Multifold phase space path integral synthetic seismograms

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SUMMARY

We consider a medium consisting of homogeneous layers separated by curved interfaces. In order to evaluate the response of a single generalized ray, the source and receiver wavefields are expanded in a series of plane waves. The coupling of these plane waves at each point of the surfaces of material discontinuity is determined by means of a Kirchhoff integral using generalized reflection and transmission coefficients. The resulting integral, called the multifold phase space path integral (PSPI) consists of a series of integrals over ray parameters and over interfaces touched by the generalized ray on its way from the source to the receiver. This approach is a generalization of the multifold configuration space path integral (CSPI) to which it reduces by successive application of the stationary phase point method over the ray-parameter integrals.

The PSPI like the CSPI automatically includes diffractions from corners. In addition classical head waves are included, although for curved interfaces the head waves are only approximate. 2-D synthetic seismograms are converted to equivalent approximate point-source responses by assuming cylindrical symmetry about source and/or receiver. The waveforms and amplitude of PSPI synthetic seismograms compare well with those computed by generalized ray theory for a 1-D model, and with finite difference synthetics for a 2-D model.

Key words: diffractions, head waves, Kirchhoff, phase space, synthetic seismograms.

1 INTRODUCTION

Kirchhoff–Helmholtz (KH) integrals, especially ray-Kirchhoff formulations, have been used both in their acoustic (Trorey 1977; Hilterman 1970, 1975) and elastic forms (Frazer & Sen 1985; Sen & Frazer 1985) for computing reflection and refraction seismograms in laterally inhomogeneous media. Rays are traced from both source and receiver to a surface of material discontinuity and the KH theory is applied to determine the coupling between each source ray and each receiver ray at the interface. The assumption inherent in the KH formulation is the tangent plane approximation which assumes that at the point of intersection wavefronts are locally plane and the interface is planar, allowing the use of plane wave reflection and transmission coefficients. However, real sources are finite and actual wavefronts at the interface are spherical. Brekhovskikh (1960) established a relation between the spherical and plane wave reflection coefficients and showed that the error term grows in the post-critical region. Thus, the Kirchhoff formulation will be in error for near and past critical reflections.

The ray-Kirchhoff formulation is valid even when the receiver is located at a caustic and the diffractions from any corner on the surface of integration are automatically included. However, caustics of the source and receiver wavefields caused on the surface of integration are not correctly treated, head waves are not included, and the diffractions from corners at intermediate interfaces are not included.

Recently, Zhu (1988) developed a ray-Kirchhoff formulation that avoids the singularity at the caustics by using a modified Eikonal equation to derive the geometrical ray theory Green's functions. This approach gives a solution that is regular even when there are caustics on the surfaces of integration. However, as the ray-field becomes multivalued (due to multipathing) on the surface of integration, the integration has to be carried out for each branch of the ray-field. This gives rise to diffraction like phases from each endpoint of the surface of integration (Frazer & Sinton 1984; Sen & Frazer 1985).

A generalization of the Kirchhoff-Helmholtz procedure which overcomes some of these problems is the multifold configuration space path integral (CSPI Frazer 1987; Sen &
Source Receiver

Figure 1. Formulation of a CSPI. Summing over all paths from source to the receiver via the interfaces reduces to Snell ray path at the high-frequency limit.

Frazer (1987). Here we briefly review the CSPI, for completeness. Consider a medium consisting of two layers over a half-space. The response of a single generalized ray (Fig. 1) can be written as

\[ u(x_i) = \int dx_j \int dx_k f(x_i, x_j, x_k, x) \times \exp \left\{ i \omega \left[ T(x_i, x_j, x_k, x) \right] \right\}, \]

(1)

which is derived using the elastic KH integral iteratively. The function \( f \) contains geometrical spreading terms, generalized reflection/transmission coefficients, etc. and \( T \) is the traveltime of a ray from the source to the receiver via the interfaces. The CSPI makes use of geometrical acoustics, generalized plane wave reflection and transmission coefficients and an elastodynamic form of the KH integral. Thus the approximations made are those used in geometrical ray theory and the tangent plane approximation. In essence a CSPI is a multiple Huygen's construction and the resulting integral resembles the Feynman configuration space path integral (Schulman 1981). A true Feynman path integral solves the Schrödinger equation (factorized Helmholtz equation, in the seismological context) in terms of an infinite dimensional integral, which when evaluated by stationary phase methods, gives the classical path. Thus, our CSPI given by equation (1) is not a true Feynman path integral, for the derivation of (1) is based on an entirely different context and assumptions. First, separation of the elastic wave equation into two exact one-way Schrödinger type equations has not been made. Second, we integrate only over surfaces of material discontinuity where the boundary conditions are explicit. Our integral contains the contribution of all possible paths which are piecewise straight with vertices at the interfaces, hence the name configuration space path integral. Successive application of the stationary phase method to equation (1) yields the classical or Snell ray paths.

