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Comment on the article "A note on Newton's shell-point equivalency theorem" [Am. J. Phys. 90, 394 (2022)]

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When central forces follow a power law, Cameron Reed [Am. J. Phys. **90**(5), 394–396 (2022)] showed that the resulting force on a test particle located outside an object with spherical symmetry is the same as if the source were located at the center of the sphere, if and only if the potential is either Newtonian, $F \propto r^{-2}$, or Hookean, $F \propto r$. This Note shares another simple proof of this result and comments on the result in the light of the so-called transmutation law of central forces.

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Consider sources of a central potential $V(r) \propto r^\alpha$ that are uniformly distributed on a sphere at an areal density of k. The sphere is centered on point O and has radius R. At a point P, a distance $z \geqslant R$ from the center of the sphere, the overall potential is

$$V(z) = 2\pi R^2 k \int_{-1}^{+1} du[R^2 + z^2 - 2Rzu]^{\alpha/2} + \text{constant}, \tag{1}$$

as in Eqs. (2)–(4) of Ref. 1. We define $u = \cos\theta$, and we assume that the density is in units that incorporate the force constant. The problem is to find which real exponent α could provide a potential of the form

$$V(z) = Kz^\alpha + \text{constant}, \tag{2}$$

where $K = 4\pi R^2 k$; that is, the same potential as that of a point source located at O with "charge" K. The values $\alpha = 0$ and $\alpha = -2$ will be discarded: the first one corresponds to the trivial (uninteresting) case of a constant potential, and the second one leads to the logarithmic potential

$$V(z) = \frac{K}{2zR} \ln \frac{R+z}{|z-R|}, \tag{3}$$

which, obviously, cannot be put in the form K/z^2 specified in Eq. (2), at any finite z. Note here that for $\alpha \neq 0$, the two expressions (1) and (2) take the same limiting form when $z \gg R$. However, this fact does not provide information about α. Such information can only be obtained by requiring that the variable parts of Eqs. (1) and (2) are exactly the same at each $z \geqslant R$.

The two distributions are said to be equivalent if they produce the same force in their common exterior. Hence, having the same derivative, their potentials (1) and (2) can only differ by a constant in that domain. Thus, let us write

$$\frac{1}{2}\int_{-1}^{+1} du[R^2 + z^2 - 2Rzu]^{\alpha/2} = z^\alpha + C, \tag{4}$$

where C is a constant, the equivalence being effective, *a priori*, whatever the value of $z \geqslant R$. It is important to remark here that the constant C in Eq. (4) is not arbitrary, contrary to those appearing in the definitions of potentials (1) and (2), which can be freely chosen. Here, C is the difference between two well-defined functions that do not necessarily have the same value at a given z. Let us distinguish the two cases $\alpha > 0$ and $\alpha < 0$.

For $\alpha > 0$, it is clear that the potential (1) is everywhere a continuous function of the coordinates of P, and that Eq. (4) can be extended to the whole range $0 \leqslant z < +\infty$. Then, taking $z = 0$ yields $C = R^\alpha$, and, dividing Eq. (4) by R^α and defining $a = z/R$, we obtain

$$\begin{aligned} 1 + a^\alpha &= \frac{1}{2}\int_{-1}^{+1} du[1 + a^2 - 2au]^{\alpha/2} \\ &= \frac{1}{2a(\alpha+2)}\left[(1+a)^{\alpha+2} - |1-a|^{\alpha+2}\right]. \end{aligned} \tag{5}$$

It is then sufficient to consider any convenient finite value of a to check this equation. For $a = 1$, we get the condition

$$\alpha + 2 = 2^\alpha, \tag{6}$$

which, for $\alpha > 0$, is satisfied if and only if $\alpha = 2$, i.e., only for a Hookean potential.

In our opinion, the above demonstration is the most expeditious one, involving little calculation. However, to convince the reader, let us add the following.

First, for completeness, let us remark that the last expression in Eq. (5) remains well defined when $a \to 0$ (or $z \to 0$), since then $(1 \pm a)^{\alpha+2} \simeq 1 \pm (\alpha+2)a$ and

$$\begin{aligned} &\frac{1}{2a(\alpha+2)}\left[(1+a)^{\alpha+2} - |1-a|^{\alpha+2}\right] \\ &\simeq \frac{1}{2a(\alpha+2)}[2a(\alpha+2)] = 1, \end{aligned} \tag{7}$$

which, in some way, reinforces the choice to adjust C at $z = 0$.

