Light-Cone Quantization of the Schwinger Model

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We consider constructing a canonical quantum theory of the light-cone gauge \( A_- = 0 \) Schwinger model in the light-cone representation. Quantization conditions are obtained by requiring that the translational generators \( P_+ \) and \( P_- \) give rise to Heisenberg equations which, in a physical subspace, are consistent with the field equations. A consistent operator solution with residual gauge degrees of freedom is obtained by solving initial value problems on the light cones. The construction allows a parton picture, although we have \( \theta \)-vacua in the theory.

§1.Introduction

Recently the search for nonperturbative solutions of QCD has led to extensive studies of light-cone field theory, in which the infinite-momentum limit is incorporated by introducing the variables \( x^+ = x^0 + x^3, \) \( x^- = x^0 - x^3, \) and carrying out a change of the representation space such that one is able to have a vacuum state composed only of particles with nonnegative longitudinal momenta and also to have relativistic bound-state equations of the Schrödinger-type.

Quantization has traditionally been carried out in parallel with the axial gauge formulation of QED in ordinary space-time coordinates. Thus \( x^+ \) and \( A^- = \frac{A^i - A^0}{2} = 0 \) have been chosen respectively to be the evolution parameter and the gauge fixing condition, and the minus component of the field equation, on which the gauge fixing condition is imposed, has been solved as a constraint by using the operator \((\partial^-)^{-1}\) to express the Hamiltonian in terms of the physical degrees of freedom. In what follows we call this formulation “the light-cone axial gauge formulation.” It turns out, however, that this formulation encounters inherent difficulties. The quantity \(((\partial^-)^{-1})^2\), which is needed for this construction, turns out to be ill-defined in a positive definite Hilbert space. One result of this is that the light-cone gauge formulations are not ghost free, contrary to what was originally expected and is still sometimes claimed. It was first found that in order to bring perturbative calculations done in light-cone gauge into agreement with calculations done in covariant gauges, the spurious singularity of the free gauge field propagator, which arises from applying \((\partial^-)^{-1}\) to the physical operators, has to be regularized not as a principal value, but according to the Mandelstam-Leibbrandt (ML) prescription in such a way that causality is preserved. Shortly afterwards, Bassetto et al. found that the ML form of the propagator is realized in a light-cone gauge canonical operator formalism in ordinary space-time coordinates if one introduces a Lagrange multiplier field and its conjugate as residual gauge degrees of freedom. Moreover, Morara and Soldati found...
just recently\textsuperscript{7}) that if one takes $x^+$ as the evolution parameter and $A_+ = \frac{A_0 + A_-}{2} = 0$ instead of $A_- = 0$ as the gauge fixing condition, and if one introduces the Lagrange multiplier field and its conjugate, then one can obtain fewer constraints among the canonical variables, so that one can construct a temporal gauge canonical operator formalism in light-cone coordinates (the light-cone temporal gauge formulation) that realizes the ML form of the free gauge field propagator.

We may therefore think that we can use the residual gauge degrees of freedom to overcome the difficulties inherent in the light-cone axial gauge formulation. In doing so we expect to find that the residual gauge degrees of freedom play important roles in creating vacuum structure, in breaking chiral symmetry spontaneously, and in creating the chiral anomaly. Any full operator solution must exhibit these properties, and most other mechanisms are kinematically forbidden. Those operators, because they become zero-mode operators in the light-cone axial gauge formulation, make the quantization somewhat nonstandard, so that Dirac’s canonical quantization procedure cannot be used to specify the algebra. Recently, McCartor and Robertson attempted to overcome this difficulty.\textsuperscript{8)} They found that the zero-mode operators are introduced as nondynamical integration constants (which are, however, functions of $x^+$). They also found that since, with the introduction of the integration constants, we can assume that other operators vanish in the limits $x^- \to \pm \infty$, the Hamiltonian formalism can be extended in such a way that the $x^+$ dependence of the dynamical operators can be solved for in the presence of the $x^+$ dependent integration constants. Since the integration constants are necessarily independent of $x^-$, they are residual gauge degrees of freedom in this gauge. The algebra of the zero-mode operators can be found by requiring the equivalence of the Heisenberg equations and the field equations.

In this paper we therefore consider constructing an operator solution of the light-cone axial gauge Schwinger model in McCartor and Robertson’s framework. To the best of our knowledge, all previous solutions to the Schwinger model in the light-cone gauge have made use of periodicity conditions to regulate infrared singularities. Because of this, the essential role of the zero-mode fields as infrared regulators, which we shall emphasize in the present paper, was not observed. The Schwinger model in the light-cone gauge was investigated by Bassetto, Nardelli and Vianello in ordinary discretized space-time coordinates.\textsuperscript{9)} They found the zero-mode fields, but since periodicity conditions were used, the role of these fields as infrared regulators was not visible. Several authors\textsuperscript{10)-12)} studied the Schwinger model in the light-cone representation, but again, using periodicity conditions. In that case, a careful formulation should reveal zero-mode fields,\textsuperscript{10)} but they have a different character, which leads to incorrect continuum limits for some physical operators.\textsuperscript{13)} In the present paper we work in the continuum so that the infrared problems are fully exposed. We therefore begin by constructing a solution in the continuum by gauge-transforming the Landau gauge solution given previously by one of the authors,\textsuperscript{14)} into the light-cone gauge. It turns out that the $x^-$-dependent massless constituent fields are contained in the transformed solution only as decoupled, zero-norm operators playing no role; so they are removed. As a result, only the $x^+$-dependent massless constituent fields are retained as the residual gauge degrees of freedom. Although the Landau gauge so-
solution was found by quantizing at the equal-time representation, the form we present is representation independent. The same applies to the light-cone gauge solution.

We then construct a light-cone temporal gauge operator solution by solving the field equations as an initial value problem on the characteristic surface \( x^- = 0 \). This enables us to introduce a free massless Fermion field \( \psi_-(x^+) \) and its fusion field \( \phi(x^+) \) as initial values. It turns out that the fusion field \( \phi \) and the Lagrange multiplier field \( \lambda \) represent residual gauge degrees of freedom and that these fields enable us to construct an operator solution identical to that given by gauge-transforming the Landau gauge solution.

We then proceed to solving the field equations as an initial value problem on the characteristic surface \( x^+ = 0 \). In this case we can introduce a Fermion field, \( \Psi_+(0, x^-) \), and its fusion field \( \phi \) as initial values. In contrast with the temporal case, it turns out that the fusion field \( \phi \) can be identified with the massive constituent field \( \Sigma \) and that translational generators \( P^+ \) and \( P^- \) are described as those of the \( \Sigma \). If the \( \Sigma \) field were the only field in the solution, the bare vacuum would be the physical vacuum in the axial gauge formulation, and the parton picture would be realized in its strongest possible form. We find, however, that with only the \( \Sigma \) field, Lorentz invariance is broken, and infrared divergences inherent to the axial gauge formulation in positive definite Hilbert spaces lead to ill-defined quantities, so that no solution exists. Therefore we introduce \( \phi \) and \( \lambda \) in the same manner as in the temporal gauge formulation to overcome the problem. To do this we must add to \( P^+ \) the operators necessary to properly translate the residual gauge degrees of freedom. These additions are made within the framework developed by McCartor and Robertson. These additions enable us to reconstruct an operator solution identical to that given in the temporal gauge formulation. Because \( \lambda \) turns out to be a zero-norm field, we obtain the degrees of freedom to compose, loosely speaking, degenerate vacua, which can be used to construct physical vacua invariant under large residual gauge transformations. (To be more precise, we encounter an ambiguity — an infinite family of solutions, each of which has a unique ground state which, to be gauge invariant must be chosen so as to spontaneously break chiral symmetry.\(^{15}\))

In this way we can obtain three equivalent representations for the light-cone gauge operator solution of the Schwinger model: the equal-time representation (negative frequencies of the fields along \( x^0 = 0 \) destroy the bare vacuum); the light-cone representation (negative frequencies of the fields along \( x^+ = 0 \) destroy the bare vacuum); and the light-cone representation in the light-cone-temporal gauge, or anti-light-cone gauge (in our formulation the case in which negative frequencies of the fields along \( x^- = 0 \) destroy the bare vacuum). Because we can write the solution in terms of the free fields, and because the isomorphism between the free fields in different representations is trivial, the solution makes manifest the isomorphisms between various particular representations of the light-cone gauge Schwinger model.