The CSPI avoids the problem of caustics and multipathing of rays on surfaces of integration and includes diffractions from corners at any intermediate interface. However, head waves and certain tunnelled waves are not included, and a layer pinchout will give rise to a singularity in the integrand for a ray transmitted through the pinched-out layer. Such singularities are integrable, but near the layer-pinpoints the curvature of the wavefront is so great that the Kirchhoff approximation is no longer valid. To address these problems we introduce below a generalization of the CSPI called the multifold phase space path integral (PSPI).

2 SCALAR KH INTEGRAL

Here for completeness we derive a frequency domain version of the scalar KH integral which will be needed in the sequel for the derivation of the PSPI. As shown in Fig. (2) we consider a volume \( V \) with boundary \( \partial V \) and an outward pointing unit normal \( \mathbf{n} \) and consider two scalar wave equations given by

\[ (\nabla^2 + \omega^2) \phi_1 = f_1, \quad (\nabla^2 + \omega^2) \phi_2 = f_2, \]

(2)

where \( \omega \) is the angular frequency, \( \alpha \) is the wave velocity and \( f_1 \) and \( f_2 \) have support at points \( x_1 \) and \( x_2 \), respectively. Thus the function \( f_1 \) vanishes inside \( V \) and on \( \partial V \) and \( f_2 \) vanishes outside \( V \) and on \( \partial V \). The scalar fields \( \phi_1 \) and \( \phi_2 \) can be used to represent displacement or velocity potentials in an elastic medium or pressure in an acoustic medium. By the divergence theorem, we have

\[ \int_{\partial V} \mathbf{n} \cdot (\nabla \phi_2 - \phi_2 \nabla \phi_1) \, dA = \int_V (\phi_1 \nabla^2 \phi_2 - \phi_2 \nabla^2 \phi_1) \, dV = \int_V \phi_1 f_2 \, dV. \]

(3)

Note that the surface integral reduces to a line integral in 2-D. Letting \( f_2 = \delta(x - x_2) \) this integral reduces to

\[ \phi_1(x_2) = \int_{\partial V} \mathbf{n} \cdot (\phi_1 \nabla \phi_2 - \phi_2 \nabla \phi_1) \, dA. \]

(4)

Thus the scalar wavefield \( \phi_1 \) at point \( x_2 \) due to a source \( f_1 \) outside of \( V \) is expressed as an integral of the scalar fields \( \phi_1 \) and \( \phi_2 \) and their normal derivatives over the surface \( \partial V \). For correct use of (4) \( \phi_1 \) must be due to a source with support exterior to \( V \) and the scalar field \( \phi_2 \) must be associated with a point-source at \( x_2 \) of the type \( f_2 = \delta(x - x_2) \). In the derivation of the CSPI (Frazer & Sen 1985), the integrand of the scalar KH integral is evaluated by using geometrical ray theory Green's functions. To
derive the PSPI, we will instead use the ray-parameter expansions of the source and receiver wavefields given in the next section.

3 P-INTEGRAL REPRESENTATION OF SCALAR WAVEFIELDS IN 2-D HOMOGENEOUS, ISOTROPIC MEDIA

The frequency-domain vector wave equation for displacement \( \mathbf{u} \) in a homogeneous, isotropic, elastic medium is given as

\[
(\lambda + 2\mu) \nabla(\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} + \rho \omega^2 \mathbf{u} = \mathbf{f}.
\]

Decomposing \( \mathbf{u} \) into Helmholtz potentials as

\[
\mathbf{u} = \nabla \phi - \nabla \times \mathbf{A}
\]

with

\[
\nabla \cdot \mathbf{A} = 0
\]

gives

\[
\nabla \cdot \mathbf{u} = \nabla^2 \phi
\]

and

\[
\nabla \times \mathbf{u} = \nabla \times \nabla \times \mathbf{A}.
\]

Thus from (5) and (6) we obtain

\[
\nabla(\lambda + 2\mu) \nabla^2 \phi + \mu \nabla \times (\nabla \times \mathbf{A}) = \mathbf{f}.
\]

