ state through an effective coupling to a vector meson-like state (Fig. 1(a)). The particular solution corresponds to a point-like part of the photon, which has a point-like coupling to a $q\bar{q}$ pair (Fig. 1(b)). These two parts are divided by the size of the transverse momenta of quarks relative to the target photon. At the values of $Q^2$ where the data are taken, the distributions depend on the initial value of $Q_0^2$, \(^5\) and on the distribution at $Q_0^2$ (the hadronic initial distribution). In many cases, $Q_0^2$...
can be taken sufficiently small for $F_2^\gamma(x, Q^2)$ to be hadronic, and the vector meson dominance model (VMD) is adopted as the hadronic initial distribution.

Recently, by analyzing the parton distributions of the photon, it has been shown that the hadronic initial distribution dominates at small $x$.\(^7\) This means that through the $x$-dependence of the hadronic initial distribution, extrapolation to small $x$ is possible in the framework of the DGLAP Equations in the next-to-leading order (NLO). On the other hand, it has been shown that the photon structure function is sensitive to the gluon distribution only at small $x$.\(^8\) Therefore, the behavior of the structure function at small $x$ can be assigned to the form of the initial gluon distribution. In fact, the analysis of the nucleon structure function has been performed by introducing the Lipatov behavior in the initial value of the solution of the NLO-DGLAP Equations. As a result, predictions of a steep rise at small $x$ agree with the experimental data.\(^9\)

However, the experimental data for the photon structure function at small $x$ are poor as compared with those of the nucleon structure function. Recently, at the Lepton-Photon '97 conference (LP-97), new data of the structure function at small $x$ were presented.\(^10\)

In this paper, the hadronic initial distributions of the gluon and the sea quark at small $x$ are expressed by using the Pomeron cut, and the point-like parts of the parton distributions are calculated up to the $O(\alpha_S(Q^2))$ term. It is shown that our predictions are in agreement with the new data from LP-97.

In §2, we obtain a general solution of the homogeneous NLO-DGLAP Equation as the hadron-like part of the photon distribution. In the initial distributions of the gluon and the sea quark, the Pomeron cuts are introduced. In the inhomogeneous equation, the particular solution containing up to the $O(\alpha_S(Q^2))$ term is obtained in §3. In §4, we compare the resulting parton distribution with the experimental data from the Lepton-Photon '97 conference. The differences between the effect of the Pomeron exchange and that of the Pomeron-cut exchange, and the contribution of the $O(\alpha_S(Q^2))$ term are discussed.

\section{The hadron-like part of the parton distributions}

The relation between $F_2^\gamma(x, Q^2)$ and the parton distributions $q_{NS}(x, Q^2)$, $\Sigma(x, Q^2)$ and $G(x, Q^2)$ is
\begin{align}
\frac{1}{x} F_2^\gamma(x, Q^2) &= q_{NS}(x, Q^2) + \langle e^2 \rangle \Sigma(x, Q^2) \\
&+ \frac{\alpha_S(Q^2)}{4\pi} B_q(x)^* q_{NS}(x, Q^2) + \langle e^2 \rangle \Sigma(x, Q^2) \right) \\
&+ \langle e^2 \rangle \frac{\alpha_S(Q^2)}{4\pi} B_G(x)^* G(x, Q^2) + 3f \langle e^4 \rangle \frac{\alpha}{4\pi} B_\gamma(x),
\end{align}
where $\langle e^k \rangle = 1/f \sum_i e_i^k$. Here, we have the definition in the terms of the quark distributions, $q_{i}(x, Q^2)$, $q_{NS}(x, Q^2) = \sum_i (e_i^2 - \langle e^2 \rangle)(q_i + \overline{q}_i)$ for the flavor nonsinglet case and $\Sigma(x, Q^2) = \sum_i (q_i + \overline{q}_i)$ for the flavor singlet case. Here $f$ is the number of active flavors. The Willson coefficient functions $B_{q,G,\gamma}(x)$ are calculated in the one-loop order.\(^3\) The symbol $^*$ represents a convolution of two functions.
The moment representation is defined by \( f(n, t) = \int_0^1 x^{n-1} f(x, t) dx \). This is adopted for the parton distributions \( q_i(x, t) \) and the splitting functions \( P_{ij}(x, t) \) for \( ij = (NS, qq, qG, Gq, GG) \) and \( P_{\gamma i}(x, t) \) for \( i = NS, q, G \). The moment representation of the inhomogeneous DGLAP Equations at the next-to-leading order (NLO) can be written for the nonsinglet massless quark distribution as

\[
\frac{d}{dt} q_{NS}(n, t) = \left[ 1 - \frac{\beta_1}{2\beta_0} \left( \frac{\alpha_S(t)}{2\pi} \right) \right] \left[ \frac{\alpha_S(t)}{2\pi} \right]^{-1} p_{NS}^{(0)}(n) + p_{NS}^{(1)}(n) \\
+ \left[ p_{NS}^{(0)}(n) + \frac{\alpha_S(t)}{2\pi} R_{NS}(n) \right] q_{NS}(n, t).
\]