In §2, to make this paper self-contained, we present the Landau gauge operator solution given previously by one of us.\(^{14}\) In §3 we transform to the light-cone gauge, obtaining a continuum solution similar in many ways to the periodic solution given by Bassetto, Nardelli and Vianello (periodic on \( t = 0 \)). In §4, to avoid the neces-
sity of quantizing constrained systems, we follow Ref. 8) and derive quantization conditions for canonical fields by requiring that their commutation relations with translational generators give rise to Heisenberg equations that are consistent with the field equations. In §5 we solve the initial value problems on the characteristic surfaces \( x^- = 0 \) and \( x^+ = 0 \). Section 6 is devoted to concluding remarks.

We use the following definitions:

\[
g^{++} = g^{--} = 0, \quad g^{+-} = g^{-+} = 2, \quad g^{++} = g^{--} = 0, \quad g^{+-} = g^{-+} = \frac{1}{2},
\]

\[
x^+ = x^0 + x^1, \quad x^- = x^0 - x^1, \quad \partial_+ = \frac{1}{2}(\partial_0 + \partial_1), \quad \partial_- = \frac{1}{2}(\partial_0 - \partial_1),
\]

\[
\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma^5 = -\sigma_3,
\]

\[
\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}, \quad m = \frac{e}{\sqrt{\pi}}.
\]

§2. Landau gauge

Previously, one of the authors (Nakawaki)\(^{14}\) gave the following operator solution to the Schwinger model in the Landau gauge:

\[
\Psi_+ = Z e^{A_+} e^{A^{(-)}_+},
\]

\[
A_+ = -i\sqrt{\pi} (2\phi_+(x^-) + \tilde{\eta} + \tilde{\Sigma}(x^+, x^-)),
\]

\[
Z^2 = e^{\gamma \sqrt{m(k^2/8\pi^2)}},
\]

\[
\Psi_- = Z e^{A^{(-)}_-} e^{A_+},
\]

\[
A_- = -i\sqrt{\pi} (2\phi_-(x^+) - \tilde{\eta} - \tilde{\Sigma}(x^+, x^-)),
\]

\[
A_\mu = -m^{-1} \epsilon_{\mu\nu} \partial^\nu (\tilde{\eta} + \tilde{\Sigma}).
\]

Here, \( \phi_+(x^-) \) is the right-moving component of a Klaiber-regulated\(^{16}\) (with parameter \( \kappa \)), free, massless scalar field composed of the fusion operators:

\[
\phi^{(+)}_+(x^-) = i(4\pi)^{-\frac{1}{2}} \int_{-\infty}^{0} dk_1 k_0^{-1} c(k_1) \left( e^{-ik_1 x} - \theta(\kappa - k_0) \right),
\]

\[
\phi^{(-)}_+(x^-) = (\phi^{(+)}_+)^*,
\]

where the \( c(k_1) \) are the fusion operators associated with bosonizing\(^{14}\) the Fermi field.\(^{14}\) (They satisfy the usual Boson commutation relations.) \( \phi_-(x^+) \) is the left-moving component of that field:

\[
\phi^{(+)}_-(x^+) = i(4\pi)^{-\frac{1}{2}} \int_{0}^{\infty} dk_1 k_0^{-1} c(k_1) \left( e^{-ik_1 x} - \theta(\kappa - k_0) \right),
\]

\[
\phi^{(-)}_-(x^+) = (\phi^{(+)}_-)^*.
\]
Also, $\tilde{\eta}$ is a pseudoscalar ghost field given by

$$\tilde{\eta}^{(+)} = i(4\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} dk_1 k_1^{-1} \eta(k_1) \left( e^{-i k \cdot x} - \theta(\kappa - k_0) \right), \quad (2.11)$$

where

$$[\eta(k_1), \eta^*(q_1)] = -k_0 \delta(k_1 - q_1). \quad (2.12)$$

The field $\tilde{\Sigma}$ — the field associated with the physical Schwinger particles — is a massive pseudoscalar field of mass $m = (e\sqrt{\pi})$. The spurions are given in terms of these modes as

$$\sigma_+ = \exp \left[ i \sqrt{\pi} \{\bar{Q}_5 + Q\} (4m)^{-1} + 2^{-1} \int_{-\kappa}^{\kappa} dk_1 k_1^{-2} \{ c(k_1) - c^*(k_1) \} \right], \quad (2.13)$$

$$\sigma_- = \exp \left[ i \sqrt{\pi} \{\bar{Q}_5 - Q\} (4m)^{-1} - 2^{-1} \int_{-\kappa}^{\kappa} dk_1 k_1^{-2} \{ \eta(k_1) - \eta^*(k_1) \} \right] \right], \quad (2.14)$$

To be invariant under large gauge transformations, the vacuum must be chosen to be a theta-state, formed as

$$|\Omega(\theta)\rangle \equiv \sum_{M=-\infty}^{\infty} e^{i M \theta} |\Omega(M)\rangle; \quad |\Omega(M)\rangle = (\sigma_+^* \sigma_-^*)^M |0\rangle. \quad (2.15)$$

The physical subspace is formed by applying all polynomials in $\tilde{\Sigma}$ to $|\Omega(\theta)\rangle$.*

Further details and proof that the construction is an operator solution can be found in the Ref. 14). Here we wish to point out that while the above solution is representation independent, it is straightforward to formulate the problem as an initial value problem at $t = 0$ and thus find a solution in the equal-time representation. Gauge-invariant point splitting (in a space-like direction) provides the necessary regulation for the operator products. If we contemplate the initial value problem on the characteristic surfaces $x^+ = 0$ and $x^- = 0$, the situation is not so straightforward. The problem in this case is that the Fermi products cannot be regulated by splitting in a light-like direction. The $\tilde{\eta}$ field is the sum of a function of $x^+$ and a function of $x^-$, so the operator products are not regulated by splitting in either light-like direction. Also, the $\tilde{\Sigma}$ field suffers an apparent, but spurious, infrared singularity due to the fact that its massive character is not manifest at light-like separations. Note that these problems only involve the formulation of the theory as an initial value problem on the characteristics; the light-cone representation certainly exists — that is, modes of the fields along the characteristics provide all the operators.

* It is worthwhile here to correct the inaccurate title of Ref. 14), “Non-existence of nontrivial vacuum structure in the Schwinger model.” Here, “Nontrivial vacuum structure” should be replaced by “Degenerate vacua made of physical degrees of freedom.”
necessary to generate the entire representation space. We do need modes along both
ccharacteristics, but all the physical operators except those necessary to define the
vacuum can be generated using modes along \( x^+ = 0 \). We shall return to the problem
of formulating an initial value problem on the characteristic surfaces below, but first
we wish to consider the question of the light-cone gauge.

