Three-Dimensional Black Holes and Liouville Field Theory

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(Received May 16, 1999)

A quantization of (2 + 1)-dimensional gravity with negative cosmological constant is presented, and it is used to study quantum aspects of (2 + 1)-dimensional black holes. The quantization consists of two procedures. One is related with quantization of the asymptotic Virasoro symmetry. The concept of the Virasoro deformation of 3-geometry is introduced. For a given black hole, the deformation of the exterior of the outer horizon is identified with a product of the appropriate coadjoint orbits of the Virasoro groups \( \hat{\text{diff}} \mathbb{S}^1 \pm \). Its quantization provides unitary irreducible representations of the Virasoro algebra, in which the state of the black hole becomes primary. To make the quantization complete, holonomies, the global degrees of freedom, are taken into account. By an identification of these topological operators with zero modes of the Liouville field, the aforementioned unitary representations are shown, as long as \( c \gg 1 \), to be the Hilbert space of this two-dimensional conformal field theory. This conformal field theory, living on the cylinder at infinity of the black hole and having continuous spectra, can recognize the outer horizon only as a one-dimensional object in \( SL_2(\mathbb{R}) \) and realize it as insertions of the corresponding vertex operator. Therefore it cannot be a conformal field theory on the horizon. Two possible descriptions of the horizon conformal field theory are proposed.

§1. Introduction

It has been proposed\(^1\)-\(^3\) that supergravity (and string theory) on \( AdS_{d+1} \) times a compact manifold is equivalent to a conformal field theory living on the boundary of \( AdS_{d+1} \). Supergravity (and string theory) on asymptotically \( AdS_{d+1} \) space-time is also expected\(^3\) to be realized by a boundary field theory on a suitable vacuum.

Investigations of the bulk quantum gravity, in particular, to clarify \( AdS \) black holes as the quantum gravitational states, will lead to a deeper understanding of the \( AdS/CFT \) correspondence. In three dimensions, a description of gravity in terms of gauge theory\(^4\) is known. In terms of this description, quantum gravity in three dimensions seems\(^5\) to be tractable, and thus consideration of such quantum gravitational states will become possible.

In this article we present a quantization of (2 + 1)-dimensional gravity with negative cosmological constant \( \Lambda = -1/l^2 \) and thereby investigate quantum aspects of the (2 + 1)-dimensional black holes from the perspective of the \( AdS_3/CFT_2 \) correspondence. We take a geometrical approach known as the coadjoint orbit method. Quantization of (2 + 1)-dimensional gravity along this line has not yet been realized.

Quantization consists of the following two procedures. The first is related with quantization of the asymptotic Virasoro symmetry, performed by Brown and Hen-
neaux.\textsuperscript{6) To describe it we introduce in §2 the notion of the Virasoro deformation of three-geometry. This is originally defined as a deformation of a suitable non-compact simply-connected region in $SL_2(R)$ and is given by the coadjoint action of the Virasoro groups $\text{diff}S^1_+ \times \text{diff}S^1_-$. The induced metric from $SL_2(R)$ provides a metric of the deformed region. Given a black hole, we can take, as such a non-compact simply-connected region, a covering space of the exterior of the outer horizon. Its Virasoro deformation commutes with the projection. Therefore we can obtain a family of the deformed quotients. This defines the Virasoro deformation of the exterior of the outer horizon of the black hole. The induced metric of the deformed region becomes that of its quotient, which can be understood as a deformed metric of the black hole obtained in Ref. 7). The asymptotic Virasoro symmetry turns out to be the infinitesimal form of this deformation. The family of the deformed quotients will be identified with a product of the coadjoint orbits of the Virasoro groups $\text{diff}S^1_{\pm}$ labeled by the mass and angular momentum of the black hole, together with the cosmological constant. In §3 we carry out the quantization of the asymptotic Virasoro symmetry and the Virasoro deformation of the exterior of the outer horizon. It is prescribed by quantization of the corresponding coadjoint orbits. Quantization of each orbit turns out to give a unitary irreducible representation of the Virasoro algebra with central charge $c = \frac{3l}{2G}$. This value of the central charge coincides with that obtained in Ref. 6) and used in Refs. 8) and 9). The state of the black hole becomes the primary state of the representation. All the excited states of the representation correspond to the local excitations through the Virasoro deformation, that is, as noted in Ref. 10), gravitons. $AdS_3$ is also examined. Quantization of its Virasoro deformation turns out to provide a unitary irreducible representation in which the state of $AdS_3$ is the $SL_2(R)_+ \times SL_2(R)_-$-invariant primary state.

Although quantization of the Virasoro deformations leads the unitary irreducible representations of the Virasoro algebra, it is not sufficient as a quantization of three-dimensional gravity. If one takes the perspective of $SL_2(R)_+ \times SL_2(R)_-$ Chern-Simons gravity,\textsuperscript{4) the Virasoro deformation corresponds to local degrees of freedom of the theory and, in order to make the quantization complete, we must take into account the global degrees of freedom, i.e. holonomies. We start §4 by discussing their quantization. Having quantized holonomies, it is very reasonable to expect from the perspective of the $AdS_3/CFT_2$ correspondence that, by an identification of these topological operators with the zero modes of a suitable two-dimensional quantum field, the unitary irreducible representations obtained by the quantization of the Virasoro deformations could be reproduced as the Hilbert space of the two-dimensional conformal field theory. This expectation turns out to be true, at least when $c \gg 1$. The holonomy variables will be identified with the zero modes of a real scalar field $X$, which is interpreted as the Liouville field. The state of $AdS_3$ is identified with the $sl_2(C)$-invariant vacuum of this conformal field theory. The states of the black holes are the primary states obtained from the vacuum by an operation of the corresponding vertex operators. Excitations of the Liouville field are equivalent to the generators of the Virasoro deformation. This field theory admits continuous spectra, as expected.
Having obtained the quantization of three-dimensional gravity using the Liouville field theory, the following question may arise: How can one understand the three-dimensional black holes in two dimensions where the Liouville field lives? The Liouville field theory lives on a two-sphere which is obtained by a compactification of the boundary cylinder at infinity of the black holes. To answer this question it becomes necessary to understand how the Virasoro deformation recognizes the outer horizon of the black hole. It is shown at the end of §4 that it recognizes the outer horizon not as a two-dimensional object but as a one-dimensional object in \( SL_2(\mathbb{R}) \). Therefore the boundary Liouville field \( X \) can see the outer horizon only as a one-dimensional object. Compactification of the boundary cylinder into a sphere simultaneously causes the solid cylinder to become a three-ball, by which this one-dimensional outer horizon intersects the boundary sphere at two points. The boundary Liouville field theory recognizes its intersections as insertions of the corresponding vertex operator.

A microscopic description of the three-dimensional black holes has been proposed by Carlip\(^{11}\) and Strominger\(^{8,9}\) using a conformal field theory on the horizon. Since the Virasoro deformation of the exterior of the outer horizon cannot recognize the horizon as a two-dimensional object, the boundary Liouville field theory cannot be equivalent to the horizon conformal field theory. In §5 we propose two possible descriptions of the horizon conformal field theory. One is a string theory in the background of a macroscopic string living in \( SL_2(\mathbb{R}) \). The worldvolume of the macroscopic string should be identified with the two-dimensional outer horizon, with which we can obtain the Virasoro algebra on the horizon as its fluctuation by microscopic string. The other possible description is a conformal field theory which could be obtained by the Virasoro deformation of the region between the inner and outer horizons of the black hole. The second possibility seems, as is speculated in the text, to imply an interpretation of the conjecture made by Maldacena,\(^1\) in terms of a gauge transformation.

\[\textbf{§2. More on geometry in three dimensions}\]

The BTZ black holes\(^{*,12}\) are three-dimensional black holes specified by their mass and angular momenta. The BTZ black hole with mass \( M(\geq 0) \) and angular momentum \( J \) will be denoted by \( X_{(J,M)} \). One can avoid a naked singularity by imposing the bound \( |J| \leq Ml \) on the allowed value of its angular momentum. In terms of the Schwarzschild coordinates \((t, \phi, r)\), where the ranges are taken as \(-\infty < t < +\infty\), \(0 \leq \phi < 2\pi\) and \(0 < r < +\infty\), the black hole metric \( ds^2_{X_{(J,M)}} \) has the form

\[ ds^2_{X_{(J,M)}} \equiv -N^2(dt)^2 + N^{-2}(dr)^2 + r^2(d\phi + N^\phi dt)^2. \tag{2.1} \]

\( N \) and \( N^\phi \) are functions of the radial coordinate \( r \) and given by

\(^*\) Exact solutions of the vacuum Einstein equation with a negative cosmological constant \( \Lambda = -1/l^2 \).
\[ N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{l^2 r^2}, \quad N^\phi = \begin{cases} \frac{r_+ r_-}{l r^2} & \text{when } J \geq 0, \\ -\frac{r_+ r_-}{l r^2} & \text{when } J < 0. \end{cases} \] (2.2)

The outer and inner horizons are located respectively at \( r = r_+ \) and \( r = r_- \). Information concerning the mass and angular momentum is encoded in \( r_\pm \) by

\[ r_\pm^2 \equiv 4GMl^2 \left( 1 \pm \sqrt{1 - \frac{J^2}{M^2 l^2}} \right), \] (2.3)

where \( G \) is the Newton constant. In the case \( Ml = |J| \), the black hole is called ‘extremal’. In this case, \( r_+ = r_- \) and the outer and inner horizons coincide. When \( Ml > |J| \), it is called non-extremal.

2.1. More on the black hole

The exterior of the outer horizon \( r = r_+ \) of the BTZ black hole plays an important role throughout this paper. We denote the exterior of the outer horizon of \( X_{(J,M)} \) by \( X_{(J,M)}^+ \). That is, \( X_{(J,M)}^+ \equiv \{(t, \phi, r) \in X_{(J,M)} \mid r > r_+ \} \). It is known\(^{12,13}\) that the black hole can be identified with an appropriate quotient of \( SL_2(\mathbb{R}) \). We will need a further detailed description of this relation between \( X_{(J,M)}^+ \) and \( SL_2(\mathbb{R}) \). In this subsection we provide this description in the case of non-extremal black holes, that is, the case of \( Ml > |J| \). The discussion presented here becomes a prototype for the other cases, which will be treated briefly in the next section.

\( SL_2(\mathbb{R}) \) is the three-dimensional non-compact group manifold allowed invariant metrics whose scalar curvatures are negative constants. These metrics are the Killing metric and its rescalings. Among them, let \( ds^2_{SL_2(\mathbb{R})} \) be the metric whose scalar curvature equals \(-6/l^2\). It is given by

\[ ds^2_{SL_2(\mathbb{R})} \equiv \frac{l^2}{2} \text{Tr} (g^{-1}dg)^2. \] (2.4)

This metric is an exact solution of the vacuum Einstein equation with the negative cosmological constant \( \Lambda = -1/l^2 \). Any element \( g \) of \( SL_2(\mathbb{R}) \) can be written as \( g = e^{-(\tau + \theta)}J^0 e^{\sigma J^1} e^{-(\tau - \theta)J^0} \), where \( 0 \leq \tau, \theta < 2\pi \) and \( 0 \leq \sigma < +\infty \).\(^{*4}\) With this Cartan decomposition, one can see that \( SL_2(\mathbb{R}) \) is topologically a solid torus (see Fig. 1).

