A Simple Model for Spatio-Temporal Chaos
in an Unstable Burgers Equation

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We consider an unstable Burgers equation that exhibits spatio-temporal chaos. This spatio-temporal chaos is characterized by creation and merging processes of shock structures. We attempt to construct a simple model for the dynamics of these shocks. We also study a two-dimensional unstable Burgers equation.

§1. Introduction and an unstable Burgers equation

Pattern formation and nonlinear dynamics have been investigated in many spatially extended dissipative systems. 1), 2) Spatio-temporal chaos has been studied as a typical dynamical state far from equilibrium. Many model equations that can exhibit spatio-temporal chaos have been proposed. The Kuramoto-Sivashinsky equation is one of the simplest such equations. 3) The dynamical and statistical properties of model equations of this type have been intensively studied. 4)–6) The creation and annihilation of cellular structures are characteristic of spatio-temporal chaos. We have proposed another simple model equation i.e. an unstable Burgers equation, to understand the dynamics of this creation and annihilation. 7) The Burgers equation is a model equation in which shockwave turbulence is observed, although it decays slowly into a uniform state. Our model equation has the form of the Burgers equation with an instability term,

$$\frac{\partial u}{\partial t} = \int g(x - x') \frac{\partial^2 u}{\partial x^2}(x') dx' + \frac{\partial u}{\partial x} + \nu \frac{\partial^2 u}{\partial x^2},$$

(1)

where $g(x)$ is the kernel for the integro-differential equation. The symmetry $g(-x) = g(x)$ is assumed. This equation possesses Galilean symmetry: $u(x, t) \rightarrow u(x + ct, t) + c$. A linear perturbation $u_k$ with wavenumber $k$ around the uniform solution $u = 0$ obeys

$$\frac{du_k}{dt} = -g(k)k^2 u_k - \nu k^2 u_k,$$

(2)

where $g(k)$ is the Fourier transform of $g(x)$. If $g(k) + \nu$ is negative for small $k$ and $g(k)$ decreases rapidly to zero for large $k$, the uniform solution $u = 0$ is unstable with respect to long wavelength perturbations, and shockwave turbulence is expected not to decay into the uniform state. Recently, a similar model equation was proposed by Goren et al. to discuss a scenario for the onset of spatio-temporal chaos. 8) The chaotic behavior of the model equation (1) was studied in a previous paper. 7)
For $\nu = 0$, an exact solution for the one-shock solution is given as

$$u(x) = -\frac{32g_1 L}{6\pi^2} \sin \left( \frac{\pi x}{L} \right) \quad \text{for} \quad 0 < x < L/2,$$

$$u(x) = \frac{32g_1 L}{6\pi^2} \sin \left( \frac{\pi x}{L} \right) \quad \text{for} \quad L/2 < x < L,$$

where $-k^2 g(k) = g_1$ for $k = \pm 2\pi/L$. $g(k) = 0$ for other values of $k$, and $L$ is the system size. The solution has a discontinuity at $x = L/2$, and the amplitude increases linearly with $g_1$. The numerically obtained solution for $\nu \neq 0$ is very close to the exact solution, except near the discontinuity point. It is continuous and has a finite width at $x = L/2$ for $\nu \neq 0$. To clarify the positions of the shock structures, we have plotted local maximum and minimum points as functions of time.

Figure 1 displays the time evolution of the maximum and minimum points for $-g(k)k^2 = 0.4k^2 \exp(-16k^2)$, $\nu = 0.01$ and $L = 112\pi$. There appear many shocks. The structure of the pair creation of the maximum and minimum points resembles a leaf, and it evolves into a shock, which resembles a branch. The merging of two branches into one branch corresponds to the coalescence process of nearby shocks. The coalescence of two shocks is a property of the original Burgers equation. This plot displays spatio-temporal chaos including creation and coalescence dynamics of the characteristic shock structures.

§2. A simple model for shock dynamics

We wish to understand complicated dynamics of Eq. (1) including the creation and merging processes, with a simpler model. We consider the positions of the shocks. There exists a stationary shock-train solution with wavelength $\lambda$ for the unstable Burgers equation. General aspects of phase dynamics for a spatially-periodic pattern with translational and Galilean invariance are discussed in Refs. 9) and 10). The two neutral modes are denoted as $\phi$ and $\psi$. The phase dynamics is then described by

$$\frac{\partial \phi}{\partial t} = -\psi,$$

$$\frac{\partial \psi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} + \beta \frac{\partial^2 \psi}{\partial x^2},$$