\section{Light-cone gauge}

We may attempt to realize the light-cone gauge by performing a nonlocal gauge
transformation on the Landau gauge solution. If we use the gauge function
\[ \Theta = m^{-1}(\tilde{\eta} + \tilde{\Sigma}), \]
we find that \( A_\pm = 0 \). The resulting construction almost works, but it is not quite
right. One problem is that the \( x^- \)-dependent parts of the \( \tilde{\eta} \) and \( \phi \) fields are not
natural degrees of freedom in the light-cone gauge. That is, standard quantization
methods, whether in the equal-time representation or in the light-cone representa-
tion, do not include those degrees of freedom in the representation space. Those
degrees of freedom decouple, at least formally, and we simply remove them from the
solution. The other problem is that we have performed the gauge transformation
with the Klaiber-regulated \( \tilde{\eta} \) field, but we have left the spurions, which contain the
low frequency parts of that field, unchanged. The effect is that the Klaiber regulator,
\( \kappa \), does not disappear from physical matrix elements, and the solution is no longer
translationally invariant, even in the physical subspace. To solve that problem we
must modify the spurions in addition to making the gauge transformation specified
by \( \Theta \). The correct solution in the light-cone gauge is then given by (removing the
tilde from the \( \eta \) field since we keep only the \( x^+ \) dependent part):

\[ \Psi_+ = Z_+ e^{A^{(-)}_+ \sigma_+ e^{A^{(+)}_+}}, \]
\[ A_+ = -2i\sqrt{\pi}\{\eta(x^+) + \tilde{\Sigma}(x^+, x^-)\}, \]
\[ Z_2^+ = \frac{m^2 e^\gamma}{8\pi\kappa}, \]
\[ \Psi_- = Z_- e^{A^{(-)}_- \sigma_- e^{A^{(+)}_-}}, \]
\[ A_- = -2i\sqrt{\pi}\phi(x^+), \]
\[ Z_2^- = \frac{\kappa e^\gamma}{2\pi}, \]
\[ A_+ = \frac{2}{m} \partial_+ (\eta + \tilde{\Sigma}), \]
\[ A_- = 0, \]
\[ \sigma_+ = \exp\left[ i\sqrt{\pi} (Q_5 + Q)(4m)^{-1} + \int_0^\kappa dk_1 k_1^{-1} \{\eta(k_1) - \eta^*(k_1)\}\right], \]
\[ \sigma_- = \exp\left[ i\sqrt{\pi} (Q_5 - Q)(4m)^{-1} + \int_0^\kappa dk_1 k_1^{-1} \{c(k_1) - c^*(k_1)\}\right]. \]
The chargeless combinations of spurions, $\sigma^*_+\sigma_-$ and $\sigma^*_+\sigma_+$ are generators of gauge transformations (as they are in the Landau gauge). Thus, again, gauge invariance fixes the vacuum to be of the form

$$|\Omega(\theta)\rangle \equiv \sum_{M=-\infty}^{\infty} e^{iM\theta} |\Omega(M)\rangle; \quad |\Omega(M)\rangle = (-\sigma^*_+\sigma_-)^M |0\rangle,$$

(3.14)

so that

$$\sigma^*_+\sigma_- |\Omega(\theta)\rangle = -e^{-i\theta} |\Omega(\theta)\rangle.$$  

(3.15)

Now we see that the field $\Psi_-$ is isomorphic to the left-moving component of a free massless Fermi field, and it has no dependence on the ghost field, even through the spurion. Even with the modifications of the spurions, the physics contained in the light-cone gauge solution is the same as that in the Landau gauge solution. In particular, we find the anomaly

$$\partial^\mu J^5_\mu = \frac{e}{2\pi} \varepsilon_{\mu\nu} F^{\mu\nu}$$

(3.16)

and the chiral condensate

$$\langle \Omega(\theta) | \bar{\Psi} \Psi | \Omega(\theta) \rangle = -\frac{m}{2\pi} e^\gamma \cos \theta.$$  

(3.17)

Note that the chiral condensate is particularly easy to calculate in light-cone gauge. (It is a considerably more complicated calculation in the Landau gauge.) Since $\Psi_+$ and $\Psi_-$ contain only fields which are independent of each other, the left-hand side of (3.17) reduces immediately to the spurions, after which (3.15) gives the right-hand side of (3.17).

We believe that this construction is the correct light-cone gauge solution to the continuum Schwinger model, but it does have some unexpected properties which we should discuss. Indeed, one should probably say that it is not an operator solution in the strictest sense of the word, since one of the operators included in the field $\Psi_+$ is not well-defined. Looking at (3.10) we see that the vacuum expectation value of the spurion $\sigma_+$ not only does not vanish, as it does in the Landau gauge solution, but diverges. This fact may cause one to wonder in what sense the equations of motion, particularly (4.6), are satisfied in the physical subspace. The point is that the $\Psi_+$ field is not a physical operator (since it carries a charge), and the only way the spurions enter physical operators is in the chargeless combinations $\sigma^*_+\sigma_-$ and $\sigma^*_+\sigma_+$. The chargeless combinations of spurions simply add zero norm states to the state acted upon and, in particular, act as $c$-numbers in the physical subspace. With this in mind, it is easy to use the arguments in Ref. 14) to show that the equations of motion are satisfied in the following sense: Take any physical operator and use the Lagrange equations of motion to derive an equation of motion for the physical...
operator. Then the derived equation of motion will be satisfied in the physical subspace. For example, we may use the equations of motion (given below) to derive the equation

\[ i\psi_-^* \partial_+ (\psi_+) = e\psi_-^* \psi_+ A_+ \]  

(3.18)

Neither of these operators are physical, because neither is self-adjoint; however, both operators are chargeless, and the equation is well-defined and is satisfied. Indeed any equation for chargeless operators which can be derived from the equations of motion is well-defined, and all are satisfied in the physical subspace. The physical subspace is isomorphic to the Landau gauge solution, but the entire solution is not. Equivalently the field equation of \( \psi_+ \) described below in (4.6) is satisfied if we introduce an infrared cutoff into \( \sigma_+ \). If we let the regulator vanish after all necessary calculations are finished, the equations for physical quantities will converge to well-defined and valid expressions, while expressions for nonphysical operators may become ill-defined. For example, by applying this limiting procedure, the vacuum expectation value of Eq. (4.6) vanishes for both sides, and the limit exists for physical operators defined below in (5.27). Note that the problem here is with the gauge choice and not with any particular quantization scheme. We believe this problem is unavoidable in the light-cone gauge. This problem is common in QCD, where the Dirac equations are equations for operators carrying a color index. Renormalized (perturbatively renormalized) equations have been obtained for “bleached” equations which result from the Lagrange equations but have not been obtained (and may not exist) for colored equations.