Similarly, for the singlet and the gluon distributions we have

\[
\frac{d}{dt} \Sigma(n, t) = \frac{\alpha}{2\pi} \left[ 1 - \frac{\beta_1}{2\beta_0} \left( \frac{\alpha_S(t)}{2\pi} \right) \right] \left[ \frac{\alpha_S(t)}{2\pi} \right]^{-1} p_{q\gamma}^{(0)}(n) + p_{q\gamma}^{(1)}(n) \\
+ \left[ p_{q\gamma}^{(0)}(n) + \frac{\alpha_S(t)}{2\pi} R_{q\gamma}(n) \right] \Sigma(n, t) \\
+ \left[ p_{G\gamma}^{(0)}(n) + \frac{\alpha_S(t)}{2\pi} R_{G\gamma}(n) \right] G(n, t),
\]

\[
\frac{d}{dt} G(n, t) = \frac{\alpha}{2\pi} \left[ 1 - \frac{\beta_1}{2\beta_0} \left( \frac{\alpha_S(t)}{2\pi} \right) \right] p_{G\gamma}^{(1)}(n) + \left[ p_{G\gamma}^{(0)}(n) + \frac{\alpha_S(t)}{2\pi} R_{G\gamma}(n) \right] \Sigma(n, t) \\
+ \left[ p_{G\gamma}^{(0)}(n) + \frac{\alpha_S(t)}{2\pi} R_{G\gamma}(n) \right] G(n, t),
\]

where \( t = (1/2) \ln(\alpha_S(Q_0^2)/\alpha_S(Q^2)) \) and \( \alpha = 1/137 \). Here, the \( O(\alpha_S(t^2)) \) terms are neglected in the expansion of the splitting functions \( P_{ij}(n, t) \) and \( P_{\gamma i}(n, t) \) in power series of \( \alpha_S(t) \). Also, \( R_{ij} = P_{ij}^{(1)}(n) - (\beta_1/2\beta_0) P_{ij}^{(0)}(n) \) for \( ij = (NS, qq, qG, Gq, GG) \). Furthermore, the NLO running coupling \( \alpha_S(Q^2) \) satisfies \( d\alpha_S(Q^2)/d\ln Q^2 = \alpha_S(Q^2)\{(-\beta_0(\alpha_S(Q^2))/4\pi - \beta_1(\alpha_S(Q^2)/4\pi)^2 \}, \) with \( \beta_0 = 11 - 2f/3 \) and \( \beta_1 = 102 - 38f/3 \).

The general solutions of the homogeneous equations are represented as

\[
q_{NS,H}(n, Q^2) = \left\{ 1 - \frac{2}{\beta_0} R_{NS}(n) \frac{\alpha_S(Q^2)}{2\pi} + \frac{2}{\beta_0} R_{NS}(n) \frac{\alpha_S(Q_0^2)}{2\pi} \right\} \\
\times L(Q^2) - \frac{2}{\beta_0} p_{NS}^{(0)}(n) q_{NS}(n, Q_0^2),
\]

\[
\left\{ \begin{array}{l} \Sigma_H(n, Q^2) \\ G_H(n, Q^2) \end{array} \right\} = \left\{ \begin{array}{l} B(\lambda_-, Q_0^2) \Sigma(n, Q_0^2) - A(\lambda_-, Q_0^2) G(n, Q_0^2) \\ A(\lambda_-, Q_0^2) B(\lambda_-, Q_0^2) - B(\lambda_+, Q_0^2) A(\lambda_-, Q_0^2) \end{array} \right\} \\
\times \left\{ \begin{array}{l} A(\lambda_+, Q^2) \\ B(\lambda_+, Q^2) \end{array} \right\} L(Q^2) - \frac{2}{\beta_0} \lambda_+ + [\lambda_+] \rightarrow [\lambda_-],
\]

where

\[
L(Q^2) = \frac{\alpha_S(Q^2)}{\alpha_S(Q_0^2)}.
\]
The hadronic initial distributions, \( q_{NS}(n, Q_0^2) \), \( \Sigma(x, Q_0^2) \) and \( G(n, Q_0^2) \) correspond to the initial values of these solutions and are defined in the non-perturbative region. \( A(\lambda_{\pm}, Q^2) \) and \( B(\lambda_{\pm}, Q^2) \) are discussed in the Appendix.

Recent studies of the proton structure function at small \( x \) have produced important pieces of information theoretically and experimentally. One piece of information is that the gluon and sea quark distributions increase with decreasing \( x \), and the initial parton distributions dominate them at small \( x \).