Let \( Z_0 \) be a non-compact simply-connected region of \( SL_2(\mathbb{R}) \) consisting of the group elements

\[ e^{-\varphi J^2} e^{\sigma J^1} e^{\psi J^2}, \] (2.5)

\(^{*4}\) For \( SL_2(\mathbb{R}) \), a basis of the Lie algebra over \( \mathbb{R} \) is taken by the three matrices \( J^a \):

\[ J^0 \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad J^1 \equiv \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad J^2 \equiv \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}. \]

These satisfy \([J^a, J^b] = \epsilon^{abc} J_c\), where \( \epsilon^{abc} \) is a totally anti-symmetric tensor normalized by \( \epsilon^{012} = 1 \), and the \( sl_2(\mathbb{R}) \)-indices are lowered (or raised) by \( \eta_{\alpha\beta} \equiv \text{diag}(-1,1,1) \) (or \( \eta^{\alpha\beta} \), the inverse matrix of \( \eta_{\alpha\beta} \)).
where \( \varphi, \psi \) and \( \sigma \) are real parameters. Their ranges are taken as \(-\infty < \varphi, \psi < +\infty \) and \( 0 < \sigma < +\infty \). \( Z_0 \) is illustrated in Fig. 2. Note that the line \( POQ \) (\( \sigma = 0 \)) in Fig. 2 is excluded by the definition. The invariant metric (2.4) defines an induced metric on \( Z_0 \). We call it \( ds_0^2 \). It is merely a restriction of (2.4) on \( Z_0 \). Regarding the parameters \( \varphi, \psi \) and \( \sigma \) as coordinates of \( Z_0 \), \( ds_0^2 \) can be written in the form

\[
ds_0^2 = \frac{l^2}{4} \left\{ (d\varphi)^2 + (d\psi)^2 - (e^\sigma + e^{-\sigma})d\varphi d\psi + (d\sigma)^2 \right\}. \tag{2.6}
\]

We can provide a description of the above region together with the induced metric in terms of \( SL_2(\mathbb{R})_+ \times SL_2(\mathbb{R})_- \) Chern-Simons gravity. To present the description, it is convenient to introduce the mapping

\[
(g^+, g^-) \mapsto g^+ g^-(\sigma). \tag{2.7}
\]

This defines a pairing of \( SL_2(\mathbb{R})_\pm \) which is invariant under the left diagonal \( SL_2(\mathbb{R}) \) action, \( g^\pm \mapsto gg^\pm \). It is also convenient to regard the group elements (2.5) as images of a function \( h \) which takes its values in \( SL_2(\mathbb{R}) \) as presented in (2.5). Explicitly, \( h \) is given by

\[
h(\varphi, \psi, \sigma) \equiv e^{-\varphi J_2} e^{\sigma J_1} e^{\psi J_2}. \tag{2.8}
\]

With this perspective, we can consider the following decomposition of \( h \) into a product of \( SL_2(\mathbb{R})_\pm \)-valued functions \( h^{\pm} \):

\[
h = h^+ h^- \tag{2.9}
\]

This decomposition is equivalent to constructing a pre-image of \( Z_0 \) in the mapping (2.7). It is unique up to the left diagonal \( SL_2(\mathbb{R}) \) gauge transformation.
symmetry turns out to be the three-dimensional local Lorentz symmetry. Given such $SL_2(\mathbb{R})_\pm$-valued functions, we can associate flat $SL_2(\mathbb{R})$ connections $A_0^{(\pm)}$ as follows:

$$A_0^{(\pm)} \equiv h^{(\pm)} dh^{(\pm)-1}. \quad (2.10)$$

It turns out that $A_0 \equiv (A_0^{(+)}, A_0^{(-)})$ is a desired classical solution of the Chern-Simons gravity. In fact, one can reproduce the metric (2.6) by means of the 3-vein $e^a$, which is obtained from $A_0$ through the following correspondence:

$$A_0^{(\pm)} = \pm \frac{1}{l} e + \omega. \quad (2.11)$$

Both the 3-vein $e^a$ and spin connection $\omega^a$ are realized as $sl_2(\mathbb{R})$-valued one-forms: $e = e^a J_a$ and $\omega = \omega^a J_a$. To be explicit, by use of this correspondence, the 3-vein $e$ has the expression $e = \frac{1}{2}(A_0^{(+)} - A_0^{(-)})$. It can be also expressed as $e = \frac{1}{2} h^{(-)} \cdot h^{(-)-1}$. Then the metric $e^a e_a (= 2 \text{Tr} e)$ acquires the form $\frac{l^2}{2} \text{Tr} (h^{-1} dh)^2$. This is exactly the induced metric $ds_0^2$. The correspondence (2.11) also shows that the flatness of the connection $A_0$ is equivalent to the Cartan form of the vacuum Einstein equation with the negative cosmological constant.

$$de^a + \epsilon^{abc} \omega_b \wedge e_c = 0,$$
$$dw^a + \frac{1}{2} \epsilon^{abc} \omega_b \wedge e_c + \frac{1}{2l^2} \epsilon^{abc} e_b \wedge e_c = 0. \quad (2.12)$$

The non-compact simply-connected region $Z_0$ is a covering space of $X^{(J,M)}_\pm$, the exterior of the outer horizon of the non-extremal black hole $X_{(J,M)}$. The projection $\pi$ can be described by the relations:

$$\frac{r_+ + r_-}{l} \left( \frac{t}{l} + \phi \right) = \varphi,$$
$$\frac{r_+ - r_-}{l} \left( \frac{t}{l} - \phi \right) = \psi,$$
$$\sqrt{r_+^2 - r_-^2} + \sqrt{r_+^2 - r_-^2} = e^{\sigma/2}, \quad (2.13)$$

along with the identification of the Schwarzschild angular coordinate, $\phi \sim \phi + 2\pi$ (see Fig. 3). This projection map becomes the isometry $\pi_* ds_0^2 = ds^2_{X^{(J,M)}_\pm}$. Therefore the exterior of the outer horizon can be identified with the quotient of $Z_0$:

$$X^{(J,M)}_\pm = Z_0 / \sim. \quad (2.14)$$

Here the equivalence relation “$\sim$”, which originates in the periodicity of $\phi$, is given by

$$(\varphi, \psi, \sigma) \sim \left( \varphi + 2\pi \frac{r_+ + r_-}{l}, \psi - 2\pi \frac{r_+ - r_-}{l}, \sigma \right). \quad (2.15)$$
It is convenient to introduce another parametrization of \( Z_0 \) which is slightly different from \((\varphi,\psi,\sigma)\). Let \( b_0, \tilde{b}_0 \) and \( c \) be the quantities

\[ b_0 \equiv \frac{(r_+ + r_-)^2}{16Gl}, \quad \tilde{b}_0 \equiv \frac{(r_+ - r_-)^2}{16Gl}, \quad c \equiv \frac{3l}{2G}. \]  

(2.16)

We can write any element of \( Z_0 \) in the form

\[ e^{-\sqrt{\frac{b_0}{c/24}} w J^2} e^{\left(\rho - \frac{1}{2} \ln \frac{b_0\tilde{b}_0}{(c/6)^2}\right) J^1} e^{\sqrt{\frac{b_0}{c/24}} \tilde{w} J^2}. \]  

(2.17)

where \( w, \tilde{w} \) and \( \rho \) are real parameters. To exactly cover \( Z_0 \) with this parametrization, the ranges of \( w, \tilde{w} \) and \( \rho \) must be taken as \(-\infty < w, \tilde{w} < +\infty\) and \( \frac{1}{2} \ln \frac{b_0\tilde{b}_0}{(c/6)^2} < \rho < +\infty\).

Using this parametrization or coordinates of \( Z_0 \), let us derive an explicit form of the corresponding classical solution of the Chern-Simons gravity. The \( SL_2(\mathbb{R}) \)-valued function \( h \), which is defined in (2.8), can be read in this coordinate as

\[ h(w, \tilde{w}, \rho) = e^{-\sqrt{\frac{b_0}{c/24}} w J^2} e^{\left(\rho - \frac{1}{2} \ln \frac{b_0\tilde{b}_0}{(c/6)^2}\right) J^1} e^{\sqrt{\frac{b_0}{c/24}} \tilde{w} J^2}. \]  

(2.18)

The following decomposition of (2.18) may be chosen:

\[ h^+ = e^{-\frac{1}{2} \left(\rho - \ln \frac{b_0}{c/6}\right) J^1} e^{\sqrt{\frac{b_0}{c/24}} w J^2}, \]

\[ h^- = e^{\frac{1}{2} \left(\rho - \ln \frac{b_0}{c/6}\right) J^1} e^{\sqrt{\frac{b_0}{c/24}} \tilde{w} J^2}. \]  

(2.19)
With this choice, the flat connections $A_0^{(±)}$, which are defined in (2.10), become

$$A_0^{(+)} = \begin{pmatrix} -\frac{1}{4}d\rho & -e^{\rho/2}dw \\ -\frac{b_0}{c/6}e^{-\rho/2}dw & \frac{1}{4}d\rho \end{pmatrix}, \quad A_0^{(-)} = \begin{pmatrix} \frac{1}{4}d\rho & -\frac{b_0}{c/6}e^{-\rho/2}d\tilde{w} \\ -e^{\rho/2}d\tilde{w} & -\frac{1}{4}d\rho \end{pmatrix}. \tag{2.20}$$

Then $A_0 = (A_0^{+}), A_0^{-}$ turns out to give the metric

$$ds^2_{(b_0, \tilde{b}_0)} = l^2 \left\{ \frac{b_0}{c/6}(dw)^2 + \frac{\tilde{b}_0}{c/6}(d\tilde{w})^2 - \left( e^{\rho} + \frac{b_0\tilde{b}_0}{(c/6)^2}e^{-\rho} \right) dw \tilde{d}w + \frac{1}{4}(dp)^2 \right\}. \tag{2.21}$$

After change of coordinates, which can be read from (2.5) and (2.17), it coincides with the induced metric $ds_0^2$. In terms of the coordinates $(w, \tilde{w}, \rho)$, the equivalence relation (2.15) used in the identification with the black hole becomes simple:

$$(w, \tilde{w}, \rho) \sim (w + 2\pi, \tilde{w} - 2\pi, \rho). \tag{2.22}$$

Therefore we can also say that the quotient of $Z_0$ by the identification (2.22), together with the induced metric (2.21), is identified with the exterior of the outer horizon of $X_{(J,M)}$.

2.2. Generalization of a flat connection

A generalization of the flat $SL_2(\mathbb{R})_+ \times SL_2(\mathbb{R})_-$ connection (2.20) has been examined\(^{15,7}\) to find a realization of the asymptotic Virasoro symmetry\(^6\) in terms of the Chern-Simons gravity. It has the form

$$A_b^{(+)} = \begin{pmatrix} -\frac{1}{4}d\rho & -e^{\rho/2}dw \\ -\frac{b(w)}{c/6}e^{-\rho/2}dw & \frac{1}{4}d\rho \end{pmatrix}, \quad A_b^{(-)} = \begin{pmatrix} \frac{1}{4}d\rho & -\frac{\tilde{b}(\tilde{w})}{c/6}e^{-\rho/2}d\tilde{w} \\ -e^{\rho/2}d\tilde{w} & -\frac{1}{4}d\rho \end{pmatrix}. \tag{2.23}$$

The suffixes $b$ and $\tilde{b}$ label the flat connections. To be consistent with the equivalence relation (2.22), we assume that $b$ and $\tilde{b}$ are respectively real functions of $w$ and $\tilde{w}$ with $2\pi$ periodicity.

The asymptotic Virasoro algebra may be realized by the generators of the infinitesimal gauge transformation

$$\lambda_f^{(±)} = \begin{pmatrix} \frac{b}{c/6}f - \frac{1}{2}f' & f e^{\rho/2} \\ \frac{1}{2}f' & \frac{b}{c/6}f - \frac{1}{2}f' \end{pmatrix}, \quad \lambda_{\tilde{f}}^{(±)} = \begin{pmatrix} \frac{\tilde{b}}{c/6}\tilde{f} - \frac{1}{2}\tilde{f}' & \frac{\tilde{b}}{c/6}\tilde{f} - \frac{1}{2}\tilde{f}' \\ \frac{1}{2}\tilde{f}' & \frac{\tilde{b}}{c/6}\tilde{f} - \frac{1}{2}\tilde{f}' \end{pmatrix}, \tag{2.24}$$

where, again to be consistent with (2.22), $f = f(w)$ and $\tilde{f} = \tilde{f}(\tilde{w})$ are assumed to be real functions with $2\pi$ periodicity. The gauge transformation of the connection (2.23) is identical\(^{16}\) to deformations of $b$ and $\tilde{b}$:

$$\delta_{\lambda_f^{(±)}}A_b^{(±)} = \begin{pmatrix} 0 & \frac{\delta b}{c/6}e^{-\rho/2}dw \\ -\frac{\delta \tilde{b}}{c/6}e^{-\rho/2}dw & 0 \end{pmatrix}, \quad \delta_{\lambda_{\tilde{f}}^{(±)}}A_b^{(±)} = \begin{pmatrix} 0 & -\frac{\delta \tilde{b}}{c/6}e^{-\rho/2}d\tilde{w} \\ 0 & 0 \end{pmatrix}. \tag{2.25}$$
These deformations have the forms
\[
\delta f b = f b' + 2 f' b - \frac{c}{12} f''' ,
\]
\[
\delta f b = f b' + 2 f' b - \frac{c}{12} f''' .
\] (2.26)