As with the Landau gauge solution, the light-cone gauge solution is straightforwardly quantized on \( t = 0 \). Indeed, except for the spurious, it is very similar to the periodic solution found by Bassetto, Nardelli and Vianello. However, it is not so straightforward to quantize the continuum solution on \( x^+ = 0 \), due to the fact that the current, \( \psi_+^* \psi_+ \) is not regulated by splitting in the \( x^- \) direction. We note that this problem was already present in the Landau gauge solution, and it is due to the initial value surface, not the gauge choice.

\[ \textbf{§4. Quantization in the light-cone representation} \]

The Schwinger model is defined by the Lagrangian

\[ L = 2F_+^- F_{+-} - 2\lambda A_- + 2i\psi_+^* \partial_+ \psi_- + 2i\psi_-^* \partial_- \psi_+ - 2e\psi_+^* \psi_- A_- - 2e\psi_-^* \psi_+ A_+ \]  

(4.1)

where

\[ A_\pm = \frac{1}{2}(A_0 \pm A_1), \quad F_{+-} = \partial_+ A_- - \partial_- A_+. \]  

(4.2)

The field equations and the gauge fixing condition are

\[ 2\partial_- F_+^- = -e\psi_+^* \psi_+ = -J_-, \]  

(4.3)

\[ 2\partial_+ F_-^+ - \lambda = e\psi_-^* \psi_- = J_+, \]  

(4.4)

\[ i\partial_- \psi_- = e\psi_- A_-, \]  

(4.5)
\[ i\partial_+ \Psi_+ = \partial_+ A_+ , \quad (4.6) \]
\[ A_- = 0. \quad (4.7) \]

From current conservation, \( \partial_+ J_- + \partial_- J_+ = 0 \), and Eqs. (4.3) and (4.4), we obtain

\[ \partial_- \lambda = 0. \quad (4.8) \]

The canonical energy-momentum tensor is given by

\[ T_{\mu\nu} = i\bar{\Psi} \gamma_{\nu} \partial_\mu \Psi - F_{\nu\rho} \partial_\mu A^\rho + g_{\mu\nu} \left( \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} + \lambda (A_0 - A_1) \right), \quad (4.9) \]

where we have used the field equation of the Fermion field. Components in light-cone coordinates are given explicitly by

\[ T_{++} = i\bar{\Psi}^* \partial_+ \Psi_+ - (F_{++}, \partial_+ A_+)_+, \]
\[ T_{-+} = i\bar{\Psi}^* \partial_- \Psi_+ - (F_{-+}, \partial_+ A_+)_+, \]
\[ T_{+-} = i\bar{\Psi}^* \partial_- \Psi_+ - (F_{+-}, \partial_- A_+)_+, \]
\[ T_{--} = i\bar{\Psi}^* \partial_- \Psi_+ + (F_{+-}, \partial_- A_-)_+, \]

where we have used the gauge-fixing condition and the Fermion field equation.

From these expressions we can see some of the problems we will encounter when we apply the canonical formulation to the Schwinger model in the representation generated by modes of the fields along either light-cone surface. When we construct a light-cone-temporal gauge formulation, in which \( x^- \) is chosen to be the evolution parameter, we use \( T_{++} \) and \( T_{-+} \) as densities to calculate the translational generators. We see that \( \Psi_+ \) and \( \Psi^*_+ \) are not contained in the densities, so that we cannot treat \( \Psi_+ \) as a degree of freedom. If we consider the standard light-cone gauge treatment (the light-cone axial gauge), in which \( x^+ \) is the evolution parameter and \( T_{+-} \) and \( T_{--} \) are the densities of the translational generators, we see that we need zero-mode fields (fields which are functions only of \( x^+ \)). These fields require special treatment.

We show in §5 that we can find a light-cone-temporal gauge solution by expressing \( \Psi_+ \) as a functional of \( A_+ \). (This is done the other way around in the light-cone gauge.) The problem of the zero-mode fields is principally one of recognizing them. In fact we can find the missing terms by defining the translational generators in the light-cone coordinate space by requiring that they are identical to those in ordinary coordinate space \((x^0, x^1)\). From the divergence equation

\[ \partial^\nu T_{\mu\nu} = 0, \quad (4.14) \]

we obtain

\[ \oint T_{\mu\nu} d\sigma^\nu = 0. \quad (4.15) \]
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If we perform the integral over the closed surface shown in Fig. 1, it is clear that the integral over the surface \( t = x^0 \) is the negative of that over the light-cone surfaces. Thus we obtain

\[
\int_{x^0 - L}^{x^0 + L} T_{\mu 0}(x^0, x^1)dx^1 = \int_{x^0 - L}^{x^0 + L} T_{\mu +}(x^+ = x^0, x^- = x^0)dx^+
+ \int_{x^0 - L}^{x^0 + L} T_{\mu -}(x^+- L, x^-)dx^-.
\]

Hence in the limit \( L \to \infty \) we obtain

\[
P_{\mu} = \int_{-\infty}^{\infty} T_{\mu 0}(x^0, x^1)dx^1 = \int_{-\infty}^{\infty} T_{\mu +}(x^+ = x^0, x^- = x^0)dx^+
+ \int_{-\infty}^{\infty} T_{\mu -}(x^+ = \infty, x^-)dx^- + \int_{-\infty}^{\infty} T_{\mu -}(x^+ = -\infty, x^-)dx^-.
\] (4.16)

We show in §5 that \( T_{\mu -} \) can be expressed solely in terms of the massive field, so that \( T_{\mu -} \) tends to 0 when \( x^+ \to \pm \infty \). In that case we have

\[
P_+ = \int_{-\infty}^{\infty} \{i\Psi_-^{\ast} \partial_+ \Psi_- + (\partial_+ A_- + \partial_+ A_+ \})dx^+,
\] (4.17)

\[
P_- = \int_{-\infty}^{\infty} (\partial_+ A_-)^2 dx^+.
\] (4.18)

We see from this that in the temporal gauge formulation there are no missing degrees of freedom. We also see that the canonical momenta of \( A_+ \) and \( \Psi_- \) are \( 2\partial_+ A_+ \) and \( i\Psi_-^{\ast} \), respectively.

Similarly, carrying out the contour integral shown in Fig. 2, we obtain:

\[
P_{\mu} = \int_{-\infty}^{\infty} T_{\mu 0}(x^0, x^1)dx^1 = \int_{-\infty}^{\infty} T_{\mu -}(x^+ = x^0, x^-)dx^-
+ \int_{-\infty}^{\infty} T_{\mu +}(x^- = \infty)dx^+ + \int_{\infty}^{x^0} T_{\mu +}(x^+ = -\infty)dx^+.
\] (4.19)
We show in §5 that if we choose nonvanishing initial values, then \( T_{++} \) tends to these values in the limits \( x^- \to \pm \infty \). In this case we have

\[
P_+ = \int_{-\infty}^{\infty} \{ i\Psi_+^* \partial_+ \Psi_+ - (\partial_- A_+)^2 \} dx^- + \int_{-\infty}^{\infty} T_{++}(x^+, x^- = \pm \infty) dx^+. \tag{4.20}
\]

We also show that \( T_{-+} \) vanishes in the limits \( x^- \to \pm \infty \), so that we have

\[
P_- = \int_{-\infty}^{\infty} T_{-+}(x^+, x^-) dx^- = \int_{-\infty}^{\infty} i\Psi_+^* \partial_- \Psi_+ dx^- . \tag{4.21}
\]