(4)

where $\alpha$ and $\beta$ are coefficients that depend on the wavelength $\lambda$, and the higher-order terms are neglected. The coefficients $\alpha$ and $\beta$ depend also on $g(x)$ and $\nu$, however,
we have not yet obtained the explicit forms of $\alpha$ and $\beta$ theoretically. We attempt to estimate the value of $\alpha$ and $\beta$ from a numerical analysis of linear stability for specific values of parameters. The linear stability of the phase equation with respect to long-wavelength perturbations, that is, $\phi$ and $\psi$ varying like $e^{\eta t+ikx}$ with $k \to 0$, is determined by
\[ \eta^2 + \beta k^2 \eta - \alpha k^2 = 0. \]  
(5)
The eigenvalue is expressed as
\[ \eta \sim \sqrt{\alpha k - \beta k^2/2} \]  
(6)
for small $k$, and therefore the spatially-periodic structure is unstable for $\alpha > 0$.

We have constructed a shock-train solution including 16 shocks with wavelength $\lambda = 7\pi$ by connecting numerically obtained one-shock solutions for $-g(k)k^2 = 0.4k^2 \exp(-16k^2)$, $\nu = 0.01$ and $L = 16 \times 7\pi = 112\pi$. The largest eigenvalue was numerically estimated from the linear growth rate of small perturbations around the numerically constructed periodic solution. A perturbation $\delta u(x,t)$ around the spatially-periodic solution $u_0(x)$ obeys
\[ \frac{\partial \delta u}{\partial t} = \int g(x-x') \frac{\partial^2 \delta u}{\partial x'^2} dx' + u_0 \frac{\partial \delta u}{\partial x} + \delta u \frac{\partial u_0}{\partial x} + \nu \frac{\partial^2 \delta u}{\partial x^2}. \]  
(7)
Since the solution $u_0$ is spatially periodic ($u_0(x+\lambda) = u_0(x)$), the perturbation $\delta u(x,t)$ can be expressed as $\delta u(x,t) = e^{\eta t+ikx}u_d(x)$, where $-\pi/(2\lambda) < k < \pi/(2\lambda)$, and $u_d$ is a spatially-periodic function ($u_d(x+\lambda) = u_d(x)$). We have constructed a perturbation characterized by $k$ with a linear combination of Fourier components with wavenumber $k + 2\pi n/\lambda$, where $n$ is an integer, and calculated the linear growth rate for the perturbation. The linear growth rate is measured with respect to the norm $| \delta u(x,t) | = \left( \int_0^L \delta u(x,t)^2 dx \right)^{1/2}$. Figure 2(a) displays the spatially-periodic solution $u_0(x)$, and Fig. 2(b) displays a numerically obtained eigenmode with wavenumber $k = 1/7$. Figure 3 displays the largest eigenvalue for several values of $k$.  

Fig. 2. (a) Spatially-periodic solution of the unstable Burgers equation for $-g(k)k^2 = 0.4k^2 \exp(-16k^2)$, $\nu = 0.01$ and $L = 112\pi$. (b) The most unstable eigenmode with wavenumber $k = 1/7$ around the spatially-periodic solution shown in (a).
We have estimated the coefficients as $\alpha \sim 0.00562$ and $\beta \sim 0.49$ in Eq. (6) from Fig. 3. The dashed line denotes the relation $\eta = \sqrt{\alpha k - \beta k^2}/2$. The spatially-periodic solution $u_0(x)$ is unstable, since the eigenvalues are positive. Phase modulation with respect to $\phi$ represents the modulation of the positions of the shocks. Modulation with respect to $\psi$ represents the modulation of the locally averaged value of $u(x,t)$ around the shock structure. This locally averaged value is related to the local velocity of the shock structure, owing to the nonlinear term $u\partial u/\partial x$.

7) The interpretation of the instability caused by positive $\alpha$ is that the modulation of the shock positions induces the modulation of the locally averaged $u$-value, and this induces an effect on the drift velocity of shocks to increase the modulation of the shock positions.