In the asymptotic region where \( \rho \) is very large, the non-zero off-diagonal elements of \( \delta_{\lambda f} A_b^{(\pm)} \) and \( \delta_{\lambda f} A_b^{(-)} \) become negligible. Hence the gauge transformation can be regarded as an asymptotic symmetry. To connect with the asymptotic Virasoro symmetry it is convenient to decompose the gauge transformation into the diffeomorphism and the local Lorentz transformation. Note that for a given flat connection \( A \), its infinitesimal general coordinate transform (say, generated by a vector field \( \xi \)) can be interpreted \(^4\) as its infinitesimal gauge transform. Via the Sugawara construction, the generator of the gauge transformation is given by \( \delta = \xi^\mu A_\mu \). Decompose the gauge transformation (2.25) to \( \delta_{\lambda f} A_b^{(\pm)} = \delta \xi A_b^{(\pm)} + \delta \eta A_b^{(\pm)} \) and \( \delta_{\lambda f} A_b^{(-)} = \delta \xi A_b^{(-)} + \delta \eta A_b^{(-)} \), where \( \xi \) and \( \tilde{\xi} \) are vector fields and \( \eta \) is a generator of the local Lorentz transformation (the diagonal \( SL_2(R) \) gauge transformation). The vector fields turn out to be
\[
\xi = 2 f' \partial_\rho + \left( - f + \frac{\tilde{\xi}}{c/6} f''' e^{-\rho} + \tilde{f}'' \right) \partial_w ,
\]
\[
\tilde{\xi} = 2 \tilde{f}' \partial_\rho + \left( - \tilde{f} + \frac{\xi}{c/6} f''' e^{-\rho} + f'' \right) \partial_{\tilde{w}} .
\] (2.27)

In the asymptotic region they behave as \( \xi \sim 2 f' \partial_\rho - f \partial_w \) and \( \tilde{\xi} \sim 2 \tilde{f}' \partial_\rho - \tilde{f} \partial_{\tilde{w}} \). These coincide with the Virasoro generators considered in Ref. 6).

2.3. Generalization of the black hole

Given the flat connection (2.23), one can construct a metric according to the prescription of the Chern-Simons gravity. Denote this metric by \( ds^2_{(b,b)} \). It has the form
\[
\begin{align*}
\frac{b}{c/6} (dw)^2 + \frac{\tilde{b}}{c/6} (d\tilde{w})^2 - \left( e^\rho + \frac{bb}{(c/6)^2} e^{-\rho} \right) dw d\tilde{w} + \frac{1}{4} (d\rho)^2 .
\end{align*}
\] (2.28)

It is also possible to obtain a region of \( SL_2(R) \), analogous to \( Z_0 \), on which the induced metric from \( SL_2(R) \) is precisely given by (2.28). Since \( A_b^{(\pm)} \) and \( A_b^{(-)} \) are flat connections, we can always trivialize them by some \( SL_2(R) \)-valued functions \( h_b^{(\pm)} \) and \( h_b^{(-)} \):
\[
A_b^{(\pm)} = h_b^{(\pm)} dh_b^{(\pm)-1} , \quad A_b^{(-)} = h_b^{(-)} dh_b^{(-)-1} .
\] (2.29)
Then the desired region of $SL_2(\mathcal{R})$, which we call $Z_{(b,\tilde{b})}$, consists of those obtained from $h_b^{(+)}$ and $h_b^{(-)}$ by their local Lorentz invariant pairing, $h_b^{(+)}^{-1}h_b^{(-)}$. It is easy to see that the induced metric on $Z_{(b,\tilde{b})}$ is nothing but $ds^2_{(b,\tilde{b})}$. When both $b$ and $\tilde{b}$ are constant, $Z_{(b,\tilde{b})}$ coincides with $Z_0$. In particular, $Z_{(b_0,\tilde{b}_0)} = Z_0$ holds. On the other hand, as long as either $b$ or $\tilde{b}$ is non-constant, $Z_{(b,\tilde{b})}$ becomes the region of $SL_2(\mathcal{R})$ which is different from $Z_0$. One may suspect that a reparametrization of $Z_0$ such as $\int_0^w -\frac{\ell^2}{c/6} dw J_2 \sqrt{\frac{b}{c/6}}$ might lead to the flat connection (2.23) or the metric (2.28). But it does not, as long as either $b$ or $\tilde{b}$ is non-constant.

It is an interesting but difficult task to describe the region $Z_{(b,\tilde{b})}$ explicitly. Nevertheless, the following argument shows that $Z_{(b,\tilde{b})}$ is generically different from $Z_0$ in $SL_2(\mathcal{R})$, which is sufficient for the discussion in this article. We first note that the completion of $Z_0$ at $\sigma = 0$ is achieved by adding the one-dimensional line POQ (Fig. 2) in $SL_2(\mathcal{R})$. It is a counterpart of the outer horizon in $SL_2(\mathcal{R})$. This can be read from the induced metric $ds^2_{(b_0,\tilde{b}_0)}$ of $Z_0$. The counterpart of the horizon is located at $\sigma = 0$, that is, $\rho = \frac{1}{2} \ln \frac{b_0}{(c/6)^2}$, and there the metric degenerates to

$$
ds^2_{(b_0,\tilde{b}_0)} \big|_{\rho=\frac{1}{2} \ln \frac{b_0}{(c/6)^2}} = \ell^2 \left( \sqrt{\frac{b_0}{c/6}} dw - \sqrt{\frac{\tilde{b}_0}{c/6}} d\tilde{w} \right)^2.
$$

(2.30)

With regard to a generic $Z_{(b,\tilde{b})}$, the analogue of the horizon will be read from the singularity of the vector fields $\xi$ and $\tilde{\xi}$. Looking at (2.27), they diverge at $\rho = \frac{1}{2} \ln \frac{b}{(c/6)^2}$. For the diffeomorphism to work, $\rho$ should satisfy $\rho > \frac{1}{2} \ln \frac{b}{(c/6)^2}$. Only with this understanding $Z_{(b,\tilde{b})}$ becomes a deformation of $Z_0$. (The vector fields $\xi$ and $\tilde{\xi}$ are expected to be integrated step by step from $Z_0$ to $Z_{(b,\tilde{b})}$ in $SL_2(\mathcal{R})$.)

The completion of $Z_{(b,\tilde{b})}$ at $\rho = \frac{1}{2} \ln \frac{b}{(c/6)^2}$ will be accomplished by adding a two-dimensional space-like surface in $SL_2(\mathcal{R})$. This can be also understood from the behavior of the induced metric

$$
ds^2_{(b,\tilde{b})} \big|_{\rho=\frac{1}{2} \ln \frac{b}{(c/6)^2}} = \ell^2 \left( \sqrt{\frac{b}{c/6}} dw - \sqrt{\frac{\tilde{b}}{c/6}} d\tilde{w} \right)^2 + \ell^2 \left( \frac{b'}{4b} dw + \frac{\tilde{b}'}{4\tilde{b}} d\tilde{w} \right)^2.
$$

(2.31)

The difference under their completions in $SL_2(\mathcal{R})$ clearly shows that $Z_{(b,\tilde{b})}$ is a different region from $Z_0$ as long as either $b$ or $\tilde{b}$ is non-constant.

If one identifies $w$ and $\tilde{w}$ with the coordinates of circles, each deformation $\delta_f b$ and $\delta_f \tilde{b}$ given in (2.26) can be regarded as the coadjoint action of the Virasoro algebra with the central charge $c$. Here $b$ and $\tilde{b}$ play the role of quadratic differentials on

---

If one takes the Kruskal coordinates, the horizon — not its analogue in $SL_2(\mathcal{R})$ — may be a two-dimensional null surface.
the central extension of \( \text{diff} \) of the outer horizon of the black hole.

The coadjoint representation of the Virasoro algebra consists of quadratic differen-

tmentum \( J \) of vector fields \( (b, a) \). This allows us to discuss the Virasoro deformation of the quotient space. Let \( \text{Virasoro} \) algebra are consistent with the equivalence relation used in the quotient.

parametrization (2.17)) in the manner of taking the quotient. The actions of the

angular momentum of the black hole, is encoded not in \( Z(b, a) \). Namely the coadjoint action of \( \text{Virasoro} \)

deformation generates the oscillating modes of \( \tilde{b}, \tilde{w} \) is not static, since \( w \) and \( \tilde{w} \) are originally the lightcone coordinates (2.13).

\[ \pi \text{ds}^2_{X}(J,M) = \pi_\ast \text{ds}^2_{(b,0)} \text{of } X(J,M), \]

The Virasoro groups \( \text{diff} S^1 \times \text{diff} S^1 \) deform the metric \( \text{ds}^2_{X}(J,M) \) by the coadjoint action. \( SL_2(R) \) itself plays the role of a classifying space in this argument. Namely the coadjoint action of \( \text{diff} S^1 \times \text{diff} S^1 \) deforms the region \( Z_0 \) without touching the metric of \( SL_2(R) \). The metric \( \text{ds}^2_{(b,0)} \) of each deformed region \( Z(b, a) \) is that one induced from \( \text{ds}^2_{SL_2(R)} \), and it determines the metric of \( Y(b, a) \) by the projection \( \pi \). It is the deformed metric of the black hole.

It should be emphasized that the above Virasoro deformation cannot transform the black hole \( X(J,M) \) to the black hole \( X'(J,M') \) whose mass \( M' \) and angular momentum \( J' \) are different from \( M \) and \( J \). This can be understood from the fact that no constant shift of \( b_0 \) or \( \tilde{b}_0 \) is allowed in (2-26), since \( f \) and \( \tilde{f} \) are taken to be \( 2\pi \) periodic for consistency with the equivalence relation used in the projection \( \pi \). The deformation generates the oscillating modes of \( b(w) \) and \( \tilde{b}(\tilde{w}) \). The deformed metric \( \pi_\ast \text{ds}^2_{(b,0)} \) is not static, since \( w \) and \( \tilde{w} \) are originally the lightcone coordinates (2.13).

\[ \frac{3}{3}. \text{ 3-geometries as the Virasoro coadjoint orbit} \]

3.1. Coadjoint orbits of the Virasoro group

The Virasoro algebra is the Lie algebra of the Virasoro group \( \text{diff} S^1 \), which is the central extension of \( \text{diff} S^1 \); the group of diffeomorphisms of a circle. It consists of vector fields \( f(w) \frac{d}{dw} \) on the circle together with a central element \( c \). A general element has the form \( f(w) \frac{d}{dw} - iac \), with \( a \) a real number. Let us write it as the pair \((f, a)\). The commutation relation is given by

\[
[(f, a_1), (g, a_2)] = \left( \frac{1}{2} (f g' - f' g, \frac{1}{48\pi} \int_0^{2\pi} dw (f g''' - f''' g') \right). \tag{3.1}
\]

The coadjoint representation of the Virasoro algebra consists of quadratic differentials \( b(w)(dw)^2 \) together with the dual central element \( c^\ast \). A general coadjoint vector

\[ \ast \) Several terms used in this paragraph will be explained in the next section. Here we hope to introduce our perspective in a simple manner.}
(an element of the dual of the Virasoro algebra) has the form $b(w)(dw)^2 + itc^*$, with $t$ a real number. Let us denote it by the pair $(b, t)$. The pairing between coadjoint and adjoint vectors is given by

$$\langle (b, t), (g, a) \rangle = \frac{1}{2\pi} \int_0^{2\pi} dw b(w) g(w) + ta.$$  \hfill (3.2)

This is invariant under reparametrizations of the circle. The coadjoint action of the Virasoro algebra is

$$\delta_f(b, t) = \left( f'b + 2f'b - \frac{t}{12} f''' \right),$$  \hfill (3.3)

where, the action of the central element being trivial, we write $\delta_f(g, a)$ as $\delta_f$. The dual pairing (3.2) is invariant under the action of the Virasoro algebra,

$$\delta_f(\langle (b, t), (g, a) \rangle) = \langle \delta_f(b, t), (g, a) \rangle + \langle (b, t), \delta_f(g, a) \rangle = 0,$$  \hfill (3.4)

where $\delta_f(g, a)$ is given by (3.1).