Now we derive quantization conditions for the canonical fields \( A_+, \Psi_- \) and \( \Psi_+ \), by requiring that their commutation relations with \( P_+ \) and \( P_- \) give rise to Heisenberg equations which are consistent with the field equations. (The argument is similar to that in Ref. 8.) In the temporal gauge formulation, \( P_+ \) in (4.17) is the kinematical operator, so that we obtain the Heisenberg equation

\[
i[P_+, \Psi_-(x)] = \partial_+ \Psi_-(x), \tag{4.22}
\]

if we require the equal-\( x^- \) quantization conditions:

\[
\{ \Psi_-(x^+, x^-), \Psi_+(y^+, x^-) \}_+ = \delta(x^+ - y^+),
\]

\[
\{ \Psi_-(x^+, x^-), \Psi_-(y^+, x^-) \}_+ = 0,
\]

\[
[A_+(x^+, x^-), \Psi_-(y^+, x^-)] = 0,
\]

\[
[\partial_- A_+(x^+, x^-), \Psi_-(y^+, x^-)] = 0. \tag{4.23}
\]

Furthermore, we obtain the Heisenberg equations:

\[
i[P_+, A_+(x)] = \partial_+ A_+(x), \quad i[P_+, \Psi_+(x)] = \partial_+ \Psi_+(x),
\]

if we require, in addition, the equal-\( x^- \) quantization conditions,

\[
[A_+(x^+, x^-), A_+(y^+, x^-)] = 0,
\]

\[
[\partial_- A_+(x^+, x^-), A_+(y^+, x^-)] = \frac{i}{2} \delta(x^+ - y^+),
\]

\[
\{ \Psi_+(x^+, x^-), \Psi_+(y^+, x^-) \}_+ = 0, \quad \{ \Psi_+(x^+, x^-), \Psi_-(y^+, x^-) \}_+ = 0,
\]

\[
[A_+(x^+, x^-), \Psi_+(y^+, x^-)] = 0,
\]

\[
[\partial_- A_+(x^+, x^-), \Psi_+(y^+, x^-)] = \frac{e}{4} (x^+ - y^+) \Psi_+(x). \tag{4.24}
\]

We remark that the second commutation relation in (4.28) is unusual in the canonical formalism and that at this point, nothing is known about the equal-\( x^- \) commutation relations between \( \Psi_+ \) and \( \Psi_+^* \). We also remark that, although \( A_+ \) obeys a field equation of second order in the light-cone temporal gauge formulation, as is seen from (4.3), the commutator \([\partial_- A_+(x^+, x^-), \partial_- A_+(y^+, x^-)\] is not zero but has the following nonvanishing value:

\[
[\partial_- A_+(x^+, x^-), \partial_- A_+(y^+, x^-)] = -i \frac{m^2}{16} \epsilon(x^+ - y^+). \tag{4.29}
\]
This is because consistent operator solutions are obtained if and only if we regularize the Fermi products in a gauge invariant way. (In the Schwinger model, regularizing the current operators and the Fermionic kinetic terms gauge invariantly gives rise to the chiral anomaly.) It is shown in §5 that (4.23) combined with gauge invariant point splitting for the term $i\Psi^- \partial_+ \Psi_-$ gives rise to $-\frac{m^2}{4} A_+^2$, so that (4.29) is required to produce the Heisenberg equation

$$i[P_+, \partial_- A_+(y^+, x^-)] = \partial_+ \partial_- A_+(x).$$

(4.30)

Now that we have obtained the quantization conditions in the temporal gauge formulation, we can make use of them to obtain the Heisenberg equations which the dynamical $P_-$ in (4.18) produces. Straightforward calculation gives

$$i[P_+, \Psi-(x)] = 0, \quad i[P_-, A_+(x)] = \partial_- A_+(x),$$

$$i[P_-, \partial_- A_+(x)] = -\frac{m^2}{4} (\partial_+)^{-1} \partial_- A_+(x),$$

$$i[P_-, \Psi_+(x)] = -\frac{ie}{2}((\partial_+)^{-1} \partial_- A_+(x), \Psi_+(x))_+.$$  (4.31)

We see from the Heisenberg equation for $\partial_- A_+$ that $\partial_- A_+$ behaves like a free field of mass $m$.

In the axial gauge formulation, $P_-$ in (4.21) is the kinematical operator, so that we obtain the Heisenberg equations

$$i[P_-, \Psi-(x)] = 0, \quad i[P_-, \Psi_+(x)] = \partial_- \Psi_+(x)$$  (4.32)

if we specify the following equal-$x^+$ commutation relations:

$$\{\Psi_-(x^+, x^-), \Psi_+^*(x^+, y^-)\}_+ = 0, \quad \{\Psi_-(x^+, x^-), \Psi_+(x^+, y^-)\}_+ = 0,$$  (4.33)

$$\{\Psi_+(x^+, x^-), \Psi_+^*(x^+, y^-)\}_+ = \delta(x^- - y^-),$$

$$\{\Psi_+(x^+, x^-), \Psi_-(x^+, y^-)\}_+ = 0.$$  (4.34)

We cannot obtain any other quantization conditions unless we solve Eqs. (4.3) and (4.6).

§5. Construction of operator solutions

Now that we have the algebra of the fields, we can proceed to construct the solution. We might consider the problem as an initial value problem on $x^- = 0$ or $x^+ = 0$, or proceed in a more covariant way. Here, we shall consider the initial value problem on each characteristic. First, we consider the initial value problem on the surface $x^- = 0$.

5.1. Light-cone temporal gauge solution

From (4.5) we see that when $A_- = 0$, the first component, $\Psi_-$, is a free field depending only on $x^+$. Thus we specify $\Psi_-$ as a free, massless Fermion field

$$\Psi_-(x) = \psi_-(x).$$  (5.1)
satisfying the anticommutation relations
\[ \{ \psi_-(x), \psi^*_-(y) \}_+ = \delta(x^+ - y^+), \quad \{ \psi_-(x), \psi_-(y) \}_+ = 0. \] (5.2)

Furthermore we make use of the fusion field defined by
\[ : e\psi^*_-(x)\psi_-(x) : = m\partial_+ \phi(x) \] (5.3)
to express \( \psi_\) in the following equivalent bosonized form:
\[ \psi_-(x) = Z^{-}\exp\left[-2i\sqrt{\pi}\phi^-(x)\right]\sigma^-\exp\left[-2i\sqrt{\pi}\phi^+(x)\right]. \] (5.4)

Here, \( Z^-\) is the finite normalization constant, \( \sigma^-\) is the spurion operator and \( \phi^-(\) and \( \phi^+(\) are the positive and negative frequency parts of \( \phi\) regularized following Klaiber. (The construction is exactly the same as in (3.5), (3.7) and (3.11).) By construction, \( \phi\) satisfies the following commutation relation:
\[ \{ \phi(x), \phi(y) \} = -i 4 \epsilon(x^+ - y^+). \] (5.5)

Then, carrying out a gauge invariant point splitting procedure, we find the following well-defined current:
\[ J_+ (x) = \lim_{y^+ \to x^+} e \{ \psi^*_-(x)\psi_-(y)\exp[-ie \int_{y^+}^{x^+} A_+(z^+, x^-) \, dz] + \text{h.c.} \} \]
\[ = m\partial_+ \phi(x) - \frac{m^2}{2} A_+(x). \] (5.6)

Furthermore, the kinetic term \( i\psi^* \partial_+ \psi_\) is regularized as
\[ \lim_{y^+ \to x^+} \left\{ i 2 \psi^*_-(x)\partial_+ \psi_-(y)\exp\left[-ie \int_{y^+}^{x^+} A_+(z, x^-) \, dz\right] + \text{h.c.} - \frac{1}{2\pi} \frac{1}{(x^+ - y^+)^2} \right\} \]
\[ = (\partial_+ \phi)^2 - \frac{m^2}{4} (A_+)^2, \] (5.7)
so that \( P_+\) in (4.17) is given by
\[ P_+ = \int_{-\infty}^{\infty} \left\{ (\partial_+ \phi)^2 - \frac{m^2}{4} (A_+)^2 + (\partial_- A_+, \partial_+ A_+)_+ \right\} \, dx^+. \] (5.8)