Here, according to this observation we assume the form of the hadronic initial distributions \( q_{NS}(n, Q_0^2) \), \( \Sigma(x, Q_0^2) \) and \( G(n, Q_0^2) \) at small \( x \). Then we express the hadronic initial distributions by using VMD arguments, as is commonly done. We use \( Q_0^2 = 1 \text{GeV}^2 \), for which the results using the VMD model are in good agreement with the measured total \( \gamma-\gamma \) cross section. The VMD photon structure function is defined as

\[
F_{\gamma,\text{VMD}}(x) = \frac{4\pi\alpha}{f_\rho^2} \sum e_i^2 x q_i^\rho(x, Q_0^2),
\]

(8)

where \( f_\rho^2/4\pi = 2.2 \). It is assumed that the quark distributions in the \( \rho_0 \) meson \( q_i^\rho(x, Q_0^2) \) can be represented by the sum of corresponding distributions in \( \pi^\pm \) mesons. The pion distributions are expressed in terms of valence quarks, \( V_\pi(x) \), and sea quark, \( \zeta_\pi(x) \), distributions. Thus we obtain

\[
F_{\gamma,\text{VMD}}^\pi(x) = \frac{4\pi\alpha}{f_\rho^2} x \left( \frac{5}{9} V_\pi(x) + \frac{4}{3} \zeta_\pi(x) \right),
\]

(9)

where the charm sea quark is ignored since \( Q_0^2 < m_c^2 \) (\( m_c \) is the mass of a charm quark). Then the parton initial distributions are

\[
\sum(x, Q_0^2) = \frac{4\pi\alpha}{f_\rho^2} (2V_\pi(x) + 6\zeta_\pi(x))
\]

(10)

and

\[
q_{NS}(x, Q_0^2) = \frac{4\pi\alpha}{f_\rho^2} \left( -\frac{1}{3} \zeta_\pi(x) \right).
\]

(11)

On the other hand, \( G(x, Q_0^2) \) affects the evolution to higher \( Q^2 \) at very small \( x \) and is taken here as

\[
G(x, Q_0^2) = \frac{4\pi\alpha}{f_\rho^2} G^\pi(x).
\]

(12)

For the pion parton distributions, many authors have used the form given in Ref. 3),

\[
\begin{align*}
x V_\pi(x) &= ax^{1-\alpha P(0)}(1-x), \\
x \zeta_\pi(x) &= bx^{1-\alpha P(0)}(1-x)^3, \\
x G^\pi(x) &= cx^{1-\alpha P(0)}(1-x)^3,
\end{align*}
\]

(13)
where each power of $x$ is chosen in line with Regge pole exchange theory, and the power of $(1-x)$ is in line with counting rules.\(^{13}\) Regge pole exchange is a generalization of single particle exchange. The exchange of $q\bar{q}$ bound states (Fig. 2(a)) corresponding to the exchange of mesons can be related to the Regge pole, which is characterized by quantum numbers such as charge, isospin, strangeness, G parity, etc. The Regge poles carrying the quantum numbers of the vacuum are called the Pomeron ($P$), which can be thought of as corresponding to an exchange of a pair of gluons (Fig. 2(b)). In addition to the exchange of a single Reggeon or Pomeron, simultaneous exchange of more Reggeons or Pomerons is also possible. This gives rise to Regge cuts (Fig. 2(c)). On the other hand, many soft gluon emissions appear at small $x$, and the small $x$ rise is generated as a result. Therefore, it is necessary to exchange multiple Pomerons for the gluon and the sea quark distribution. This corresponds to Pomeron-cut exchange. Thus, we adopt Pomeron-cut exchange instead of Pomeron exchange for the gluon and the sea quark distribution. Here, the properties of the VMD model have been kept in the valence quark distribution whose behavior at small $x$ is described by the $\rho$-Regge pole exchange. The high-energy behavior of the two-body amplitude due to Regge-cut exchange is expressed by $A^C(s,t) \approx s^{\alpha_C(t)}/\ln s$,\(^{14}\) as compared with $A^\text{Pole}(s,t) \approx s^{\alpha_R(t)}$, which is the high-energy behavior due to Regge-pole exchange. These $x$-representations correspond with $x^{-\alpha_C(t)}/\ln(1/x)$ and $x^{-\alpha_R(t)}$. Here we have used the relation $s = Q^2(1-x)/x$, in which $s$ is the second power of the energy of the hadronic system produced in the $\gamma-\gamma$ reaction. Accordingly, we take the following form expressed by the Pomeron-cut exchange\(^{14}\) for the gluon distribution and the sea quark distribution:

\[
\begin{align*}
xG^g(x) &= b(x^{1-\alpha_C(0)}/\ln(1-x))(1-x)^3, \\
x\zeta^g(x) &= c(x^{1-\alpha_C(0)}/\ln(1/x))(1-x)^5,
\end{align*}
\]

where $\alpha_C(t) = \{\alpha_P(t) + \alpha_P(t) - 1\}$ gives the position of the branch point trajectory. Using $\alpha_P(0) \approx 1.08$, we get Lipatov behavior.\(^{15}\) $x^{-\lambda}$. Then the value of $\lambda$ produced by $P-P$ cut is larger than that of $\lambda$ by $P$ pole. The $1/\ln(1/x)$ term arises from the integral over the phase between the two $P-P$ poles. We can see that the behavior
at the small $x$ in Eqs. (14) and (15) is similar to the behavior of the gluon distribution obtained from the solution of the Balitzkij, Fadin, Kuraev, Lipatov (BFKL) Equation,\textsuperscript{15}

$$xG^N(x, Q^2) \propto (x^{1-\alpha_S^2}/\ln(1/x)^{1/2}),$$

where $\alpha_S^2 = 1 + (12\ln(2)/\pi)$ and $\bar{\alpha}_s$ is fixed, i.e., $\bar{\alpha}_s = 0.2$ for a typical value.