As can be seen in (3.3) the dual central element $c^*$ is invariant under the coadjoint action. The value of $t$ determines the central charge. We set $t = \frac{3l}{2G}$. According to the definition (2.16) we write this $t$ as $c$. And $W_{b_0}$ written as $W_{b_0}^{(c)}$ at the end of the previous section, is the coadjoint orbit of $(b_0, c)$, that is, the $\text{diff}S^1$-orbit of $(b_0, c)$. The transformation (3.3) also shows that $W_{b_0}$ is a homogeneous space of $\text{diff}S^1$. Therefore we can parametrize the orbit by elements of $\text{diff}S^1$ (modulo the little group $H$): For $s \in \text{diff}S^1$, letting $w_s = s(w)$, the integrated form of (2.26) or (3.3) becomes

$$b_s(w_s)(dw_s)^2 = \left[ b_0 + \frac{c}{12} \{w_s, w\} \right] (dw)^2,$$  \hfill (3.5)

where $\{w_s, w\}$ is the Schwarzian derivative, $\{w_s, w\} \equiv \frac{d^3w_s/dw^3}{d^2w_s/dw^2} - \frac{3}{2} \left( \frac{d^2w_s/dw^2}{d^w_s/dw} \right)^2$. Then $b^s$ provides the corresponding element of $W_{b_0}$.

According to the Kirillov-Souriau-Kostant theory, 17) each coadjoint orbit admits a canonical symplectic structure defined by the dual pairing. If one regards tangent vectors at $b^s$ as vector fields on a circle, the symplectic form $\Omega$ can be described as

$$\Omega|_{bs}(u, v) = \langle (b^s, c), [u, 0), (v, 0)] \rangle,$$  \hfill (3.6)

where $u = u(w_s)$ and $v = v(w_s)$ denote tangent vectors at $b^s$.

Generators of the Virasoro algebra are given by Hamiltonian functions on $W_{b_0}$. If one takes $L_m = ie^{imw} \frac{d}{dw}$, the commutation relation (3.1) becomes

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12} m^3 \delta_{m+n, 0}.$$  \hfill (3.7)

The corresponding Hamiltonian functions are given by

$$l_m(s) = \langle (b^s, c), (e^{imw_s}, 0) \rangle.$$  \hfill (3.8)
where \( s \in \text{diff}S^1 \) is thought of as the coordinate of \( W_{b_0} \). Using the prescribed symplectic structure,\(^{17}\) they turn out to satisfy the (classical) Virasoro algebra:

\[
\{l_m, l_n\} = im(m - n)l_{m+n} + \frac{ie}{12}m^3 \delta_{m+n,0}.
\]

\[
(3.9)
\]

3.2. 3-geometries as the Virasoro coadjoint orbit

We now generalize the argument given in the previous section to the cases of massive-extremal and massless black holes. For each of them there appears an appropriate region \( Z_0 \) of \( SL_2(\mathbb{R}) \). The Virasoro groups \( \hat{\text{diff}}S^1_+ \times \hat{\text{diff}}S^1_- \) can deform each \( Z_0 \) in \( SL_2(\mathbb{R}) \). This will be described by the coadjoint action of the Virasoro group. The quotient of the deformed region, obtained in a similar manner as in the non-extremal case, provides a deformation of the exterior of the outer horizon. Each quotient has a metric induced from \( SL_2(\mathbb{R}) \), which is the deformed metric of the black hole considered in Ref. 7). The family of these quotients is identified with the product of the coadjoint orbits \( W_+^{(\pm)} \times W_-^{(\pm)} \). The values of \( b_0 \) and \( \tilde{b}_0 \) can be read from the values \((J, M)\) of the black hole by using (2.16). We also examine the case of \( \text{AdS}_3 \). In this case it becomes necessary to study \( \tilde{SL}_2(\mathbb{R}) \), the universal cover of \( SL_2(\mathbb{R}) \). Taking an appropriate region of \( \tilde{SL}_2(\mathbb{R}) \), it becomes possible to repeat an argument similar to that used for the black holes. In particular, the family of the deformed regions will be identified with \( W_+^{(\pm)} \times W_-^{(\pm)} \).

3.2.1. Extremal black hole

Massive case

Let \( Z_0 \) be a region of \( SL_2(\mathbb{R}) \) consisting of the group elements\(^*\)

\[
e^{-\varphi \gamma^2}e^{\sigma \gamma^1}e^{\tilde{w} \gamma^0}e^{\tilde{w} \gamma^0} e^{\tilde{w} \gamma^1},
\]

\[
(3.10)
\]

where \( \varphi, \tilde{w}, \sigma \) are real parameters. Their ranges are taken as \(-\infty < \varphi, \tilde{w}, \sigma < +\infty\). This region is a non-compact simply-connected region of \( SL_2(\mathbb{R}) \). Regarding these parameters as coordinates of \( Z_0 \), the induced metric from \( SL_2(\mathbb{R}) \) can be written in the form

\[
ds_0^2 = \frac{r^2}{l^2} \left\{ (d\varphi)^2 - 2e^\sigma d\varphi d\tilde{w} + (d\sigma)^2 \right\}.
\]

(3.11)

This non-compact simply-connected region of \( SL_2(\mathbb{R}) \) is a covering space of \( X_+^{(M, l, M)} \), the exterior of the outer horizon of the massive extremal black hole \( X_+^{(M, l, M)} \). The projection \( \pi \) can be described by

\[
\frac{2r_+}{l} \left( \frac{t}{l} + \phi \right) = \varphi, \quad \frac{t}{l} - \phi = \tilde{w}, \quad \frac{r^2}{r_+^2} = e^\sigma,
\]

(3.12)

with the identification \( \phi \sim \phi + 2\pi \). Since \( \pi \) turns out to be the isometry, we can identify \( X_+^{(M, l, M)} \) with the quotient of \( Z_0 \):

\[
X_+^{(M, l, M)} = Z_0 / \sim.
\]

\[
(3.13)
\]

\(^*\) \( J^\pm \equiv \pm J^0 + J^2 \).
In this derivation we first decompose the flat $SL_2(R)_{+} \times SL_2(R)_{-}$ connection

$$A_0^{(+)} = \left( \begin{array}{cc} -\frac{1}{4} \frac{d\rho}{\rho} & -e^{\rho/2} dw \\
-4GM e^{-\rho/2} dw & \frac{1}{4} \frac{d\rho}{\rho} \end{array} \right), \quad A_0^{(-)} = \left( \begin{array}{cc} \frac{1}{4} \frac{d\rho}{\rho} & 0 \\
e^{-\rho/2} d\bar{w} & -\frac{1}{4} \frac{d\rho}{\rho} \end{array} \right).$$

In this derivation we first decompose the $SL_2(R)$-valued function $h$ (3.14), according to the rule (2.9), into the pair of $SL_2(R)_{\pm}$-valued functions $h^{(+)} = e^{-\frac{1}{2} (\rho - \ln 4GM)} e^J$ and $h^{(-)} = e^{\frac{1}{2} \rho J^1} e^{\bar{w} J^2}$, and then we apply the prescription (2.10). The connection (3.15) coincides with (2.23) at $(b, \tilde{b}) = (Ml, 0)$.

The equivalence relation used in the quotient (3.13) acquires the form (2.15) in these new coordinates. Thus our argument completely reduces to that given in the previous section. In particular, deformations of the massive-extremal black hole $X_{(ML, ML)}^{+}$ are realized by the quotients $Y_{(b, \tilde{b})}$, which are connected with it by the Virasoro group. The family of these $Y_{(b, \tilde{b})}$ can be identified with $W_{ML}^{(+)} \times W_{0}^{(-)}$.

### Massless case

Let $Z_0$ be a region of $SL_2(R)$ consisting of the group elements

$$e^{-w J^1 + \rho J^1} e^{\bar{w} J^2}, \quad (\equiv h(w, \bar{w}, \rho))$$

(3.16)

where $w, \bar{w}$ and $\rho$ are real parameters. Their ranges are taken as $-\infty < w, \bar{w}, \rho < +\infty$. This region is also a non-compact simply-connected region of $SL_2(R)$. Regarding these parameters as coordinates of $Z_0$, the induced metric from $SL_2(R)$ can be written as

$$ds^2_0 = \frac{t^2}{4} \left\{ -4e^{\rho} dw d\bar{w} + (d\rho)^2 \right\}.$$  

(3.17)

This non-compact simply-connected region is a covering space of $X_{(0, 0)}^{+}$, the exterior of the horizon of the massless black hole. The projection $\pi$ is prescribed by

$$\frac{t}{l} + \phi = w, \quad \frac{t}{l} - \phi = \bar{w}, \quad \frac{r}{l} = e^{\rho/2},$$

(3.18)

with the identification $\phi \sim \phi + 2\pi$. Hence the exterior of the horizon of the massless black hole can be identified with the quotient of $Z_0$:

$$X_{(0, 0)}^{+} = Z_0/\sim.$$  

(3.19)

Here the equivalence relation is $(w, \bar{w}, \rho) \sim (w + 2\pi, \bar{w} - 2\pi, \rho)$.

Decomposition of (3.16) by $h^{(+)} = e^{-\frac{1}{2} \rho J^1} e^{w J^1}$ and $h^{(-)} = e^{\frac{1}{2} \rho J^1} e^{\bar{w} J^2}$ leads to the flat $SL_2(R)_{+} \times SL_2(R)_{-}$ connection

$$A_0^{(+)} = \left( \begin{array}{cc} -\frac{1}{4} \frac{d\rho}{\rho} & -e^{\rho/2} dw \\
0 & \frac{1}{4} \frac{d\rho}{\rho} \end{array} \right), \quad A_0^{(-)} = \left( \begin{array}{cc} \frac{1}{4} \frac{d\rho}{\rho} & 0 \\
e^{-\rho/2} d\bar{w} & -\frac{1}{4} \frac{d\rho}{\rho} \end{array} \right).$$

(3.20)
This is the connection (2.23) at \((b, \tilde{b}) = (0, 0)\). Hence our argument reduces to the previous one. Deformations of the massless black hole \(X_{(0,0)}^+\) are realized by the quotients \(Y_{(b,\tilde{b})}\), which are connected with it by the Virasoro group. The family of these \(Y_{(b,\tilde{b})}\) can be identified with \(W_0^{(+)} \times W_0^{(-)}\).

### 3.2.2. Anti de-Sitter space

\(AdS_3\) is the universal cover of \(SL_2(\mathbb{R})\). In terms of the Schwarzschild type coordinates \((t, \phi, r)\), where the ranges are \(-\infty < t < +\infty, 0 \leq \phi < 2\pi\) and \(0 \leq r < +\infty\), the metric \(ds^2_{AdS_3}\) acquires the form

\[
 ds^2_{AdS_3} = -\left(1 + \frac{r^2}{l^2}\right)^2 (dt)^2 + \left(1 + \frac{r^2}{l^2}\right)^{-2} (dr)^2 + r^2 (d\phi)^2. \tag{3.21}
\]

Note that \(r = 0\) is not a true singularity. It is merely a coordinate singularity. Nevertheless, to provide a Virasoro deformation of this space which is analogous to that for the black holes, we must consider the complement of \(r = 0\) rather than the whole of \(AdS_3\). Actually this complement is an analogue of \(X_{(J,M)}^0\) for the black hole. The complement of the line at \(r = 0\) will be called \(AdS_3^+_0\).