Now note that owing to (5.6), Eq. (4.4) can be written as
\[ (4\partial_+ \partial_- + m^2)A_+(x) = 2\{ \lambda(x) + m\partial_+ \phi(x) \}. \] (5.9)

Applying \( \partial_-\) to this leads to
\[ (4\partial_+ \partial_- + m^2)\partial_- A_+(x) = 0, \] (5.10)
due to the fact that \( \phi(x)\) and \( \lambda(x)\) depend only on \( x^+\). Thus we see that \( \partial_- A_+\) is a free field of mass \( m\). Since \( \partial_- A_+\) is gauge invariant and is equal to \( -\frac{m}{2} \Sigma\) in the
Landau gauge operator solution, our present result is in agreement with the earlier results. Here we set
\[ \partial_- A_+(x) = -\frac{m}{2} \tilde{\Sigma}, \quad (5.11) \]
where the normalization is determined by (4.29). Then we note that replacing \( \partial_- A_+ \) in (5.9) with \( \tilde{\Sigma} \) enables us to express \( A_+ \) in terms of \( \phi, \lambda \) and \( \tilde{\Sigma} \) as
\[ A_+(x) = \frac{2}{m^2} \{ \lambda + m\partial_+ (\phi + \tilde{\Sigma}) \}. \quad (5.12) \]
The commutation relation (4.29) is rewritten as that of \( \tilde{\Sigma} \)
\[ [\tilde{\Sigma}(x^+, x^-), \tilde{\Sigma}(y^+, x^-)] = -\frac{i}{4} \epsilon(x^+ - y^+), \quad (5.13) \]
and the commutation relations of \( \lambda \) are obtained by rewriting (5.9) for \( \lambda \) as
\[ \lambda = \frac{1}{2} \{ m^2 A_+ + 4\partial_+ (\partial_- A_+) - m\partial_+ \phi \} \quad (5.14) \]
and by making use of the commutation relations (4.26), (4.29), (5.5) and
\[ [\phi(x), A_+(y)] = 0, \quad [\phi(x), \partial_- A_+(y)] = 0, \quad (5.15) \]
which result from (4.24), (5.1) and (5.3). Combining these results we obtain
\[ [\lambda(x), \lambda(y)] = 0, \quad [\lambda(x), \tilde{\Sigma}(y)] = 0, \quad [\lambda(x), \phi(y)] = \frac{i}{4} \delta(x^+ - y^+). \quad (5.16) \]
Now we see that \( \lambda \) is a zero norm field and that Maxwell’s equations are recovered in the physical subspace formed by factoring the zero norm field \( \lambda \) out of the representation space. If we rewrite \( \lambda \) as
\[ \lambda(x) = m\partial_+ (\eta(x) - \phi(x)) \quad (5.17) \]
and take account of the third commutation relation of \( \lambda \) in (5.16) we find that \( \eta \) is a negative norm field depending only on \( x^+ \) and satisfies the commutation relations
\[ [\eta(x), \eta(y)] = \frac{i}{4} \epsilon(x^+ - y^+), \quad [\eta(x), \tilde{\Sigma}(y)] = 0, \quad [\eta(x), \phi(y)] = 0. \quad (5.18) \]
As a result \( A_+ \) in (5.12) may be written as
\[ A_+ = \frac{2}{m} \partial_+ (\tilde{\Sigma} + \eta). \quad (5.19) \]
In terms of these fields, \( P_+ \) in (5.8) and \( P_- \) in (4.18) are diagonalized as
\[ P_+ = \int_{-\infty}^{\infty} \{ (\partial_+ \phi)^2 - (\partial_+ \eta)^2 - (\partial_+ \tilde{\Sigma})^2 \} dx^+, \quad (5.20) \]
\[ P_- = \int_{-\infty}^{\infty} \frac{m^2}{4} (\tilde{\Sigma})^2 dx^+, \quad (5.21) \]
which shows that \( \phi, \eta \) and \( \tilde{\Sigma} \) are constituent free fields of the light-cone temporal gauge Schwinger model.

Let us turn to specifying \( \Psi_+ \), which satisfies (4.3) and (4.6). Because in the temporal gauge formulation there is no dynamical equation which allows us to determine \( \Psi_+ \) as an initial value problem, we make use of the fact that using (5.19), we can write Eq. (4.3) as

\[
J^- = e\Psi^*_+ \Psi_+ = m \partial_\Sigma(x).
\]  

(5.22)

We see from this that the \( \tilde{\Sigma} \) field can be identified as a fusion field composed of \( \Psi^* \) and \( \Psi_+ \) and that \( \Psi_+ \) in turn can be expressed in an equivalent bosonized form. (This result will be obtained in the axial gauge formulation.) As a matter of fact, \( A_+ \) given in (5.19) is identical with the electromagnetic field given in (3.8), so that the Fermion operator (3.2) satisfies Eq. (4.6). Therefore we specify \( \Psi_+ \) as in (3.2) and show that it also satisfies Eq. (4.3). To this end we also regularize the bilinear product \( e\Psi^* \Psi_+ \) by the gauge-invariant point splitting procedure. We see immediately that if we split only in the \( x^+ \) direction, then the sum \( \tilde{\Sigma} + \eta \) in the exponent behaves like a zero norm operator, so that the procedure does not work. We also see that if we split only in the \( x^- \) direction, then the \( \eta \) field gives rise to a divergence at high frequencies. Therefore we have to split in another direction. The following two-step limit, with \( \epsilon \) being a time-like vector, gives us the desired result,

\[
J_-(x) = \frac{e}{2} \lim_{\epsilon^+ \to 0} \left\{ \lim_{\epsilon^- \to 0} \left( \Psi^+_+(x+\epsilon)\Psi_+(x)\exp \left[ -ie \int_{x^-}^{x^+} dz^\nu A_\nu(z) + \text{h.c.} \right] \right) \right\}
= m \partial_\Sigma(x).
\]  

(5.23)

The axial-vector current \( J^5_\mu = \epsilon_{\mu\nu} J^\nu \), where \( \epsilon_{-+} = -\epsilon_{++} = \frac{1}{2} \) and \( \epsilon_{--} = \epsilon_{+-} = 0 \), satisfies (3.16). In addition, the conserved axial-vector current

\[
J^5_{C\mu} = \epsilon_{\mu\nu}(J^\nu + m^2 A^\nu),
\]  

(5.24)

is obtained by regularizing \( e\overline{\Psi}\gamma_\mu\gamma^5\Psi \), with \( \Psi_- \) and \( \Psi_+ \) being \( \psi_- \) and (3.2) respectively, in the same manner as the vector current:

\[
J^5_{C+}(x) = -\frac{e}{2} \lim_{\epsilon^+ \to 0} \left\{ \lim_{\epsilon^- \to 0} \left( \psi^+_+(x^+ + \epsilon^+) \psi_-(x^+) \exp \left[ ie \int_{x^-}^{x^+} dz^\nu A_\nu(z) + \text{h.c.} \right] \right) \right\}
= -m \partial_\Sigma(\phi + \eta + \tilde{\Sigma}),
\]  

(5.25)

\[
J^5_{C-}(x) = \frac{e}{2} \lim_{\epsilon^+ \to 0} \left\{ \lim_{\epsilon^- \to 0} \left( \Psi^+_+(x+\epsilon)\Psi_+(x)\exp \left[ ie \int_{x^-}^{x^+} dz^\nu A_\nu(z) + \text{h.c.} \right] \right) \right\}
= m \partial_\Sigma(x).
\]  

(5.26)

It can be shown furthermore that the Fermion operator (3.2) satisfies the anticommutation relations in (4.34) if we define them in the \( y^+ \to x^+ \) limit so as to avoid any divergences.
We end the temporal gauge construction by defining the physical space $V$ by

$$V = \{ |\text{phys}\rangle | \lambda^{(+)}(x)|\text{phys}\rangle = 0 \}, \quad (5.27)$$

where $\lambda^{(+)}(x)$ denotes the positive frequency part of $\lambda$.