Second, the constant C has been found to be R^α at $z = 0$. Adjusting it instead at $z = R$ would give

$$\begin{aligned} C &= -R^\alpha + R^\alpha 2^{\alpha/2-1}\int_{-1}^{1} (1-u)^{\alpha/2} du \\ &= R^\alpha\left[\frac{2^{\alpha+1}}{\alpha+2} - 1\right]. \end{aligned} \tag{8}$$

However, for coherence, this second value must be equal to the former, which requires

$$\frac{2^{\alpha+1}}{\alpha+2} = 2 \quad \text{or} \quad 2^{\alpha} = \alpha + 2, \tag{9}$$

i.e., the same equation as Eq. (6), yielding $\alpha = 2$. Clearly, this amounts to fixing C at $z = R$ and then checking Eq. (4) at $z = 0$.

Third, to check Eq. (5), let us try the other value $a = 2$. We obtain

$$(\alpha + 2)[2^{\alpha+2} + 4] + 1 = 3^{\alpha+2}, \tag{10}$$

that is, an equation a bit more complicated. Yet, again for coherence, if we combine it with Eq. (6), we obtain the simpler one

$$3^{\alpha+2} = [1 + 2(\alpha + 2)]^2, \tag{11}$$

which is found to be satisfied if and only if $\alpha = 2$.

A final check is the following calculation, which is similar to that of Ref. 1. Generate a Taylor expansion of the right-hand side of Eq. (5) in the variable $1/a$ and subtract a^{α}. Setting $\nu = \alpha + 2$, this leads to the condition

$$RHS - a^{\alpha} = 1 = \frac{(\nu-1)(\nu-2)}{3!} a^{\nu-4} + \frac{(\nu-1)(\nu-2)(\nu-3)(\nu-4)}{5!} a^{\nu-6} + \cdots, \tag{12}$$

which forces us to take $\nu = 4$ (or $\alpha = 2$), because then $a^{\nu-4} = 1$, $(\nu-1)(\nu-2)/3! = 1$, and all the following terms of the expansion disappear, having zero coefficients.

Let us now turn to the case $\alpha < 0$. We have to be cautious with it because the potential at the left-hand side of Eq. (4) may be singular for $z = R$. However, its integrand remains finite for any $z > R$ and we can safely take the limit $z \to +\infty$. This leads to $C = 0$. Then, dividing the resulting equation by z^{α} and setting $b = R/z$, we obtain

$$1 = \frac{1}{2}\int_{-1}^{+1} du[1 + b^2 - 2bu]^{\alpha/2} = \frac{1}{2b(\alpha+2)}\left[(1+b)^{\alpha+2} - |1-b|^{\alpha+2}\right]. \tag{13}$$

Clearly, that equation cannot be satisfied if $\alpha + 2 < 0$, since in this case, its right hand member diverges when $b \to 1$. Thus, assuming $\alpha + 2 > 0$ and taking $b = 1$, we get the condition

$$2(\alpha + 2) = 2^{\alpha+2}, \tag{14}$$

which, for $\alpha \neq 0$, is satisfied if and only if $\alpha = -1$, i.e., only for a Newtonian potential.

The shell-point equivalency theorem is, thus, proved for *both* the Newtonian and the Hookean potentials. These potentials, already associated by Bertrand's famous theorem,[2] are found, once more, to be the only ones satisfying this new property. In fact, this is not very surprising if we remember their link through the law of transmutation of central forces.[3–5] This law establishes a connection between homogeneous potentials with positive exponents $\alpha_a = \beta_a^2 - 2$ and those having negative exponents $\alpha_b = \beta_b^2 - 2$, which results in the simple relation $\beta_a\beta_b = 2$ (see Ref. 5). Here, $\beta_a = 2$ for the Hookean potential and $\beta_b = 1$ for the Newtonian one. If some distribution of Hookean point sources produces in a given domain of space a potential similar to that of an individual source, it seems natural that the Newtonian companion will also produce in another domain a potential similar to that of a point source. This is so because, as we know, both point sources provide the same closed trajectories of a test particle on which they are acting, namely, circles and ellipses.

Finally, let us make the following remarks: While the equivalence is valid at all distances in the Hookean case, it only applies outside the sphere in the Newtonian case. In addition, both potentials are well defined and continuous at $z = R$, as they should be. Of course, the result also extends to a uniform volumic distribution inside a solid sphere, by simply superposing surface distributions like the one considered here, as is done usually (see references in Ref. 1).

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