Let \(Z_0\) be a region of \(SL_2(\mathbb{R})\) consisting of the group elements

\[
e^{-wJ_0^0}e^{\sigma J_1^0}e^{-\tilde{w}J_0^0}, \tag{3.22}
\]

where \(w, \tilde{w} \) and \(\sigma\) are real parameters. Note that (3.22) has the form of the Cartan decomposition. In order to avoid a double parametrization, the ranges of \(w\) and \(\tilde{w}\) should be taken within \(0 \leq w \pm \tilde{w} < 4\pi\). Therefore we choose \(0 \leq w \pm \tilde{w} < 4\pi\) for their ranges. The value of \(\sigma\) is set to satisfy \(0 < \sigma < +\infty\). If one includes \(\sigma = 0\) in the definition, \(Z_0\) coincides with \(SL_2(\mathbb{R})\) itself. Due to its exclusion, \(Z_0\) becomes the complement of the circle at \(\sigma = 0\). If one regards the time of \(AdS_3\) to be \(2\pi l\)-periodic, \(Z_0\) can be identified with \(AdS_3^+_0\). The relation with the Schwarzschild-type coordinates can be read as

\[
w = \frac{t}{l} + \phi, \quad \tilde{w} = \frac{t}{l} - \phi, \quad e^{\sigma/2} = \sqrt{1 + \frac{r^2}{l^2} + \frac{r}{l}}. \tag{3.23}
\]

Let us introduce a slightly different parametrization of \(Z_0\) with the form

\[
e^{-wJ_0^0}e^{(\rho + 2\ln 2)J_1^0}e^{-\tilde{w}J_0^0}, \quad (\equiv h(w, \tilde{w}, \rho)) \tag{3.24}
\]

where real parameter \(\rho\) satisfies \(-2\ln 2 < \rho < +\infty\), due to the shift. Decomposition of the \(SL_2(\mathbb{R})\)-valued function \(h\) (3.24) by the pair of \(SL_2(\mathbb{R})_+\)-valued functions \(h^{(+)} = e^{-\frac{1}{2}(\rho + 2\ln 2)J_1^0}e^{wJ_0^0}\) and \(h^{(-)} = e^{\frac{1}{2}(\rho + 2\ln 2)J_1^0}e^{-\tilde{w}J_0^0}\) leads to the flat \(SL_2(\mathbb{R})_+ \times SL_2(\mathbb{R})_-\) connection

\[
 A^{(+)}_0 = \left(\begin{array}{cc}
 -\frac{1}{3}\frac{d\rho}{e^{\rho/2}dw} & -e^{\rho/2}dw \\
 \frac{1}{3}e^{\rho/2}dw & \frac{1}{3}\frac{d\rho}{d\tilde{w}}
\end{array}\right), \quad A^{(-)}_0 = \left(\begin{array}{cc}
 \frac{1}{3}\frac{d\rho}{e^{\rho/2}d\tilde{w}} & \frac{1}{3}e^{-\rho/2}d\tilde{w} \\
 -e^{\rho/2}dw & -\frac{1}{3}\frac{d\rho}{d\tilde{w}}
\end{array}\right). \tag{3.25}
\]

This is the connection (2.23) at \((b, \tilde{b}) = (-c/24, -c/24)\).
$SL_2(\mathbb{R})$ has a topologically nontrivial circle. The universal cover $\tilde{SL}_2(\mathbb{R})$ can be obtained by making this $S^1$ a line. This is equivalent to a simple change of the identification of $w$ and $\tilde{w}$ in the Cartan decomposition into the identification $(w, \tilde{w}) \sim (w + 2\pi, \tilde{w} - 2\pi)$. The covering of $Z_0$ taken in $\tilde{SL}_2(\mathbb{R})$ is the complement of the line at $\sigma = 0$ and can be identified with $AdS^3$.

For a given element $(b, \tilde{b}) \in W^{(+)}_{-c/24} \times W^{(-)}_{-c/24}$, the flat connection $(2.23)$ may be thought of as a connection on a solid cylinder, where $\rho$ measures its radial direction and $(w, \tilde{w})$ is identified with the lightcone coordinates of the cylinder. Trivializations of $A^{(z)}_{b,\tilde{b}}$ and their local Lorentz invariant pairing define a map from the solid cylinder to $SL_2(\mathbb{R})$. The solid cylinder wraps around $SL_2(\mathbb{R})$ infinitely many times. We may unfold this wrapping by considering $\tilde{SL}_2(\mathbb{R})$ instead of $SL_2(\mathbb{R})$. Therefore, letting $Z_0$ be the complement of the line in $\tilde{SL}_2(\mathbb{R})$ we can construct its deformation by the Virasoro group. The family of the deformations can be identified with $W^{(+)}_{-c/24} \times W^{(-)}_{-c/24}$.

3.3. Quantization of the Virasoro coadjoint orbits

For a given coadjoint orbit $W_{b_0}$, the Hamiltonian function $l_0(s)$ generates the $S^1$ action and can be considered as an energy function of the orbit. The value $b_0$ is a fixed point of this circle action and corresponds to the classical vacuum. We discuss the stability of this vacuum. Let us write $s \in \text{diff} S^1$ in the form $s(w) = w + \sum_n s_n e^{-inw}$, where the $s_n$ are complex numbers satisfying $\hat{s}_n = s_{-n}$. These $s_n$ (with $n \neq 0$) provide local coordinates in the neighborhood of $b_0$. $s_0$ acts trivially on $b_0$ (cf. (3.3)). The behavior of the energy function in the neighborhood of $b_0$ can be seen by inserting the above $s$ into (3.8) and expanding it in terms of $s_n$. The expansion turns out to be of the form

$$l_0(s) = b_0 + \sum_n n^2 \left( b_0 + \frac{c}{24} n^2 \right) |s_n|^2 + O(s_n^3).$$

(3.26)

The stability of the classical vacuum is assured by the condition that $l_0$ is bounded from below at $b_0$. This is achieved only when $b_0$ satisfies

$$b_0 \geq -\frac{c}{24}. \quad (3.27)$$

All the coadjoint orbits corresponding to the 3-geometries under consideration satisfy this stability condition.

The orbit $W_{b_0}$ which satisfies the condition (3.27) can be quantized. This provides a unitary irreducible representation of the Virasoro algebra. Actually it can be quantized by the Kähler quantization or geometric quantization method. The coadjoint orbit is topologically the homogeneous space $\text{diff} S^1/H$. The little group $H$ is $S^1$ for black holes and $SL_2(\mathbb{R})$ for $AdS_3$. For these $H$, the homogeneous space $\text{diff} S^1/H$ becomes a complex manifold. The complex structure turns out to be compatible with the symplectic structure (3.6). Thereby, the orbit $W_{b_0}$

---

*Generators of $H$ can be seen easily by solving $\delta f(b_0, c) = 0$ (3.3) for each case.
which we need, becomes Kähler. The complex line bundle on the orbit with its first
Chern class \( \Omega \) provides the unitary irreducible representation on the space of the holomorphic sections.

To describe the representations it is convenient to shift the Virasoro generator
\( L_0 \) to \( L_0 + \frac{c}{24} \). With this shift the algebra (3.7) takes the standard form

\[
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}.
\]

(3.28)

Any representation of the Virasoro algebra with the central charge \( c \) can be specified by its highest weight state \( |h\rangle \). The state \( |h\rangle \), which is called “primary”, satisfies
\( L_0 |h\rangle = h|h\rangle \) and \( L_n |h\rangle = 0 \) for \( \forall n \geq 1 \). In addition to these conditions, the primary
state \( |0\rangle \) satisfies \( L_{-1} |0\rangle = 0 \). Thus it is an \( SL_2(\mathbb{R}) \) invariant state.

Unitary representations are the representations in which the \( L_n \) satisfy the condition
\( L_n^\dagger = L_{-n} \). The unitary irreducible representations which are obtained by the
quantization of the orbits can be summarized as follows.\(^1\)

For the orbit
\( |b_0 + \frac{c}{24}\rangle \), the corresponding unitary irreducible representation is given by the
Verma module \( V_{h=b_0+\frac{c}{24}} \). It is a module obtained by successive actions of \( L_{-n} \)
\( (n \geq 1) \) on \( |b_0 + \frac{c}{24}\rangle \). It has the form

\[
V_{b_0+\frac{c}{24}} = \bigoplus_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \mathbb{C} L_{-n_1} L_{-n_2} \cdots L_{-n_k} \left| b_0 + \frac{c}{24} \right\rangle.
\]

(3.29)

For the orbit \( W_{-c/24} \), the corresponding unitary irreducible representation is given by an analogue of the Verma module \( V_0 \). It is obtained by successive actions of \( L_{-n} \)
\( (n \geq 2) \) on \( |0\rangle \). We call this module \( V_0 \). It has the form

\[
V_0 = \bigoplus_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 2} \mathbb{C} L_{-n_1} L_{-n_2} \cdots L_{-n_k} |0\rangle.
\]

(3.30)

Gathering these results regarding quantization of the orbits, we can carry out
quantization of the asymptotic Virasoro symmetry of the 3-geometries and their
Virasoro deformations in the following manner. For the BTZ black hole \( X_{(I,M)} \),
the deformations \( Y_{(b,\hat{b})} \) of the exterior of the outer horizon can be identified with
the product of the coadjoint orbits \( W_{b_0}^{(+)} \times W_{b_0}^{(-)} \). The quantizations of the orbits
\( W_{b_0}^{(+)} \) and \( W_{b_0}^{(-)} \) provide the unitary irreducible representations \( V_{b_0+\frac{c}{24}}^{(+)} \) and \( V_{b_0+\frac{c}{24}}^{(-)} \).
Therefore the quantization of the deformations leads to the representation \( V_{b_0+\frac{c}{24}}^{(+)} \times V_{b_0+\frac{c}{24}}^{(-)} \). In particular, a state of the black hole can be identified with the primary
state \( |b_0 + \frac{c}{24}\rangle \otimes |b_0 + \frac{c}{24}\rangle \) of the representation . For \( AdS_3 \), the deformations of \( AdS_3^+ \)
can be identified with the product of the coadjoint orbits \( W_{-\frac{c}{24}}^{(+)} \times W_{-\frac{c}{24}}^{(-)} \). These
orbits provide the unitary irreducible representations \( V_0^{(+)} \) and \( V_0^{(-)} \). Therefore
quantization of the deformation leads to the representation \( V_0^{(+)} \times V_0^{(-)} \). The state

\(^1\) Here the condition \( c \geq 1 \) is required.
of $AdS_3$ can be identified with the primary state $|0\rangle \otimes |0\rangle$. This is the state invariant under $SL_2(\mathbb{R})_+ \times SL_2(\mathbb{R})_-$, which is the isometry of $AdS_3$.

Excitations by $L_{-n}$ correspond to the Virasoro deformation of $Y_{(b_0,\tilde{b}_0)} = Z_0/\sim$. Originally this is the deformation of $Z_0$ in $SL_2(\mathbb{R})$ which by no means provides a transformation of the black hole to another one having different mass and angular momentum.\(^4\) Nevertheless, these degrees of freedom provide the unitary representations of the Virasoro algebra. This shows that these degrees of freedom can be included in the physical spectrum as (massive) gravitons.\(^{21}\)

\section*{§4. Quantization of three-dimensional gravity}

As we have seen, quantizations of the Virasoro deformations of BTZ black holes and $AdS_3$ lead to unitary irreducible representations of the Virasoro algebra at least when $\frac{3l^2}{2G} \geq 1$. One may wonder whether quantization of three-dimensional gravity with negative cosmological constant becomes complete with these quantizations. In fact it does not. If one takes the Chern-Simons gravity viewpoint, these deformations correspond to the local degrees of freedom of the theory (gravitons), that is, oscillating modes of $b$ and $\tilde{b}$ in the flat connection (2.23). To make the quantization** complete, one must take into account the global degrees of freedom, i.e., holonomies. The holonomy variables should be dynamical in the Chern-Simons gauge theory. In our context this requires an introduction of their conjugates. Given a suitable Poisson structure on the holonomy variables and their quantizations, it is very reasonable to expect from the perspective of the $AdS_3/CFT_2$ correspondence that, with the identification of these quantum operators with the zero modes of an appropriate two-dimensional quantum field, the unitary representations obtained in the previous section could be reproduced as the Hilbert space of the two-dimensional conformal field theory. This expectation turns out to be true, at least when $\frac{3l^2}{2G} \gg 1$. The holonomy variables can be identified with the zero modes of a real scalar field $X$. This scalar field should be interpreted as the Liouville field ultimately.

We start with the description of the Poisson structure of the holonomies. Then we turn to construction of the unitary irreducible representations in the framework of the Liouville field theory or the Coulomb gas formalism. Finally, we make identifications between the states of three-dimensional gravity and those of the Liouville field theory.

\subsection*{4.1. Holonomy variable}

For a given closed path $\gamma$, the holonomy of a connection $A = (A^+, A^-)$ along the path is given by

$$H^{(\pm)}[\gamma] \equiv P e^{\int_{\gamma} A^{(\pm)}},$$

(4.1)

where $P$ denotes the path ordering. Since we can assume $A$ is a flat connection, $H[\gamma] = (H^+[\gamma], H^-[\gamma])$ is invariant up to conjugation under smooth deformations of $\gamma$. In particular, $\text{Tr} H^{(\pm)}$ are homotopy invariant. The BTZ black holes originally

\footnote{\(^4\) It is not a reparametrization of $Z_0$.}

\footnote{** It would give quantum gravity in the non-topological phase.\(^4\)}
admit a nontrivial holonomy around a closed path connecting \((w, \bar{w}, \rho)\) and \((w + 2\pi, \bar{w} - 2\pi, \rho)\) in the constant \(\rho\) surface. Let us call this path \(\gamma_1\). If there is only one nontrivial holonomy, there appears no symplectic structure. In order to make it a dynamical variable we need to introduce another “closed” path. Let \(\gamma_2\) be a path connecting \((w, \bar{w}, \rho)\) and \((w + \sqrt{c/\pi} \Sigma, \bar{w} + \sqrt{c/\pi} \bar{\Sigma}, \rho)\) in the constant \(\rho\) surface.