For the pion parton distributions with forms obtained above, we now shall determine the constants $a$, $b$ and $c$. For the valance quark, we use the form of the $\rho$ Regge pole exchange in Eq. (13), and thus the constant $a$ is found to be $a = 0.75$ from the net number of quarks in the meson. This means that the valance quarks carry 40% of the meson momentum. For the sea quarks and the gluons, the form of the Pomeron-cut exchange in Eqs. (14) and (15) is adopted. Here the constants $b$ and $c$ are determined by the momentum conservation,\textsuperscript{5} as the sea quarks take 10% of the momentum of the meson and the gluons take 50%.

Next, by performing the Mellin transformation of $V^x(x)$, $\zeta^x(x)$ and $G^x(x)$, these moment-$n$ space representations are obtained. Using them, we can obtain moment-$n$ space representations of Eqs. (10)–(12). We use these for $q_{NS}(n, Q^2_0)$, $\Sigma(x, Q^2_0)$ and $G(n, Q^2_0)$ in Eqs. (5) and (6).

In this section, we modified the initial distributions of the gluon and of the sea quark by using the Pomeron-cut exchange instead of the traditional Pomeron exchange. As these initial distributions have no $Q^2$ dependence, the difference between the effect of the Pomeron-cut exchange and that of the Pomeron exchange does not depend on $Q^2$. We obtained the representations with the $x^{1-\alpha_S(0)}$ term describing the increase in $x\zeta^x(x)$ and $xG^x(x)$ at small $x$ and also the $1/\ln(1/x)$ term suppressing their growth in the small $x$ limit.

§3. The point-like part

In this section, we obtain the particular solution of the inhomogeneous DGLAP Equations (2)–(4), where we retain the $O(\alpha_S(Q^2))$ terms in inhomogeneous terms.

The solution for the nonsinglet equations is expressed as

$$q_{NS,PL}(n, Q^2) = \frac{\alpha}{2\pi} \left[ a_{11}(n) \frac{1 - L^{-1/2} P_{NS}^{(0)}(n)}{1 - \frac{2}{\alpha_S} P_{NS}^{(0)}(n)} \right] \left( \frac{\alpha_S(Q^2)}{2\pi} \right)^{-1}$$

$$+ \frac{\alpha}{2\pi} \left[ a_{12}(n) \frac{1 - L^{-1/2} P_{NS}^{(0)}(n)}{1 - \frac{2}{\alpha_S} P_{NS}^{(0)}(n)} + a_{21}(n) \frac{1 - L^{-1/2} P_{NS}^{(0)}(n)}{1 - \frac{2}{\alpha_S} P_{NS}^{(0)}(n)} \right]$$

$$+ \frac{\alpha}{2\pi} \left[ a_{22}(n) \frac{1 - L^{-1/2} P_{NS}^{(0)}(n)}{1 - \frac{2}{\alpha_S} P_{NS}^{(0)}(n)} + a_{31}(n) \frac{1 - L^{-1/2} P_{NS}^{(0)}(n)}{1 - \frac{2}{\alpha_S} P_{NS}^{(0)}(n)} \right] \left( \frac{\alpha_S(Q^2)}{2\pi} \right)^{1/2},$$

(16)

where $L = \alpha_S(Q^2)\alpha_S(Q^2_0)$. Also the solutions for the singlet distribution, Eq. (3)
and the gluon distribution, Eq. (4) are

$$\left\{ \begin{array}{l} \Sigma_{PL}(n, Q^2) \\ G_{PL}(n, Q^2) \end{array} \right\} = \left( \frac{\alpha}{2\pi} \right) \left\{ \begin{array}{l} b_{11}(n) \\ c_{11}(n) \end{array} \right\} \frac{1 - L^{-\frac{2}{\beta_0}\lambda_+(n)}}{1 - \frac{2}{\beta_0}\lambda_+(n)} \left( \frac{\alpha S(Q^2)}{2\pi} \right)^{-1}$$

$$+ \left( \frac{\alpha}{2\pi} \right) \left\{ \begin{array}{l} b_{12}(n) \\ c_{12}(n) \end{array} \right\} \frac{1 - L^{-\frac{2}{\beta_0}\lambda_+(n)}}{1 - \frac{2}{\beta_0}\lambda_+(n)} + \left\{ \begin{array}{l} b_{21}(n) \\ c_{21}(n) \end{array} \right\} \frac{1 - L^{-\frac{2}{\beta_0}\lambda_+(n)}}{1 - \frac{2}{\beta_0}\lambda_+(n)}$$