As a simple model for the shock dynamics, we use the linear phase dynamics. We introduce two variables, $x_i$ and $z_i$, where $x_i$ represents the position of the $i$th shock and $z_i$ represents the locally averaged $u$-value around the $i$th shock, which is a quantity such as $(u_{\text{max}} + u_{\text{min}})/2$. The two variables $x_i$ and $z_i$ are related to $\phi$ and $\psi$ as $x_i = \lambda i + \phi(x_i)$ and $z_i = \psi(x_i)$. The model equations for $z_i$ and $x_i$ are written as

$$\frac{dx_i}{dt} = -z_i,$$
$$\frac{dz_i}{dt} = D_x(x_{i+1} - 2x_i + x_{i-1}) + D_z(z_{i+1} - 2z_i + z_{i-1}),$$

where $D_x = \alpha/\lambda^2$ and $D_z = \beta/\lambda^2$. The linear dynamics provides a good approximation if the arrangement of the shocks is fairly regular, that is $x_i \sim \lambda i$ and $z_i \sim 0$. Since the linear equation is unstable, the modulation of the shock positions grows. For the nonlinear evolution of the shock dynamics, we assume two processes. One is the creation process of a new shock: a new shock is created in the middle of two neighboring shocks if the distance between the two shocks is larger than a critical value $l_c$. New variables $x_i'$ and $z_i'$ are then introduced as $x_i' = (x_i + x_{i+1})/2$ and $z_i' = (z_i + z_{i+1})/2$, where $x_i$ and $x_{i+1}$ are the positions of the two neighboring shocks. The other assumed process is a merging process of two neighboring shocks. The two variable sets for the two neighboring shocks are replaced by one variable set as $x_i' = (x_i + x_{i+1})/2$ and $z_i' = (z_i + z_{i+1})/2$ if the neighboring shocks
collide with each other, that is, if \( x_i = x_{i+1} \). The number of dynamical variables changes in the time evolution of our simple model. We have estimated numerically the critical length \( l_c \) from the distance just before the creation of the new pair of locally maximum and minimum points. The average value of the length is 42.5 in the time evolution of Fig. 1, and the standard deviation is 3.1. We have used the average value for the critical length \( l_c \). Figure 4 displays the time evolution of the shock positions for the simple model with \( D_x = \alpha/\lambda^2 = 2.86 \times 10^{-5}, D_z = 2.5 \times 10^{-3} \) and \( l_c = 42.5 \). The initial conditions are \( x_i = 3.5\pi(2i - 1), i = 1, 2 \cdots, 16 \) and \( z_i = 0.01\sin\{(2i - 1)\pi/2\} + 0.01\sin\{(2i - 1)\pi/8\} + 0.01\sin\{(2i - 1)\pi/16 + \pi/5\} \), where the system size \( L = 112\pi \). The initial conditions for Fig. 1 consist of the unstable stationary periodic solution \( u_0(x) \) in Fig. 2(a) perturbed with \( \delta u(x) = 0.01\sin(16\pi x/L) + 0.01\sin(4\pi x/L) + 0.01\sin(2\pi x/L + \pi/5) \). Therefore, Fig. 4 displays time evolution for the simplified model equation starting from the initial conditions corresponding to those for Fig. 1. Creation and merging processes occur chaotically also in Fig. 4. The regular spatially periodic configuration of shocks is linearly unstable, and the modulation of shock positions grows. When two neighboring shocks collide with each other, they merge into one shock. If the interval between two neighboring shocks becomes sufficiently long, a new shock appears between these two shocks. Although the rule governing the time evolution is simple, it generates a complicated pattern. The first merging processes occur near \( t \sim 300 \) in Fig. 1. However, they occur near \( t \sim 400 \) in Fig. 4. The difference is probably due to the fact that the linear equation (8) is not a good approximation if the deviation of the shock positions from the initial regular arrangement becomes larger. The time evolution of the shock positions described by the simple model is quantitatively different from that described by Eq. (1). This is partly due to the chaotic dynamics themselves, since the chaotic time evolution depends sensitively on the initial conditions. However, the main reason is probably that our model is too simple. However, although our simple model may not be a quantitatively good model, we think it is a qualitatively good model.

§3. A two-dimensional model for the unstable Burgers equation

Spatio-temporal chaos in two dimensions has been studied in experiments of Benard convection and the electrohydrodynamic convection in liquid crystals. The creation and annihilation of defect structures are characteristic of spatio-temporal chaos in two dimensions. As a model equation in two dimensions, we can construct
an unstable Burgers equation in two dimensions as

\[
\frac{\partial u}{\partial t} = \int g(x - x') \frac{\partial^2 u}{\partial x'^2} (x') dx' + 2u \frac{\partial u}{\partial x} + \nu_x \frac{\partial^2 u}{\partial x^2} + \nu_y \frac{\partial^2 u}{\partial y^2},
\]