If one regards it as a closed path and then considers a holonomy around this path, the quantities \(\Sigma\) and \(\bar{\Sigma}\) together with \(r_{\pm}\) may become dynamical. In this extended “phase” space, the BTZ black holes themselves become a Lagrangian submanifold \(\Sigma = \bar{\Sigma} = 0\). This extension is a generalization of the off-shell extension of the Euclidean black holes examined by Carlip and Teitelboim.\(^{22}\)

The flat connection which describes \(X^{+}_{(J,M)}\), the exterior of the outer horizon of the non-extremal BTZ black hole, is given by (2.20). The holonomy of this connection around the path \(\gamma_1\) can be evaluated as

\[
H^{(+)}[\gamma_1] = h^{(+)}(w + 2\pi, \bar{w} - 2\pi, \rho)h^{(-1)}(w, \bar{w}, \rho) = e^{-\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1} e^{2\pi \sqrt{b_0/c/24} J^2} e^{\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1},
\]

\[
H^{(-)}[\gamma_1] = h^{(-)}(w + 2\pi, \bar{w} - 2\pi, \rho)h^{(-1)}(w, \bar{w}, \rho) = e^{\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1} e^{-2\pi \sqrt{b_0/c/24} J^2} e^{-\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1},
\]

where \(h^{(\pm)}\) are those used in its trivialization, and their explicit forms are given in (2.19). The trace becomes

\[
\text{Tr}H^{(+)}[\gamma_1] = 2\cosh \pi \sqrt{\frac{b_0}{c/24}}, \quad \text{Tr}H^{(-)}[\gamma_1] = 2\cosh \pi \sqrt{\frac{b_0}{c/24}}.
\]

Similarly, its holonomy around the path \(\gamma_2\) has the form

\[
H^{(+)}[\gamma_2] = e^{-\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1} e^{2\Sigma J^2} e^{\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1},
\]

\[
H^{(-)}[\gamma_2] = e^{\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1} e^{-2\Sigma J^2} e^{-\frac{1}{2}(\rho - \ln \frac{b_0}{c/\pi})J^1}.
\]

Therefore we obtain

\[
\text{Tr}H^{(+)}[\gamma_2] = 2\cosh \Sigma, \quad \text{Tr}H^{(-)}[\gamma_2] = 2\cosh \bar{\Sigma}.
\]

To describe the Poisson structure of these holonomies in Chern-Simons gravity, we follow the recipe developed by Nelson, Regge and Zertuche.\(^{23}\) It dictates that holonomies in a constant time surface have non-vanishing Poisson brackets when the underlying paths intersect. If one regards the radial coordinate \(\rho\) as a time, the following Poisson algebra of \(\text{Tr}H^{(\pm)}[\gamma_i](i = 1, 2)\) can be obtained:

\[
\{\text{Tr}H^{(\pm)}[\gamma_1], \text{Tr}H^{(\pm)}[\gamma_2]\} = \mp \frac{6\pi}{c} \left[ - \left(\text{Tr}H^{(\pm)}[\gamma_1]\right) \left(\text{Tr}H^{(\pm)}[\gamma_2]\right) + 2\text{Tr}H^{(\pm)}[\gamma_2\gamma_1]\right].
\]
Substitution of the explicit forms (4.3) and (4.5) simplifies the expression (4.6) to
\[
\begin{align*}
\{ \cosh \frac{\pi r_+}{l}, \cosh \frac{\pi r_+ - r_-}{l}, \cosh \Sigma \} &= - \frac{4\pi G}{l} \sinh \frac{\pi r_+ - r_-}{l} \sinh \Sigma, \\
\{ \cosh \frac{\pi r_+ + r_-}{l}, \cosh \frac{\pi r_+ - r_-}{l}, \cosh \tilde{\Sigma} \} &= - \frac{4\pi G}{l} \sinh \frac{\pi r_+ + r_-}{l} \sinh \tilde{\Sigma}.
\end{align*}
\] (4.7)

These indicate that \( r_+ + r_-, \Sigma \) and \( \tilde{\Sigma} \) become canonical variables with the Poisson algebra
\[
\begin{align*}
\{ r_+ + r_-, \Sigma \} &= -4G, \\
\{ r_+ - r_-, \tilde{\Sigma} \} &= -4G.
\end{align*}
\] (4.8)

Having obtained the symplectic structure, one can quantize these variables. Let \( \hat{r}_+ + \hat{r}_- \), \( \hat{\Sigma} \) and \( \hat{\tilde{\Sigma}} \) be the corresponding operators. Their nontrivial commutation relations are
\[
\begin{align*}
[\hat{r}_+ + \hat{r}_-, \hat{\Sigma}] &= -4iG, \\
[\hat{r}_+ - \hat{r}_-, \hat{\tilde{\Sigma}}] &= -4iG.
\end{align*}
\] (4.9)

4.2. Realization through the Liouville field theory

Let \( X \) be a real scalar field on \( P^1 \).\(^{\text{a)}} \) The action is given by
\[
S[X] = \frac{1}{4\pi i} \int_{P^1} \bar{X} \wedge \partial X + \frac{\alpha_0}{2\pi} \int_{P^1} RX,
\] (4.10)
where \( R \) is the Riemann tensor of a fixed Kähler metric on \( P^1 \).\(^{\text{b)}} \) Here \( \alpha_0 \) is a real number, and its value is specified below.

If one takes the expansion
\[
X(z) = x - ip \ln z + i \sum_{n \neq 0} \frac{\alpha_n}{n} z^{-n}
\] (4.11)
of \( X \) at \( z = 0 \), \(^{\text{c)}} \) the nontrivial commutation relations among the mode operators become
\[
[x, p] = i, \quad [a_m, a_n] = m \delta_{m+n, 0},
\] (4.12)
where \( a_0 = p \). Using these operators we define the in-Fock vacuum \(|k\rangle_{\text{in}} \) \((k \in C) \) in the following manner: Let \(|0\rangle_{\text{in}} \) be the state defined by the conditions
\[
a_n|0\rangle_{\text{in}} = 0 \quad \text{for } \forall n \geq 0.
\] (4.13)

Then \(|k\rangle_{\text{in}} \) is introduced as the state obtained from \(|0\rangle_{\text{in}} \) by the relation
\[
|k\rangle_{\text{in}} = e^{ikx}|0\rangle.
\] (4.14)

\(^{\text{a)}} \) To use the conventional technique of 2d CFT we consider the Euclidean version.

\(^{\text{b)}} \frac{1}{2\pi} \int_{P^1} R = \chi(P^1) = 2.

\(^{\text{c)}} \) For simplicity we only describe the holomorphic part of the theory. The operator product expansion of \( X(z) \) (holomorphic part of \( X(z, \bar{z}) \)) becomes \( X(z)X(w) \sim -\ln(z - w) \).
This state satisfies \( p|k\rangle_{\text{in}} = k|k\rangle_{\text{in}} \). The in-Fock space \( \mathcal{F}_{k}^{\text{in}} \) is the Fock space built on \( |k\rangle_{\text{in}} \). It has the form
\[
\mathcal{F}_{k}^{\text{in}} = \bigoplus_{n_{1} \geq n_{2} \geq \cdots \geq n_{p} \geq 1} C a_{-n_{1}} a_{-n_{2}} \cdots a_{-n_{p}} |k\rangle_{\text{in}}.
\]
This space is located at \( z = 0 \).

The stress tensor \( T \) has the form
\[
T(z) = -\frac{1}{2} \partial X \partial X(z) + \alpha_{0} \partial^{2} X(z).
\] (4.15)

Expansion of the stress tensor in the form \( T(z) = \sum_{n} L_{n} z^{-n-2} \) provides the generators of the Virasoro algebra (3.28) with the central charge \( 1 + 12\alpha_{0}^{2} \). In terms of the oscillator modes, their expressions become
\[
L_{0} = \sum_{k \geq 1} a_{-k} a_{k} + \frac{1}{2} p^{2} + i\alpha_{0} p,
\]
\[
L_{n} = \frac{1}{2} \sum_{k} a_{-k} a_{k+n} + i(n+1)\alpha_{0} a_{n} \quad \text{ for } \forall n \neq 0.
\] (4.16)

The actions of \( L_{n} \) on the in-Fock vacua satisfy the following properties: On the vacuum \( |0\rangle_{\text{in}} \), using the expressions (4.16), one can read
\[
L_{n}|0\rangle_{\text{in}} = 0 \quad \text{ for } \forall n \geq -1,
\] (4.17)

while on the vacuum \( |k\rangle_{\text{in}} \) with \( k \neq 0 \), one obtains
\[
L_{0}|k\rangle_{\text{in}} = \left( \frac{(k+i\alpha_{0})^{2}}{2} + \frac{\alpha_{0}^{2}}{2} \right) |k\rangle_{\text{in}},
\]
\[
L_{n}|k\rangle_{\text{in}} = 0 \quad \text{ for } \forall n \geq 1.
\] (4.18)

These properties, together with a comparison between the commutation relations \([L_{0}, a_{-n}] = na_{-n}\) and \([L_{0}, L_{-n}] = nL_{-n}\), show that the in-Fock space \( \mathcal{F}_{k}^{\text{in}} \) is equivalent to the Verma module \( \mathcal{V}_{h} = \mathcal{V}_{\frac{1}{2}(k+i\alpha_{0})^{2} + \frac{1}{2}\alpha_{0}^{2}} \) with \( c = 1 + 12\alpha_{0}^{2} \) or \( \mathcal{V}_{0}' \) if \( k = 0 \).

Hermitian conjugation of the mode operators becomes
\[
a_{n}^{\dagger} = a_{-n} \quad \text{ for } \forall n \neq 0,
\]
\[
p^{\dagger} = p + 2i\alpha_{0},
\]
\[
x^{\dagger} = x.
\] (4.19)

Note that \( p \) cannot be hermitian due to the existence of the background charge \( \alpha_{0} \). Since the action (4.10) is real, it is expected that the realization (4.16) of the Virasoro algebra becomes unitary. In fact, under this conjugation the realization becomes unitary; that is, the \( L_{n} \) satisfy \( L_{n}^{\dagger} = L_{-n} \). Unitarity imposes a constraint on the allowed value \( k \) of the in-Fock vacuum. The condition \( L_{0}^{\dagger} = L_{0} \) together with the requirement that its eigenvalue be non-negative, restricts \( |k\rangle_{\text{in}} \) to the form (Fig. 4)
\[
k = \beta - i\alpha_{0}, \quad \beta \in \mathbf{R}_{\geq 0},
\] (4.20)
or
\[ k = i(\gamma - 1)\alpha_0, \quad \gamma \in (0, 1]. \] (4.21)

Now we introduce the out-Fock space \( \mathcal{F}_{k'}^{\text{out}} \). Let \( \text{out} \langle 0 \rangle \) be the state which satisfies the conditions \( \text{out} \langle 0 \rangle a_n^\dagger = 0 \) (for \( \forall n \geq 0 \)). The out-Fock vacuum \( \text{out} \langle k' \rangle \) can be introduced by \( \text{out} \langle k' \rangle = \text{out} \langle 0 \rangle e^{-ik'x} \). It satisfies \( \text{out} \langle k' \rangle |p = \text{out} \langle k' \rangle |(k' - 2i\alpha_0) \). The out-Fock space \( \mathcal{F}_{k'}^{\text{out}} \) is the Fock space built on \( \text{out} \langle k' \rangle \). Using (4.19) it has the form
\[ \mathcal{F}_{k'}^{\text{out}} = \bigoplus_{n_1 \geq n_2 \geq \cdots \geq n_p \geq 1} C_{\text{out}} \langle k' \rangle |a_{n_p} \cdots a_{n_2} a_{n_1}. \]

This space is located at \( z = \infty \).