$$+ \left( \frac{\alpha}{2\pi} \right) \left\{ \begin{array}{l} b_{22}(n) \\ c_{22}(n) \end{array} \right\} \frac{1 - L^{-\frac{2}{\beta_0}\lambda_+(n)}}{1 - \frac{2}{\beta_0}\lambda_+(n)} + \left\{ \begin{array}{l} b_{31}(n) \\ c_{31}(n) \end{array} \right\} \frac{1 - L^{-\frac{2}{\beta_0}\lambda_+(n)}}{1 - \frac{2}{\beta_0}\lambda_+(n)} \left( \frac{\alpha S(Q^2)}{2\pi} \right)$$

$$+ \left( \frac{\alpha}{2\pi} \right) [\lambda_+ \leftrightarrow \lambda_-], \quad (17)$$

where $a_{ij}(n)$, $b_{ij}(n)$ and $c_{ij}(n)$ are given in the Appendix and $\lambda_+$, $\lambda_-$ are defined in Eq. (7). Now we show that the above solutions are reasonable. First, we confirmed numerically that the point-like parts of the parton distributions (except for the $O(\alpha_S(Q^2))$ terms) produce the result given in Ref. 3) and that the $n$-moment of the distributions in the $Q^2 \to \infty$ limit are independent of the values $Q^2_0$. Next, the well-known solutions to leading order can be obtained from the above expressions by taking $R_{ij}(n) \to 0$, $\beta_1(n) \to 0$ and $P_{ij}^1(n) \to 0$ in Eqs. (16) and (17).

On the other hand, our solutions have simple poles at the value $n_M$ of $n$ for each order term of $\alpha_S(Q^2)$. For example, $n_M$ is the value of $n$ which satisfies $1 - \frac{2}{\beta_0}\lambda_+(n) = 0$ for the $O((\alpha_S)^{-1})$ term. These values for $f = 4$ are given in Ref. 16). Then, we calculated the position of the $n$-pole $n_M$ for $f = 3$, and we found the following same tendency: $n_M = 0.385$ (NS) and $n_M = 1.725$ (S,G) for the $O((\alpha_S)^{-1})$ term, $n_M = 1.0$ (NS) and $n_M = 2.0$ (S,G) for $O((\alpha_S)^0)$, and $n_M = 6.0$ (NS), $n_M(\lambda_+) = 7.0$ (S,G) and $n_M(\lambda_-) = 3.0$ (S,G) for $O((\alpha_S)^1)$. As mentioned in Ref. 16), when the solutions are transformed into $x$-space by the inverse Mellin transform, these $n$-poles correspond to the origin singularities of the form $(1/x)^{n_M}$ in $x$-space. Thus the introduction of the $O(\alpha_S(Q^2))$ terms makes the behavior of the power singularities more severe. One can, however, find that in Eqs. (16) and (17) these singularities are canceled by vanishing numerators at the same points, and they are replaced in the $\alpha_S(Q^2) \to 0$ limit by a logarithmic divergence, since $\lim_{d \to 0} \left(1 - L^d/d\right) = -\ln L$. Also, from Eqs. (16) and (17), one can find $g_{NS,PL}(n, Q_0^2) = \Sigma_{PL}(n, Q_0^2) = \Sigma_{PL}(n, Q_0^2) = G_{PL}(n, Q_0^2) = 0$. Thus, we can assume that the contribution of the hadron-like parts dominates in the region of small $Q^2$. In fact, we have demonstrated this numerically, as shown by the graph in the next section.

In this section, we obtained the particular solution of the inhomogeneous DGLAP Equations taking account of up to $O(\alpha_S(Q^2))$ terms for the point-like parts. We have discussed that the added $O(\alpha_S(Q^2))$ terms have more severe behavior at small $x$. Generally, DGLAP Equations are used to sum up terms like $(\alpha_S \ln(Q^2/L^2))^n$. In small $x$ region, terms like $(\alpha_S \ln(Q^2/s))^n \approx (\alpha_S \ln(x/(1-x)))^n$ should also be resummed. This is due to the infrared divergence. The coefficients of $O(\alpha_S(Q^2))$ terms in the distribution functions become large at small $x$, as discussed above. Hence we introduced only the $O(\alpha_S(Q^2))$ terms in the distribution functions in NLO;
that is $O(\alpha_S(Q^2))$ terms arising through the combination of the Willson coefficient functions ($B_{q,G,\gamma}(x)$) and the distribution functions have been dropped.

### §4. Results and discussion

In this section, first we show that the Pomeron cut exchange in the hadron-like part and the $O(\alpha_S(Q^2))$ term in the point-like part discussed in the previous sections make important contributions to $F_2^\gamma(x, Q^2)$. Next, our predictions are compared with the new data from PH-97, and we analyze the contributions of the terms introduced here.