where an additional viscous term in the y-direction is formally introduced, and the coefficient of the nonlinear term has been changed for convenience. We assume periodic boundary conditions both in the x and y directions: \( u(x + L_x, y) = u(x, y) \), \( u(x, y + L_y) = u(x, y) \). An instability occurs only in the x-direction in this model equation. There is a one-dimensionally chaotic solution \( u_1(x, t) \) which is uniform in the y-direction. The solution \( u_1(x, t) \) obeys Eq. (9) without the last term, \( \nu_y \frac{\partial^2 u}{\partial y^2} \). We consider the stability of the one-dimensional solution with respect to perturbations in the y-direction. The perturbation \( \delta u_k(x, t) e^{iky} \) with wavenumber \( k_y \) obeys the linearized equation

\[
\frac{\partial \delta u_k}{\partial t} = \int g(x - x') \frac{\partial^2 \delta u_k}{\partial x'^2} (x') dx' + 2u \frac{\partial \delta u_k}{\partial x} + 2\delta u_k \frac{\partial u_0}{\partial x} + \nu_x \frac{\partial^2 \delta u_k}{\partial x^2} - \nu_y k^2 \delta u_k.
\]

The linear growth rate \( \lambda(t) \) at time t for the norm of \( \delta u_k(x, t) \) fluctuates in time owing to the chaotic dynamics. The averaged value of the linear growth rate is estimated as \( \log(e^{\int_0^T \lambda(t) dt})/T \). The average value for \( k = 0 \) represents the largest Lyapunov exponent of the chaotic solution \( u_1(x, t) \) for the one-dimensional equation (9) without the last term. The Lyapunov exponent \( \lambda_0 \) was numerically calculated as 0.0102 for \( -g(k)k^2 = 0.2k^2 \exp(-k^2) \), \( \nu = 0.02 \) and \( L_x = 12\pi \). The linear growth rate at time \( t \) for a perturbation with finite wavenumber \( k \) is decreased to \( \lambda(t) - \nu_y k^2 \).

Therefore, its time average is also decreased to \( \log(e^{\int_0^T \lambda(t) dt})/T - \nu_y k^2 = \lambda_0 - \nu_y k^2 \). The linear growth rate is positive for perturbations with wavenumber \( k < \sqrt{\lambda_0/\nu_y} \), and in this case the modulation in the y-direction grows. If the viscous coefficient \( \nu_y \) is larger than \( \lambda_0(L_y/2\pi)^2 \), one-dimensional chaos that is uniform in the y-direction is linearly stable. The critical value of \( \nu_y \) is approximately 0.37 for \( -g(k)k^2 = 0.2k^2 \exp(-k^2) \), \( \nu = 0.02 \) and \( L_x = L_y = 12\pi \). Figure 5 displays a snapshot pattern of \( u(x, y, t) \) for \( \nu_y = 0.6 \). The shaded region corresponds to \( \partial u/\partial x > 0 \), which is the shock region. The shock lines are straight, which implies that the one-dimensional solution \( u_1(x, t) \) is stable. Figure 6 displays three snapshot patterns of \( u(x, t) \) for \( \nu_y = 0.3 \). Nonuniform patterns are seen in the figures. The creation of the shock region occurs locally, and it extends in the y-direction. The merging process of the neighboring shocks also occurs locally. There appear defect-like structures.
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Fig. 6. Three snapshot patterns obtained from Eq. (9) for \(-g(k)k^2 = 0.2k^2 \exp(-k^2)\), \(\nu_x = 0.02\), \(\nu_y = 0.3\), \(L_x = 12\pi\) and \(L_y = 12\pi\). The time interval between two snapshots is 10.

Some modulation in the \(y\)-direction grows and defect structures appear.

that connect the two-shock regions and the merged region. These defect structures are characteristic of spatio-temporal chaos in two dimensions.

As a simple model for two-dimensional chaos, we consider the dynamics of shock lines which extend in the \(y\)-direction. The model equation is a simple generalization of Eq. (8):

\[
\begin{align*}
\frac{\partial x_i(y)}{\partial t} &= -2z_i(y) + \nu_y \frac{\partial^2 x_i}{\partial y^2}, \\
\frac{\partial z_i(y)}{\partial t} &= D_x(x_{i+1}(y) - 2x_i(y) + x_{i-1}(y)) + D_z(z_{i+1}(y) - 2z_i(y) + z_{i-1}(y)) + \nu_y \frac{\partial^2 z_i}{\partial y^2},
\end{align*}
\]