### 5.2. Light-cone axial gauge solution

In the axial gauge formulation, $x^+$ is taken to be the evolution parameter, so that we can solve Eq. (4.6) as an initial value problem on $x^+ = 0$. As an initial value of $\Psi_+$ we take the free Fermi field $\Psi_R(x^-)$ and define the fusion field $\tilde{\phi}$ by

$$e^{\Psi_R^*(x^-)\Psi_R(x^-)} = m\partial_-\tilde{\phi}(x^-). \quad (5.28)$$

Then, by construction $\tilde{\phi}$ satisfies the commutation relation

$$[\tilde{\phi}(x^-), \tilde{\phi}(y^-)] = -i\frac{4}{\epsilon}(x^- - y^-), \quad (5.29)$$

and Eq. (4.3) becomes

$$J_-(0, x^-) = m\partial_-\tilde{\phi}(x^-) = -2\partial_-^2 A_+(0, x^-), \quad (5.30)$$

where point splitting in the $x^-$ direction has enabled us to utilize the fact that $A_- = 0$. From (5.30) we obtain

$$\partial_- A_+(0, x^-) = -\frac{m}{2}\tilde{\phi}(x^-). \quad (5.31)$$

As a consequence, if we neglect the second unspecified term of (4.20) for the moment, then we can express $P_+$ solely in terms of $\tilde{\phi}$ as

$$P_+ = \int_{-\infty}^{\infty} T_+(x^-)dx^- = \int_{-\infty}^{\infty} \{ J_-(0, x^-)A_+(0, x^-) - (\partial_- A_+)^2 \}dx^-$$

$$= \int_{-\infty}^{\infty} (\partial_- A_+)^2dx^- = \frac{m^2}{4} \int_{-\infty}^{\infty} (\tilde{\phi}(x^-))^2dx^- \quad (5.32)$$

Furthermore, by making use of the equivalent bosonized form of $\Psi_R$ we can express $P_-$ in (4.21) as

$$P_- = \int_{-\infty}^{\infty} i\Psi_+^*(0, x^-)\partial_-\Psi_+(0, x^-)dx^- = \int_{-\infty}^{\infty} (\partial_-\tilde{\phi})^2dx^- \quad (5.33)$$

It follows from (5.32) and (5.33) that the fusion field $\tilde{\phi}$ is again a constituent free field of mass $m$.

The temporal evolution of $\Psi_+(0, x^-)$ is defined by making use of $P_+$ in (5.32) by

$$\Psi_+(x^+, x^-) \equiv e^{iP_+ x^+} \Psi_R(0, x^-) e^{-iP_+ x^+}. \quad (5.34)$$

Then by making use of the equivalent bosonized form we can write

$$\Psi_+(x^+, x^-) = Z \exp[-2i\sqrt{\pi}\tilde{\phi}^-(x)] \sigma_R \exp[-2i\sqrt{\pi}\tilde{\phi}^+(x)], \quad (5.35)$$
where
\[ Z^2 = \frac{\tilde{\kappa} e^\gamma}{2\pi}, \] (5.36)

\[ \tilde{\phi}(+) (x) = \frac{i}{\sqrt{4\pi}} \int_0^\infty \frac{dp_- c(p_-)(e^{-ip\cdot x} - \theta(\tilde{\kappa} - p_-))}{p_-}, \]

\[ \tilde{\phi}(-)(x) = (\tilde{\phi}(+)(x))^*, \] (5.37)

with \( p_+ = \frac{m^2}{4p_-} \), and \( \sigma_R \) is the spurion operator:
\[ \sigma_R = \exp \left[ \int_0^{\tilde{\kappa}} \frac{dp_- c(p_-) - c^*(p_-)}{p_-} \right]. \] (5.38)

Here we have omitted the Klein transformation factor. It is evident that \( \Psi^+ \) satisfies the anticommutation relations (4.34).

Now we note that on the surface \( x^+ = \text{constant} \), \( \Psi^+ \) behaves like a free fermion field and that both \( P_+ \) and \( P_- \) are diagonalized in terms of the fusion field. Thus we see that the common hope in the light-cone quantization that the light-cone bare states are closer to partons than the ordinary equal-time bare states is realized in its strongest possible form. At the same time, we also see that the initial value problem which we have considered also gives rise to the well-known problem common to axial gauge quantizations: We come to have ill-defined equal-\( x^+ \) commutation relations of \( A^+ \), because \( A^+ \) is obtained from (5.31) as
\[ A^+ = -\frac{m}{2} (\partial_-)^{-1} \phi = \frac{2}{m} \partial_+ \phi. \] (5.39)

Note that this difficulty results from the fact that the antiderivative \( (\partial_-)^{-1} \) is not well-defined in any positive definite Hilbert space. Therefore we introduce the \( \eta \) field as in (5.19) to regularize (5.39), although doing so obscures the parton picture. To obtain a consistent solution we also introduce the Fermi field \( \psi_-(x^+) \), as well as the fusion field \( \phi(x^+) \), and solve Eqs. (4.4) and (4.5) in the same manner as in the temporal gauge formulation. This enables us to identify the \( \tilde{\Sigma} \) field with the fusion field \( \tilde{\phi} \) and thus hereafter we denote \( \tilde{\phi} \) as \( \tilde{\Sigma} \). Furthermore, by assuming that the massive degrees of freedom of \( T^+\pm \) contained in (5.20) vanish as \( x^- \to \pm \infty \), we obtain
\[ T^+\pm (x^+, x^- = \pm \infty) = (\partial_+ \phi)^2 - (\partial_+ \eta)^2, \] (5.40)

so that \( P_+ \) in (4.20) is fixed to be
\[ P_+ = \frac{m^2}{4} \int_{-\infty}^{\infty} \tilde{\Sigma}^2 dx^- + \int_{-\infty}^{\infty} \{(\partial_+ \psi)^2 - (\partial_+ \eta)^2\} dx^+. \] (5.41)

In this way we can reconstruct, in the axial gauge formulation, the \( P_+ \) given in (5.20).

Now that \( A_+ \) possesses zero mode fields (fields independent of \( x^- \)), we must take this fact into account when we solve Eq. (4.6) as the initial value problem on the surface \( x^+ = 0 \). As an alternative initial value satisfying the equal-\( x^+ \) anticommutation
relations we choose
\[ \Psi_+(0, x^-) = \exp[-2i\sqrt{\pi} \eta(0)] \Psi_R(x^-). \] (5.42)

Note that (5.42) has a diverging vacuum expectation value, which is inevitable as long as we respect the equal-\(x^+\) anticommutation relations. In fact, when we rewrite the exponential function of \( \eta(0) \) as a normal product, divergences appear at low frequencies and at high frequencies, but the divergence at high frequencies is canceled by the zero from \( \Psi_R \), whereas the divergence at low frequencies remains.