Each moment-$n$ space distribution, $q_{NS}(n, Q^2)$, $\Sigma(n, Q)$ and $G(n, Q^2)$, can be expressed as the sum of the hadron-like part and the point-like part of the parton distribution obtained in the previous sections. Using these $n$-space distributions and the coefficient functions $B_q(n)$, $B_G(n)$ and $B_\gamma(n)$ in Ref. 3), we obtain the $n$-space structure function $F_2^\gamma(n, Q^2)$ from Eq. (1). Then, in order to obtain the $x$-space structure function $F_2^\gamma(x, Q^2)$, $F_2^\gamma(n, Q^2)$ is inverted according to

$$F_2^\gamma(n, Q^2) = \frac{1}{\pi} \int_0^\infty dz \text{Im}[e^{i\varphi}x^{-h-ze^{i\varphi}}F_2(n = h + ze^{i\varphi}, Q^2)].$$  \hspace{1cm} (18)

Here the contour of integration is chosen as $3)^{,17)$

\[
\varphi = \frac{3}{4}\pi,
\]

\[
h = 0.80 \text{ for nonsinglet},
\]

\[
h = 1.80 \text{ for singlet},
\]

\[
h = 1.75 \text{ for gluon},
\]

where the values of $h$ are taken to the right of all singularities of each distribution in the complex $n$ plane.

In Fig. 3(a), the experimental data\(^{18)}\) from LEP1 and the predictions found with the parametrization by GRV (Glück, Reya and Vogt)\(^{19)}\) and by SaS (Schuler and Sjostrand)\(^{20)}\) are shown. Our results are shown for $Q^2 = 1.86$ GeV$^2$ and $Q^2 = 3.76$ GeV$^2$ in Fig. 3(b). One can see that our predictions are in agreement with the experimental data. The differences between the small $x$ photon structure function $F_2^\gamma(x, Q^2)$ obtained from the Pomeron exchange (short-dashed curve) and that from the Pomeron-cut exchange (solid curve) are clear at $Q^2 = 1.86$ GeV$^2$, $Q^2 = 3.76$ GeV$^2$ and $Q^2 = 14.7$ GeV$^2$ in Fig. 4. There, we have also shown the contributions of the $O(\alpha_S(Q^2))$ term to the small $x$ $F_2^\gamma(x, Q^2)$ using the difference between the long-dashed curve and the solid curve. As mentioned in the previous section, the difference between the Pomeron exchange and the Pomeron-cut exchange is also seen at the previous three values of $Q$. However, the effect of the $O(\alpha_S(Q^2))$ term in the point-like part increases with $Q^2$. We can also see that it is nearly zero, as two lines coincide at $Q^2 = 1.86$ GeV$^2$. In Fig. 4, evidence of the Pomeron-cut exchange in the hadron-like part of the photon at small $x$ is clearly shown. Here the Pomeron-cut exchange makes the rise weak through the term $1/\ln(1/x)$. In Fig. 5, it is shown.
Photon Structure Function at Small $x$

Fig. 3. The LEP1 data of $F_2^\gamma(x)/\alpha$ at small $x$ and the predictions of the GRV (HO) parameterization, the GRV (LO) parameterization and the SaS parameterization are shown in (a). Our result is shown in (b). This includes the contributions of the Pomeron-cut exchange and the $O(\alpha_s(Q^2))$ term.

that the hadron-like part dominates at small $x$ and small $Q^2$. On the other hand, the point-like part is the characteristic part of the photonic distribution. In this part, the $O(\alpha_s(Q^2))$ term has been added. It is shown in Fig. 4 that this term has also made the rise weak. As a result, the rise of the photon structure function at small $x$ is projected less than that of the proton structure function. In fact it has been found experimentally that the rise of the photon structure function at small $x$ is not as strong as that in the proton structure function. Here, we have worked with the MS scheme and used $Q_0^2 = 1 \text{GeV}^2$, $\Lambda = 200 \text{MeV}$ and $f = 4$.

The scale $Q_0^2 = 1 \text{GeV}^2$ is the value where the results using the VMD model are in good agreement with the measured total $\gamma-\gamma$ cross section. We have assumed that the parton distribution functions at $Q_0^2 = 1 \text{GeV}^2$ can be represented by the corre-
Fig. 4. Comparison of our predictions for $F_2^x(x)/\alpha$ with a different type of distribution function at $Q^2 = 1.86 \text{ GeV}^2$, $Q^2 = 3.76 \text{ GeV}^2$ and $Q^2 = 14.7 \text{ GeV}^2$. The prediction using the Pomeron exchange for the gluon and the sea quark, that includes the $O(\alpha_S(Q^2))$ term in the point-like part is represented by the short-dashed curve. The long-dashed curve is the result including no $O(\alpha_S(Q^2))$ term in the point-like part and using the Pomeron-cut exchange instead of the Pomeron exchange. The contributions of the Pomeron-cut exchange and the $O(\alpha_S(Q^2))$ term are contained in the prediction of the solid curve.

Fig. 5. $F_2^x(x)/\alpha$ obtained from the hadronic part of the Pomeron-cut exchange is indicated by the short-dashed curve at $Q^2 = 1.86 \text{ GeV}^2$ and $Q^2 = 3.76 \text{ GeV}^2$. The solid curve and the long-dashed curve correspond to the same cases as in Fig. 4.