where \(x_i(y)\) and \(z_i(y)\) denote the shock position at \(y\) and the locally averaged value of \(u\) around the shock line \(x_i(y)\). Only the viscous terms are introduced in the \(y\)-direction. The coefficients \(D_x\) and \(D_z\) are determined from a linear stability analysis using a method similar to the one-dimensional model. A simple discretization method \((\partial^2 x_i/\partial y^2 \rightarrow (x_{i,j+1} - 2x_{i,j} + x_{i,j-1})/(\Delta y)^2, \ \partial^2 z_i/\partial y^2 \rightarrow (z_{i,j+1} - 2z_{i,j} + z_{i,j-1})/(\Delta y)^2)\) was used for the numerical simulation. Figure 7 displays a snapshot pattern for the shock lines at \(D_x = 0.000103\), \(D_z = 0.00344\), \(\nu_y = 0.6\), \(L_x = L_y = 12\pi\), \(l_c = 10.5\) and \(\Delta y = 12\pi/50\). These parameters correspond to Eq. (9) at \(-g(k)k^2 = 0.2k^2 \exp(-k^2)\) and \(\nu_y = 0.02\). Here, the shock lines are straight. This corresponds to a state shown in Fig. 5. Figure 8 displays three snapshot patterns of the shock lines.

Fig. 7. A snapshot of shock lines obtained from Eq. (11) at \(\nu_y = 0.6\). The shock lines are straight, since the perturbation in the \(y\)-direction decays.
Fig. 8. Three snapshot patterns of shock lines in the time evolution obtained from Eq. (11) with \( \nu_y = 0.3 \). The interval between two snapshots is 10. Shock lines are locally merged, and defect-like structures appear. A shock line is also created locally and extends in the \( y \)-direction.

lines at \( \nu_y = 0.3 \). Here, the shock lines are curved and the mergings of two shocks occur locally and extend in the \( y \)-direction, as in Fig. 6. There naturally appear defect structures in the merging processes. The creation of a shock line occurs also locally and extends in the \( y \)-direction. In our simulation, no diffusion coupling is assumed at the edges of such shock lines with endpoints. Our model for shock-line chaos may be too simple, but it includes the mechanism of defect creation. The model equation (9) can be written as

\[
\frac{\partial U}{\partial t} = \int g(x' - x, y' - y) \nabla^2 U(x', y') dx' dy' + (\nabla \cdot U)^2 + \nu \nabla^2 U,
\]

(12)

where \( U(x) = \int u(x') dx' \) or \( u(x, t) = \partial U / \partial x \). Equations (9) and (12) are anisotropic models, and the shock structures extend in the \( y \)-direction. We can construct an isotropic model as a modification of Eq. (12):

\[
\frac{\partial U}{\partial t} = \int g(x' - x, y' - y) \nabla^2 U(x', y') dx' dy' + (\nabla \cdot U)^2 + \nu \nabla^2 U.
\]

(13)

Here \( g(x' - x, y' - y) = g(r - r') \); that is, the integral kernel depends only on the distance \( r - r' = \sqrt{(x - x')^2 + (y - y')^2} \) between \((x, y)\) and \((x', y')\). Figure 9 displays

Fig. 9. Three snapshot patterns obtained from the isotropic model equation (13). The time interval between two snapshots is 25. Creation and annihilation of cellular structures occur here. Each side of the cell corresponds to shock structure.
three snapshot patterns of $\nabla^2 U$ for $-g(k)k^2 = 0.2k^2 \exp(-k^2)$ and $\nu = 0.02$. In the shaded regions, $\nabla^2 U > 0$. There appears a honeycomb-like pattern. It is dynamically fluctuating. Each side of the honeycomb cell corresponds to a shock region. In this isotropic model, the creation and annihilation processes of cells are characteristic of spatio-temporal chaos. The annihilation and creation of cells can be interpreted in terms of the shock dynamics. The collapse of a small cell is interpreted as the merging process of shock regions surrounding the cell. A large cell breaks up into two cells through the creation process of a new shock region. The shock dynamics may be qualitatively understood with our simple model Eq. (11).

§4. Summary

We have studied an unstable Burgers equation to better understand spatio-temporal chaos. In the behavior described by this equation, shock structure plays the role of a building block for the spatio-temporal chaos. We have proposed a simple model to understand the shock dynamics qualitatively. It is composed of an unstable linear equation governing the shock positions and nonlinear processes, that is, the creation and merging of shocks. This may be only a qualitative model, but with it spatio-temporal chaos can be more easily understood. We have constructed a two-dimensional model and found defect chaos. The defect dynamics were also described by a simple model for shock lines. We have also found chaotic dynamics of cellular structures in an isotropic model of the two-dimensional unstable Burgers equation.

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References

2) P. C. Hohenberg and M. Cross, Rev. Mod. Phys. 65 (1993), 51.
7) H. Sakaguchi, Physica D129 (1999), 57.