To investigate the connection between (5.42) and (3.2), we rewrite the exponential function of \( \eta(0) \) as
\[ e^{-2i\sqrt{\pi} \eta(0)} = \exp\left[ \frac{1}{2} \int_0^{\infty} \frac{dk_+}{k_+} \right] \exp[-2i\sqrt{\pi} \eta(-)(0)] \sigma_+ \exp[-2i\sqrt{\pi} \eta(+)(0)]. \] (5.43)

At the same time, we rewrite the spurion operator \( \sigma_R \) in the normal product form
\[ \sigma_R = \exp\left[ -\frac{1}{2} \int_0^{\infty} \frac{dp_-}{p_-} \right] \exp\left[ -\int_0^{\infty} \frac{dp_-}{p_-} c^*(p_-) \right] \exp\left[ \int_0^{\infty} \frac{dp_-}{p_-} c(p_-) \right]. \] (5.44)

Then we see that because
\[ \int_0^{\infty} \frac{dp_-}{p_-} = \int_0^{\infty} dp_+ = \int_0^{\infty} dp_+, \] (5.45)

the divergence from the former is canceled by one from the latter if we require
\[ \tilde{\kappa} = \frac{m^2}{4\kappa}. \] (5.46)

In this case, the normalization factor \( Z \) in (5.36) is identical with \( Z_+ \) in (3.4), so that (5.42) agrees exactly with (3.2) at \( x^+ = 0 \).

Now we note that the Fermion operator (3.2) is obtained from the initial value (5.42) with \( \tilde{\kappa} = \frac{m^2}{4\kappa} \) as a result of the temporal evolution
\[ \Psi_+(x^+, x^-) \equiv e^{iP_+x^+} \Psi_+(0, x^-) e^{-iP_+x^+} \] (5.47)

and that the zero mode fields do not prevent us from obtaining
\[ J_- = m\partial_- \tilde{\Sigma}, \quad T_{--} = i\Psi_+^* \partial_- \Psi_+ = (\partial_- \tilde{\Sigma})^2. \] (5.48)

Hence both \( P_+ \) and \( P_- \) are already diagonal, so that the parton picture is realized even when there exist zero mode fields in the formulation. This is the main finding of this paper.

We end this section by pointing out that we cannot change the order of integrations and differentiations in the evaluation of the commutator \([A_+(x), A_+(y)]\). This is because the physical and the ghost contributions are ill-defined separately, so that this commutator has to be calculated in such a way that divergences from the physical contributions and ghost contributions cancel each other. Actually, we have
\[ [\eta(x) + \tilde{\Sigma}(x), \eta(y) + \tilde{\Sigma}(y)] = IE(x - y), \] (5.49)
where

\[ E(x) = \frac{1}{2\pi} \int_0^\infty \frac{dk_+}{k_+} \left\{ \sin k_+ x^+ - \sin \left( k_+ x^+ + \frac{m^2}{4k_+} x^- \right) \right\} \]

\[ = \frac{1}{4} \epsilon(x^+) - \frac{\epsilon(x^+)}{4} J_0(m \sqrt{x^2}). \tag{5.50} \]

Here \( J_0 \) is the Bessel function of order 0. Consequently, the commutator \([A_+(x), A_+(y)]\) is obtained unambiguously by differentiating \(iE(x - y)\) with respect to \(\partial_+^x\) and \(\partial_+^y\).

§6. Concluding remarks

In this paper we have obtained the light-cone gauge operator solution of the Schwinger model by gauge-transforming the Landau gauge solution into the light-cone gauge and also by solving the initial value problems on the characteristic surfaces \(x^- = 0\) and \(x^+ = 0\). By so doing we have shown that the modes along both characteristics are needed to generate the entire representation space and that the zero-mode fields can be introduced in the light-cone axial gauge formulation in such a way that the solution is independent of any particular representation. The zero-mode part of \(P_+\) has been restored in the light-cone axial gauge formulation as integration constants (the boundary surface, \(x^- \to \pm \infty\), contributions). With these boundary values explicitly included, we can assume that \(\tilde{\Sigma}\) vanishes in the limits \(x^- \to \pm \infty\). This is consistent with the fact that we do not need boundary surface contributions \((x^+ \to \pm \infty)\) to the translational generators in the light-cone temporal gauge formulation and that the boundary surface contributions can be calculated from the energy-momentum densities \(T_{+\pm}\) and \(T_{-\pm}\) described only in terms of the \(\tilde{\Sigma}\) in the form

\[ T_{+\pm} = i\Psi_+^* \partial_+ \Psi_+ - (\partial_- A_+)^2 = \frac{m^2}{4} \tilde{\Sigma}^2, \]

\[ T_{-\pm} = i\Psi_+^* \partial_- \Psi_+ = (\partial_- \tilde{\Sigma})^2. \]

We have in particular investigated the roles that the fields \(\phi\) and \(\eta\) play. These fields appear as dynamical fields if quantized on \(x^0 = 0\), zero-mode fields if quantized on \(x^+ = 0\), and static fields if quantized on \(x^- = 0\). These fields can constitute zero-norm states which mix to form physical vacua, and without them vacua invariant under the large residual gauge transformations cannot be given. This would lead, among other things, to an incorrect value for the chiral condensate and a failure of the cluster property. We have shown that even when we have zero-mode fields, a parton picture is realized. We have also shown that the zero-mode fields are essential to properly regularize Fermi products. Without them the Fermi products cannot be regularized in a fully covariant, gauge invariant way, and the solution would fail. The zero-mode fields are important for these reasons, but we emphasize these two in the present paper. Any quantization procedure must include the zero-mode fields or it will not give the correct solution.
It is most likely that zero-mode fields play similar roles in more realistic field theories. In fact, we have preliminary evidence of this. The light-cone gauge solution of 4-dimensional free electromagnetic field equations is independent of the representation, and infrared divergences are regularized by zero-mode fields.\footnote{Furthermore} Furthermore the infinite degeneracy of the vacuum state due to spontaneous breakdown of chiral symmetry has been analogously found in QCD\textsuperscript{4}. Of course 2-dimensional abelian gauge fields have no physical degrees of freedom, whereas 4-dimensional nonabelian gauge fields possess physical transverse components so that it is likely that the mechanism of spontaneous breakdown of the chiral symmetry in the Schwinger model is different from that in QCD\textsuperscript{4}.

In DLCQ, the $p^t = 0$ singularity is regulated with periodicity conditions. In that case the continuum result is not recovered for physical matrix elements, even if the problem is solved exactly at finite $L$ and the limit $L \to \infty$ is then taken.\footnote{It may well be that for many problems, it is necessary to carefully regulate the theory prior to imposing periodicity conditions. The DLCQ grid would then be just a numerical device, not a regulator. Such procedures have been suggested in Ref. 19. We do not think that the techniques necessary to carry out this procedure are known for all cases, but knowledge on this subject is growing.} It may well be that for many problems, it is necessary to carefully regulate the theory prior to imposing periodicity conditions. The DLCQ grid would then be just a numerical device, not a regulator. Such procedures have been suggested in Ref. 19. We do not think that the techniques necessary to carry out this procedure are known for all cases, but knowledge on this subject is growing.

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