We shall study the effect of slightly changing the value of $Q_0^2$ chosen as the hadronic scale. The Pomeron-cut-exchange prediction with $Q_0^2 = 1 \text{ GeV}^2$ and the Pomeron-exchange predictions with $Q_0^2$ ranging between 0.5 GeV$^2$ and 1.5 GeV$^2$ are shown in Fig. 6. We can see in Fig. 6 that the Pomeron-
Fig. 6. Predictions for $F_2^\gamma(x)/\alpha$ at $Q^2 = 1.86 \text{ GeV}^2$. The Pomeron-cut-exchange input with $Q_0^2 = 1.0 \text{ GeV}^2$ (solid curve); Pomeron-exchange input with $Q_0^2 = 0.5 \text{ GeV}^2$ (long-dashed curve); Pomeron-exchange input with $Q_0^2 = 1.0 \text{ GeV}^2$ (short-dashed curve); Pomeron-exchange input with $Q_0^2 = 1.5 \text{ GeV}^2$ (dotted curve).

Fig. 7. Predictions for $F_2^\gamma(x)/\alpha$ in the $\overline{\text{MS}}$ scheme (solid curve) and AGF scheme in Ref. 7) (dotted curve) with Pomeron-cut-exchange input using $Q_0^2 = 1.0 \text{ GeV}^2$.

cut-exchange prediction, being in good agreement with the experimental data,$^{18}$ cannot be produced from the Pomeron-exchange though the value of $Q_0^2$ is changed as noted above.

It is well known that the unphysical behavior near $x = 1$ is caused by the direct-photon coefficient function $B_\gamma(x)$ given as

$$B_\gamma(x) = 4 \left( [x^2 + (1 - x)^2] \ln \frac{1 - x}{x} - 1 + 8x(1 - x) \right).$$

In many cases, these problems are circumvented by adopting the DIS-scheme.$^3,^{19}$ Recently, another approach$^7$ in the $\overline{\text{MS}}$ scheme was proposed. It shows that the dominant part of $B_\gamma(x)$ should in fact be included in the non-perturbative input, so that one has $q^{\text{had}}(x, Q_0^2) = q^{\text{VMD}}(x, Q_0^2) - B_0(x)$, with $B_0(x)$ a known function such that $B_\gamma(x) - B_0(x)$ is regular as $x \to 1$. $B_0(x)$ is assumed to tend to zero as $x \to 0$.

We will attempt to obtain an improvement of the behavior near $x = 1$ by using $B_0(x)$ in the forthcoming publication, but the behavior at small $x$ is affected only slightly by this improvement. In fact, we have shown in Fig. 7 that the difference between our results and the result using the scheme in Ref. 7) vanishes with decreasing $x$. Therefore the improvement of the problem for $x \to 1$ will affect the results in this paper very little.

We believe that we have obtained a reasonable expression at small $x$ in the framework of the NLO-DGLAP Equations by taking account of the Pomeron-cut exchange in the initial gluon distribution and the initial sea quark distribution. Of course there is such a case that one directly uses the BFKL Equation for the photonic gluon distribution in the region of small $x$. In our case, the properties of the VMD model have been kept by the $\rho$-Regge pole exchange in the valence quark distribution. Also, many soft gluon emissions at small $x$ are described by
the Pomeron-cut exchange in the gluon distribution and the sea quark distribution. According to the QCD factorization theorem, the long distance part described by the VMD model should be universal. Henceforth we will attempt to apply our method to other processes involving vector mesons.

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Appendix

For the singlet quark and the gluon distributions we have to solve the simultaneous inhomogeneous evolution equations (3) and (4). Then we eliminate \( G_H(n,t) \) from the homogeneous equations for Eqs. (3) and (4) and obtain the equation

\[
\frac{d^2}{dt^2} \Sigma_H(n,t) - p(n,t) \frac{d}{dt} \Sigma_H(n,t) + q(n,t) \Sigma_H(n,t) = 0. \tag{A.1}
\]

Here

\[
p(n,t) = p_0(n) + p_1(n) \frac{\alpha_s(t)}{2\pi},
\]

with

\[
p_0(n) = P_{qq}^{(0)}(n) + P_{GG}^{(0)}(n) \quad \text{and} \quad p_1(n) = \frac{2}{\beta_0}(R_{qq}(n) + R_{GG}(n)) - (P_{qG}^{(0)}(n))^{-1} R_{qG}(n).
\]

Also

\[
q(n,t) = q_0(n) + q_1(n) \frac{\alpha_s(t)}{2\pi},
\]

with

\[
q_0(n) = P_{qq}^{(0)}(n) P_{GG}^{(0)}(n) - P_{qG}^{(0)}(n) P_{Gq}^{(0)}(n),
\]

\[
q_1(n) = R_{qq}(n) - (P_{qG}^{(0)}(n))^{-1} R_{qG}(n) + \frac{2}{\beta_0} (P_{GG}^{(0)}(n) R_{qq}(n) + P_{qq}^{(0)}(n) R_{GG}(n) - P_{qG}^{(0)}(n) R_{Gq}(n) - P_{Gq}^{(0)}(n) R_{qG}(n)).
\]

As the solution of Eq. (A.1) we have

\[
\Sigma_H(n,t) = A(\lambda_+, t)e^{\lambda_+(n)t} + A(\lambda_-, t)e^{\lambda_-(n)t},
\]

where

\[
A(\lambda, t) = K_1(\lambda) + K_2(\lambda) \frac{\alpha_s(t)}{2\pi}, \tag{A.2}
\]

with

\[
K_1(\lambda) = \frac{\beta_0}{2} - 2\lambda(n) + p_0(n),
\]

\[
K_2(\lambda) = \lambda(n) p_1(n) - q_1(n).
\]