The actions of \( L_n \) on the out-Fock vacua satisfy the following properties: On the vacuum \( \text{out} \langle 0 \rangle \), using the expressions (4.16), one can obtain
\[ \text{out} \langle 0 \rangle |L_{-n} = 0 \quad \text{for } \forall n \geq -1, \] (4.22)
while on the vacuum \( \text{out} \langle k' \rangle \) with \( k' \neq 0 \),
\[ \text{out} \langle k' \rangle |L_0 = \text{out} \langle k' \rangle \left\{ \frac{(k' - i\alpha_0)^2}{2} + \frac{\alpha_0^2}{2} \right\}, \]
\[ \text{out} \langle k' \rangle |L_{-n} = 0 \quad \text{for } \forall n \geq 1. \] (4.23)

These properties, along with the action of \( p \) on the out-Fock vacuum, show that the out-Fock space \( \mathcal{F}_{k'=k+2i\alpha_0}^{\text{out}} \) can be identified with \( \ast \mathcal{F}_k^{\text{in}} \), the dual of \( \mathcal{F}_k^{\text{in}} \). With this identification the pairing between \( \mathcal{F}_{k+2i\alpha_0}^{\text{out}} \) and \( \mathcal{F}_k^{\text{in}} \) becomes non-degenerate. In particular, if one takes the allowed value (4.20) or (4.21) of \( k \), the representation becomes unitary. Choosing the value of \( \alpha_0 \) appropriately, it becomes the unitary irreducible representation of the Virasoro algebra obtained by the quantization of the coadjoint orbit.

\[ ^{(1)} \text{We use a convention such that the state } \gamma_k \langle k \rangle, \text{ which is dual to } |k\rangle_{\text{in}}, \text{ satisfies } \gamma_k \langle k \rangle |p = \gamma_k \langle k \rangle |k. \]
4.3. Quantization of 3d gravity based on the Liouville field theory

Given the commutation relations (4.9) of the quantum holonomy operators \( \hat{r}_+, \hat{r}_-, \hat{\Sigma} \) and \( \hat{\tilde{\Sigma}} \), one is tempted to identify these topological operators with the zero modes of the Liouville field \( X(z, \bar{z}) \). Let \( x(\tilde{x}) \) and \( p(\tilde{p}) \) be the zero modes of the holomorphic part \( X(z) \) (the anti-holomorphic part \( X(\bar{z}) \)) of the Liouville field. The identification is precisely given by

\[
x = \sqrt{\frac{l}{2G}} \hat{\Sigma}, \quad \tilde{x} = \sqrt{\frac{l}{2G}} \hat{\tilde{\Sigma}},
\]

\[
p = \frac{\hat{r}_+ + \hat{r}_-}{\sqrt{8Gl}}, \quad \tilde{p} = \frac{\hat{r}_+ - \hat{r}_-}{\sqrt{8Gl}},
\] (4.24)

and it becomes consistent with the commutation relations \([x, p] = i\) and \([\tilde{x}, \tilde{p}] = i\).

4.3.1. Identification of states

At the end of the previous section, the state of the BTZ black hole \( X^+_{(J,M)} \) was identified with the primary state \( |b_0 + \frac{c}{24}\rangle \otimes |\tilde{b}_0 + \frac{c}{24}\rangle \). The values of the weights can be expressed in terms of the geometrical data:

\[
b_0 + \frac{c}{24} = \frac{1}{16Gl} (r_+ - r_-)^2 + \frac{l}{16G},
\]

\[
\tilde{b}_0 + \frac{c}{24} = \frac{1}{16Gl} (r_+ + r_-)^2 + \frac{l}{16G}.
\] (4.25)

A comparison of these weights with those of the in-Fock vacuum given in (4.18) forces us to adjust the background charge of the Liouville field to

\[
\alpha_0 = \sqrt{\frac{l}{8G}},
\] (4.26)

and, using this background charge, it also leads to the identification of the black hole state with the following in-Fock vacuum:

\[
X^+_{(J,M)} \Longleftrightarrow |J, M\rangle_{in} \equiv |k_{(J,M)}\rangle_{in} \otimes |\tilde{k}_{(J,M)}\rangle_{in},
\] (4.27)

where \( k_{(J,M)} \) and \( \tilde{k}_{(J,M)} \) are given by

\[
k_{(J,M)} = -i \sqrt{\frac{l}{8G}} \frac{r_+ - r_-}{\sqrt{8Gl}},
\]

\[
\tilde{k}_{(J,M)} = -i \sqrt{\frac{l}{8G}} \frac{r_+ + r_-}{\sqrt{8Gl}}.
\] (4.28)

These are the values allowed by the unitarity condition (4.20). The region (a) in Fig. 4 corresponds to the black holes. All the excitations in the Fock space \( F^+_{k_{(J,M)}} \times F^+_{\tilde{k}_{(J,M)}} \) can be identified with the excitations owing to the Virasoro deformation of \( X^+_{(J,M)} \). These correspond to gravitons.
To employ the above formalism completely, the central charge (2.16) of the Virasoro deformation of the 3-geometries should be matched with that of the two-dimensional conformal field theory. This consistency requires the condition $c \gg 1$, or equivalently $\sqrt{c} \gg 1$. Under this condition the central charge of the Liouville field theory can be regarded as $1 + 12\alpha_0^2 = 1 + \frac{3\ell}{c^2} \sim \frac{3\ell}{c^2}$. Therefore these two coincide in this limit. The deviation at finite $c$ is of order $1/c$.

The state of $AdS_3^+$ has been identified with the primary state $|0\rangle \otimes |0\rangle$. This is invariant under $SL_2(R)_+ \times SL_2(R)_-$. For this reason we can identify this state with the $sl_2(C)$-invariant vacuum of the Liouville field theory:

$$AdS_3 \iff \langle \text{vac} \rangle \equiv |0\rangle_{\text{in}} \otimes |0\rangle_{\text{in}}. \quad (4.29)$$

The $sl_2(C)$-invariant vacuum is the Fock vacuum with $\gamma = 1$ ($\tilde{\gamma} = 1$) in (4.21) (the origin in Fig. 4). From the viewpoint of the unitary representation theory, other values of $\gamma$ and $\tilde{\gamma}$ are also allowed as long as they satisfy $0 < \gamma, \tilde{\gamma} \leq 1$. Note that $(\gamma, \tilde{\gamma}) = (0,0)$; that is, $(\beta, \tilde{\beta}) = (0,0)$ in (4.20) corresponds to the state of the massless black hole $X^+_{(0,0)}$. Consider a state $|k_\gamma\rangle_{\text{in}} \otimes |\tilde{k}_{\tilde{\gamma}}\rangle_{\text{in}}$, where $k_\gamma$ and $\tilde{k}_{\tilde{\gamma}}$ have the form given in (4.21). The weights of this state are respectively $-\frac{2\gamma}{16c^2} + \frac{l}{16c^2}$ and $-\frac{2\tilde{\gamma}}{16c^2} + \frac{l}{16c^2}$. Using (4.25) one can read off the mass and angular momentum as $M = -\frac{\gamma^2 + \tilde{\gamma}^2}{16c^2}$ and $J = -(\gamma^2 - \tilde{\gamma}^2)$). Therefore the corresponding 3-geometry is conic. This implies that the state $|k_{\gamma}\rangle_{\text{in}} \otimes |\tilde{k}_{\tilde{\gamma}}\rangle_{\text{in}}$ is that of a conical singularity. The region (b) in Fig. 4 corresponds to the conical singularities. Rather surprisingly, its Virasoro deformation gives rise to the unitary representation. This might imply that conical singularities with the allowed mass and momentum get mild quantum mechanically. Our understanding of singularities in classical relativity may be required to change in quantum theory.

The values of $b_0$ and $\tilde{b}_0$ which correspond to $k_\gamma$ and $\tilde{k}_{\tilde{\gamma}}$ are $-\frac{2\gamma}{16c^2}$ and $-\frac{2\tilde{\gamma}}{16c^2}$. The three-dimensional metric becomes

$$ds^2_{(b_0,\tilde{b}_0)} = -\frac{2\gamma}{16c^2} - \frac{2\tilde{\gamma}}{16c^2} = \frac{1}{4} \left\{ -d(\gamma w)^2 - d(\tilde{\gamma} \tilde{w})^2 - (e^{\rho + \ln \frac{4}{\gamma}} + e^{-\rho + \ln \frac{4}{\tilde{\gamma}}})d(\gamma w)d(\tilde{\gamma} \tilde{w}) + (d\rho)^2 \right\}. \quad (4.30)$$

In this case $Z_0$ will be taken in $\widetilde{SL}_2(R)$. If one considers it in $SL_2(R)$, it consists of the elements

$$e^{-\gamma w J_0} e^{(\rho + \ln \frac{4}{\gamma}) J_1} e^{-\tilde{w} J_0}. \quad (4.31)$$

The three-geometry which we wish to describe is $Y_{(-\frac{2\gamma}{16c^2}, \frac{2\tilde{\gamma}}{16c^2})}$. It is obtained from $Z_0$ by the projection $\pi$. A comparison of (4.31) with the Cartan decomposition of $SL_2(R)$ given by $e^{-\frac{1}{2}(1+\theta) J_0} e^{\sigma J_1} e^{-\frac{1}{2}(1-\theta) J_0}$ shows that $Z_0$ can be taken in $\widetilde{SL}_2(R)$ as a “cheese cake” with angle $\pi(\gamma + \tilde{\gamma})$ (see Fig. 5). The ranges are $-\infty < t < +\infty$, $0 \leq \theta < \pi(\gamma + \tilde{\gamma})$ and $0 < \sigma < +\infty$. The projection $\pi$ causes a conical singularity at
σ = 0 (Fig. 6). Mathematically speaking, \( \pi : Z_0 \to Y(\gamma, \tilde{\gamma}, \lambda) \) is regular, but once we make the completion of \( Z_0 \) at \( \sigma = 0 \) by adding the line \( PP' \) and extend \( \pi \) to this line, it becomes singular. This singular mapping causes the conical singularity. In particular, when \( \gamma = \tilde{\gamma} \), the appearance of the conical singularity is due to a particle with mass \( \frac{1}{4} \gamma^2 \) sitting at \( \sigma = 0 \).

4.3.2. How can we understand 3d black holes in two dimensions?

Having obtained the correspondence between the states of three-dimensional quantum gravity and two-dimensional conformal field theory, one may ask how three-dimensional black holes can be understood in two dimensions, where the conformal field lives.

To investigate this question, it is useful to discuss first the case of conical singularities. The state \(|\gamma, \tilde{\gamma}\rangle_{in} \otimes |\hat{k}_\gamma, \hat{k}_{\bar{\gamma}}\rangle_{in} \) of a conical singularity can be obtained by an operation of the corresponding vertex operator on the \( sl_2(C) \)-invariant vacuum,  

\[
|\gamma, \tilde{\gamma}\rangle_{in} = \lim_{z, \bar{z} \to 0} e^{ik_\gamma X(z)} e^{i\tilde{k}_{\bar{\gamma}} X(\bar{z})} |\text{vac}\rangle_{in}. \tag{4.32}
\]

Putting the holomorphic and anti-holomorphic pieces together, we can rewrite the above operator in the form  

\[
e^{ik_\gamma X(z)} e^{i\tilde{k}_{\bar{\gamma}} X(\bar{z})} = e^{\sqrt{\frac{2l}{8G}} \left(1 - \frac{\gamma^2}{2} \right) X(z, \bar{z}) - \frac{\gamma^2}{4} \tilde{X}(z, \bar{z})}, \tag{4.33}
\]

where \( X(z, \bar{z}) = X(z) + X(\bar{z}) \) and \( \tilde{X}(z, \bar{z}) \equiv X(z) - X(\bar{z}). \) If one takes the path-integral formulation of 2d CFT, the norm of this state, which should be normalized
to unity, can be written as
\[
\langle \gamma, \tilde{\gamma} | \gamma, \tilde{\gamma} \rangle_{\text{in}} = \int DX e^{-S} \cdot e^{\sqrt{\frac{l}{8G}} (1 - \frac{\gamma + \tilde{\gamma}}{2}) X(\infty) + \sqrt{\frac{l}{8G}} \frac{\gamma - \tilde{\gamma}}{2} \tilde{X}(\infty)} \cdot e^{\sqrt{\frac{l}{8G}} (1 - \frac{\gamma + \tilde{\gamma}}{2}) X(0) - \sqrt{\frac{l}{8G}} \frac{\gamma - \tilde{\gamma}}{2} \tilde{X}(0)},
\]