Substituting the above solution for the gluon homogeneous equation, we obtain the homogeneous solution

\[
G_H(n,t) = B(\lambda_+, t)e^{\lambda_+(n)t} + B(\lambda_-, t)e^{\lambda_-(n)t},
\]
where

$$B(\lambda, \tau) = L_1(\lambda) + L_2(\lambda) \frac{\alpha_s(t)}{2\pi},$$  \hspace{1cm} (A.3)

with

$$L_1(\lambda) = (P_{qG}^{(0)}(n))^{-1} K_1(\lambda)(\lambda(n) - P_{qq}^{(0)}(n)),$$

$$L_2(\lambda) = -(P_{qG}^{(0)}(n))^{-1} \left[ \left( \frac{\beta_0}{2} - \lambda(n) + P_{qq}^{(0)}(n) \right) K_2(\lambda) + (P_{qG}^{(0)}(n)R_{qq}(n) + \lambda(n)R_{qG}(n) - P_{qq}^{(0)}(n)R_{qG}(n)) \right].$$

Here \(A(\lambda_0, Q_0^2)\) and \(B(\lambda_\pm, Q^2)\) in Eq. (6) are defined in (A.2) and (A.3). Also, the coefficients \(a_{ij}(n)\) for \(q_{NS,PL}(n, Q^2)\) in Eq. (16) are represented as

$$a_{11}(n) = \frac{2}{\beta_0} P_{NS,PL}^{(0)}(n),$$

$$a_{12}(n) = -\left( \frac{2}{\beta_0} \right) R_{NS}(n),$$

$$a_{21}(n) = -\left( \frac{2}{\beta_0} \right) \left( T(n)P_{NS,PL}^{(0)}(n) - P_{NS,PL}^{(1)}(n) \right),$$

$$a_{22}(n) = \frac{2}{\beta_0} R_{NS}(n)(T(n)P_{NS,PL}^{(0)}(n) - P_{NS,PL}^{(1)}(n)),$$

$$a_{31}(n) = -\left( \frac{2}{\beta_0} \right) (R_{NS}(n)P_{NS,PL}^{(0)}(n) + T(n)P_{NS,PL}^{(1)}(n)),$$

where \(T(n) = \beta_1/\beta_0 - (2/\beta_0)R_{NS}(n)\).

Then \(b_{ij}(n)\) and \(c_{ij}(n)\) in Eq. (17) are defined as

$$b_{ij} = H_i(\lambda_-(n))K_j(\lambda_+(n)),$$

$$c_{ij} = H_i(\lambda_-(n)L_j(\lambda_+(n)),$$

where

$$H_1(\lambda) = P_{qG}^{(0)}(n)L_1(\lambda),$$

$$H_2(\lambda) = P_{qG}^{(0)}(n)[L_2(\lambda) - S(\lambda_+, \lambda_-)L_1(\lambda) + P_{qG}^{(1)}(n)\lambda(n) - P_{qG}^{(1)}(n)K_1(n)];$$

$$H_3(\lambda) = -P_{qG}^{(0)}(n) \left[ S(\lambda_+, \lambda_-)L_2(\lambda) + \left( \frac{Z(\lambda_+, \lambda_-)}{X(\lambda_+, \lambda_-)} - \frac{Y(\lambda_+, \lambda_-)}{X(\lambda_+, \lambda_-)} S(\lambda_+, \lambda_-) \right) L_1(\lambda) \right]
+ P_{qG}^{(0)}(n)[L_2(\lambda) - S(\lambda_+, \lambda_-)L_1(\lambda)]
- P_{qG}^{(1)}(n)[K_2(\lambda) - S(\lambda_+, \lambda_-)K_1(\lambda)].$$

with

$$X(\lambda_+, \lambda_-) = K_1(\lambda_+)L_1(\lambda_-) - (\lambda_+ \leftrightarrow \lambda_-),$$

$$Y(\lambda_+, \lambda_-) = K_1(\lambda_+)L_2(\lambda_-) + L_1(\lambda_-)K_2(\lambda_+) - (\lambda_+ \leftrightarrow \lambda_-),$$

$$Z(\lambda_+, \lambda_-) = K_1(\lambda_+)L_2(\lambda_-) - L_1(\lambda_-)K_2(\lambda_+) - K_1(\lambda_+)L_1(\lambda_-) + L_1(\lambda_-)K_2(\lambda_+).$$
\[ Z(\lambda_+ , \lambda_-) = K_2(\lambda_+ \tilde{\lambda}_2(\lambda_-) - (\lambda_+ \leftrightarrow \lambda_-) , \]
\[ S(\lambda_+ , \lambda_-) = \frac{Y(\lambda_+ , \lambda_-)}{X(\lambda_+ , \lambda_-)} + \frac{\beta_1}{2\beta_0} . \]

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