(4.34)

where \( S \) is the action (4.10) with \( \alpha_{0} = \sqrt{\frac{l}{8G}} \). The vertex operators inserted at 0 and \( \infty \) respectively create states of the conical singularity from the \( sl_{2}(C) \) invariant in- and out-vacua. Choosing the background metric so that its curvature is concentrated at \( \infty \), we rewrite the expression (4.34) in the form
\[
\langle \gamma, \tilde{\gamma} | \gamma, \tilde{\gamma} \rangle_{\text{in}} = \int DX e^{-\frac{1}{16\pi i} \int \partial X \wedge \partial X} \cdot e^{\sqrt{\frac{l}{8G}} (1 - \frac{\gamma + \tilde{\gamma}}{2}) X(\infty) + \sqrt{\frac{l}{8G}} \frac{\gamma - \tilde{\gamma}}{2} \tilde{X}(\infty)}
\]

\[
\cdot e^{\sqrt{\frac{l}{8G}} (1 - \frac{\gamma + \tilde{\gamma}}{2}) X(0) - \sqrt{\frac{l}{8G}} \frac{\gamma - \tilde{\gamma}}{2} \tilde{X}(0)}.
\]

(4.35)

The Liouville field \( X \) treats the conical singularity as an insertion of the corresponding vertex operator. In three dimensions, the origin of this vertex operator can be understood as follows. To explain this, we first remark that the \( P^{1} \) where the Liouville field theory lives can be regarded as that obtained by a compactification of the boundary cylinder at infinity. This compactification also makes the solid cylinder become a three-dimensional ball. In such a compactification to a three-ball, the conical singularity located at the center must intersect with \( P^{1} \) precisely at the two points 0 and \( \infty \) (see Fig. 7). Now, returning to the path-integral expression given in (4.35), we can say that these intersections of the conical singularity with the boundary sphere are realized as the vertex operators in the boundary Liouville field theory. In particular, the gravitational state of a particle with mass \( \frac{1}{4\pi G} \) positioned at the center of AdS3 can be treated by an insertion of the corresponding vertex operator.

Next let us discuss the case of black holes. Basically we follow the same argument as above. The black hole state (4.27) can be obtained by an operation of the corresponding vertex operator on the \( sl_{2}(C) \)-invariant vacuum,
\[
|J, M\rangle_{\text{in}} = \lim_{z, \bar{z} \to 0} e^{\left( \sqrt{\frac{l}{8G}} + i \omega \frac{\gamma - \tilde{\gamma}}{4} \right) X(z, \bar{z}) + \sqrt{\frac{l}{8G}} \frac{\gamma - \tilde{\gamma}}{2} \tilde{X}(z, \bar{z})} \langle \text{vac} \rangle_{\text{in}}.
\]

(4.36)

If one takes the path-integral formulation, the norm of the black hole state can be written in the following manner:
\[
\langle J, M | J, M \rangle_{\text{in}}
\]
This path-integral representation also provides some idea about our interpretation of black holes in two dimensions. We first remark that the outer horizon of a black hole cannot be recognized as a two-dimensional object under the Virasoro deformation. (See Fig. 3 for the non-extremal case.) This horizon is treated as a one-dimensional object for a massive black hole and as a point for a massless black hole. The Virasoro deformation is originally introduced as a deformation of the non-compact simply-connected region $Z_0$. This region of $SL_2(\mathbb{R})$ is the covering space of the exterior of the outer horizon of the black hole. Under the Virasoro deformation, what one can recognize as the outer horizon is the counterpart of the outer horizon on $SL_2(\mathbb{R})$. It is obtained by the completion of $Z_0$ at $\sigma = 0$. We can easily see that these completions for the cases of massive extremal and massless black holes are respectively given by adding one-dimensional and zero-dimensional objects. The zero-dimensional nature in the massless case may be understood as a degeneration from the massive case. We hope to explain the geometrical origin of the insertion of the vertex operator in (4.37). Let us first examine the compactification of the boundary cylinder of the black hole $X^{+}_{(J,M)}$ to $P_1$. The $t = \pm \infty$ circles of the boundary cylinder are mapped to the two points 0 and $\infty$ of $P_1$. This compactification of the boundary also provides a compactification of the bulk geometry. Now we ask to what it gives rise. We can assume reasonably that $t = \pm \infty$ slices of $X^{+}_{(J,M)}$ are compactified to 0 and $\infty$ of $P_1$. If this assumption holds, the outer horizon must be compactified to 0 and $\infty$ of $P_1$. This is because the $t = \pm \infty$ slices of $X^{+}_{(J,M)}$ include the outer horizon after the completion (see Fig. 3). This shows that the intersections of the black hole with the boundary sphere are realized by the vertex operators, as described in (4.37) (see Fig. 8).

§5. Towards conformal field theory on the horizon

Obtaining a microscopic description of black holes in quantum gravity or string theory would be a very impressive and enlightening accomplishment. For three-dimensional black holes, such a microscopic description has been proposed by Carlip\(^{11}\) and Strominger\(^{8,9}\). It asserts that microscopic states of black holes are the states of a conformal field theory on the horizon. Although at present we do not
know exactly what conformal fields exist on the horizon, their proposal makes it possible to give statistical mechanical explanations of the thermodynamical properties of the black hole.

Considering their proposal, many questions arise. First of all, can one identify the boundary Liouville field theory with a conformal field theory on the horizon? Our answer is ‘no’. As explained at the end of the last section, the Virasoro deformation of the exterior of the outer horizon cannot recognize the horizon as a two-dimensional object: it can only recognize the horizon as a one-dimensional object. This means that the Virasoro algebra obtained from the deformation cannot be the Virasoro algebra on the horizon. On the other hand, this deformation provides the asymptotic Virasoro algebra, whose generators are gravitons in the bulk and are identified with the excitations of the boundary Liouville field. This field theory has continuous spectra. Therefore this conformal field theory is not equivalent to a conformal field theory on the horizon which describes microscopic states of black holes. These two field theories, having the same central charge, are different points in the moduli space of 2-dimensional conformal field theory.

If one accepts this answer, one might be led to another question: What conformal field theory lives on the horizon? To this question we cannot give a definite answer. Here we would like to propose two possible descriptions of this conformal field theory. The first one is based on a simple hypothesis: The Virasoro algebra of the horizon conformal field theory should generate the Virasoro deformation of the horizon. To work this hypothesis, we need, first of all, to treat the outer horizon as a two-dimensional object. If one takes the Chern-Simons gravity viewpoint, it is doubtful whether one can handle the outer horizon as a two-dimensional object. To be successful, presumably one needs the string theory viewpoint. More precisely, if the two-dimensional horizon is identified with a macroscopic string in $SL_2(\mathbb{R})$, the horizon conformal field theory would be string theory on this background. This theory describes quantum fluctuations of the macroscopic string and will lead to a microscopic description of the black holes. But it is not clear, at least for us, to what extent this string theory is analyzable.

The second possible description of the horizon conformal field theory is based on the hypothesis that the Virasoro deformation of the region between the inner and outer horizons leads to the Virasoro algebra on the horizon conformal field theory. This hypothesis can be regarded as an analogue to that which we used in the construction of the boundary Liouville field theory. In particular, it may allow us to follow steps similar to those we took in the study of the exterior of the outer horizon.

To be explicit, let us consider the region between the inner and outer horizons of the non-extremal BTZ black hole $X_{(J,M)}$. We will call this region $X_{(J,M)}^{-}$. A covering space of $X_{(J,M)}^{-}$ is given by a non-compact simply-connected region of $SL_2(\mathbb{R})$ consisting of the group elements

$$e^{-\varphi J^2} e^{\sigma J^0} e^{\psi J^2},$$

(5.1)

Explicitly, $X_{(J,M)}^{-} \equiv \{(t, \phi, r) \in X_{(J,M)} | r_- < r < r_+\}$. 

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\(^{a)}\) Explicitly, $X_{(J,M)}^{-} \equiv \{(t, \phi, r) \in X_{(J,M)} | r_- < r < r_+\}$. 

---
where the ranges of $\varphi$, $\psi$, and $\sigma$ are taken as $-\infty < \varphi, \psi < +\infty$ and $0 < \sigma < \pi$. Let us call this region $Z_1$. The projection can be described by

$$ \frac{r_+ - r_-}{l} \left( \frac{l}{l} + \phi \right) = \varphi, \quad \frac{r_+ - r_-}{l} \left( \frac{l}{l} - \phi \right) = \psi, \quad \frac{2r_+^2 - r_+^2 - r_-^2}{r_+^2 - r_-^2} = \cos \sigma, \quad (5.2) $$

together with the identification $\phi \sim \phi + 2\pi$. The induced metric of $Z_1$ has the form

$$ ds^2 = \frac{l^2}{4} \left( (d\varphi)^2 + (d\psi)^2 - 2\cos \sigma \, d\varphi d\psi - (d\sigma)^2 \right), \quad (5.3) $$

and it is isometric to $ds^2_{X(J,M)}$. Therefore $Z_1$ is indeed the covering space. $X_{(J,M)}$ is identified with the quotient of $Z_1$

$$ X_{(J,M)} = Z_1/\sim, \quad (5.4) $$

where the equivalence relation is $(\varphi, \psi, \sigma) \sim (\varphi + 2\pi \frac{r_+ + r_-}{l}, \psi - 2\pi \frac{r_+ - r_-}{l}, \sigma)$. As is done in the study of $Z_0$, it is convenient to rewrite the group element (5.1) in the form

$$ e^{-\sqrt{\frac{b_0}{c/24}} w J^2} e^{(\rho - \frac{1}{2} \ln \frac{b_0}{c/24}) J^0} e^{\sqrt{\frac{b_0}{c/24}} \tilde{w} J^2} \cdot e^{h_1(w, \tilde{w}, \rho)}, \quad (5.5) $$

Description of $Z_1$ in terms of the Chern-Simons gravity is given by the flat connection $A_1 = (A_1^{(+)}, A_1^{(-)})$, which is determined by the prescription (2.10) using a decomposition of $h_1$ into the following $SL_2(\mathbb{R})_{\pm}$-valued functions $h_1^{(\pm)}$:

$$ h_1^{(+)} = e^{-\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^0} e^{\sqrt{\frac{b_0}{c/24}} w J^2}, \quad h_1^{(-)} = e^{\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^0} e^{\sqrt{\frac{b_0}{c/24}} \tilde{w} J^2}. \quad (5.6) $$

A quick comparison between $h_1^{(\pm)}$ and $h^{(\pm)}$ given in (2.19) shows that the flat connection $A_1$ is gauge-equivalent to $A_0$, given by (2.20), the flat connection which describes $Z_0$, which is the covering space of the exterior of the horizon:

$$ A_1 = A_0^g. \quad (5.7) $$

Here the gauge transformation $g = (g^{(+)}, g^{(-)})$ is given by $g^{(+)} = e^{-\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^0} e^{\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^1}$ and $g^{(-)} = e^{\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^0} e^{-\frac{1}{2} (\rho - \ln \frac{b_0}{c/24}) J^1}$. This gauge equivalence should be understood as a formal one, because the allowed ranges of the coordinates differ. (For $A_0$ given by (2.20) the range of $\rho$ is $\frac{1}{2} \ln \frac{b_0 b_1}{(c/6)^2} < \rho < +\infty$, while for $A_1$ it is $\frac{1}{2} \ln \frac{b_0 b_1}{(c/6)^2} < \rho < \pi + \frac{1}{2} \ln \frac{b_0 b_1}{(c/6)^2}$.) Nevertheless, if one takes the existence of such a gauge transformation seriously, one may arrive at the idea that a deformation of $Z_1$ could be constructed in such a manner that it is related with the Virasoro deformation of $Z_0$ by an appropriate gauge transformation. The deformation so obtained will define the Virasoro deformation of $Z_1$ and thereby that of $X_{(J,M)}$, the region between the inner and outer horizons of the black hole. The corresponding conformal field theory can be regarded as the horizon conformal field theory.
The second description of the horizon conformal field theory may provide another interpretation of the surprising conjecture made by Maldacena, which may be understood in our situation as a correspondence between the boundary and horizon conformal field theories under a suitable renormalization group flow. The second description, if it is correct, implies the possibility of a realization of this correspondence in terms of gauge transformations.

Acknowledgements

We thank S. Mano for useful discussions.

References

5) S. Carlip, Quantum Gravity in 2+1 Dimensions (Cambridge, 1998).
7) M. Banados, hep-th/9901148.

Note added: After this work was completed we received a preprint in which the content of 3.3 is also discussed.