The source term \( \mathbf{f} \) can also be expressed in terms of Helmholtz potentials as

\[
\mathbf{f} = \nabla \Phi_0 - \nabla \times \mathbf{L}_0.
\]

For a purely compressional source, \( \mathbf{f} = \nabla \Phi_0 \), and \( \mathbf{L}_0 = 0 \), so that (8) reduces to

\[
\left( \nabla^2 + \frac{\omega^2}{\alpha^2} \right) \Phi = -\frac{\Phi_0}{(\lambda + 2\mu)} = \Phi_0.
\]

We now apply a spatial Fourier transform given by

\[
\Phi(p, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\omega x) \Phi(x, z) \, dx
\]

for \( \omega > 0 \), with inverse

\[
\Phi(x, z) = \frac{\omega}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega p) \Phi(p, z) \, dp.
\]

From (10) and (11), we have

\[
\frac{\partial^2 \Phi}{\partial \xi^2} + \omega^2 \left( \frac{1}{\alpha^2} - p^2 \right) \Phi = \Phi_0.
\]

Homogeneous solutions of (13) can be written as \( \Phi = \exp(\pm i\omega qz) \), where \( q = \sqrt{1/\alpha^2 - p^2} \) is the vertical slowness. In order to obtain the inhomogeneous solutions, which are determined uniquely by the source term on the right-hand side of (13), further transformation is necessary. Applying the following transformation

\[
\tilde{\Phi} = \int_{-\infty}^{\infty} \Phi \exp(-i\gamma z) \, dz
\]

we obtain

\[
(-\gamma^2 + \omega^2 q^2) \tilde{\Phi} = \tilde{\Phi}_0
\]

or

\[
\frac{\tilde{\Phi}}{-\gamma^2 + \omega^2 q^2} = \frac{\tilde{\Phi}_0}{-\gamma^2 + \omega^2 q^2}
\]

which can be evaluated after we evaluate \( \tilde{\Phi}_0 \). This is given as

\[
\tilde{\Phi}_0 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dz \exp(-i\omega x - i\gamma z) \Phi_0(x, z).
\]

For a point explosion source,

\[
\mathbf{f} = \nabla \delta(x - \mathbf{x}_0)
\]

and we have

\[
\Phi_0 = \frac{\Phi_0}{\rho \alpha^2}.
\]

Without loss of generality we can take \( \mathbf{x}_0 = \mathbf{z}_0 = 0 \) so that

\[
\tilde{\Phi}_0 = \frac{1}{\rho \alpha^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dz \exp(-i\omega x - i\gamma z) \delta(x) \delta(z).
\]

Then substitution for \( \tilde{\Phi}_0 \) in (15) gives

\[
\tilde{\Phi} = \frac{1}{\rho \alpha^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(i\gamma z) \tilde{\Phi} \, dz,
\]

\[
\tilde{\Phi} = \frac{1}{2\pi \rho \alpha^2} \int_{-\infty}^{\infty} \left( \frac{1}{\omega^2 q^2 - \gamma^2} \right) \exp(i\gamma z) \, dy.
\]

We now use residue theorem to evaluate (19) and apply inverse Fourier transform to obtain \( \Phi \). This is given as

\[
\Phi = \int_{-\infty}^{\infty} \exp[i\omega r(z + qz)] \, dq,
\]

\[
\phi = \frac{1}{2\pi \rho \alpha^2} \int_{-\infty}^{\infty} \exp[i\omega \mathbf{P} \cdot \mathbf{x}] \, d\mathbf{p},
\]

where \( \mathbf{P} = \mathbf{r} \cdot \mathbf{z} \) is the slowness vector. Equation (20) expresses the P-wave displacement potential due to an explosion source in a 2-D homogeneous, isotropic medium as an integral over ray-parameter \( p \).

4 POINT-FORCE IN A 2-D MEDIUM

The elastic equations of motion for a line force acting in an infinite medium can be written as (e.g., Pilant, 1979)

\[
\alpha^2 \nabla(\nabla \cdot \mathbf{u}) - \beta^2 \nabla \times \nabla \times \mathbf{u} + \omega^2 \mathbf{u} = -\frac{\delta(r)}{\rho 2\pi r}
\]

where \( \alpha \) and \( \beta \) are compressional and shear wave velocities, \( \rho \) is the density, \( r = \sqrt{x^2 + z^2} \) and \( \mathbf{u} \) is a unit vector in the direction of the point-force. First, we write

\[
\mathbf{u} = \nabla(\nabla \cdot \mathbf{A}_0) - \nabla \times (\nabla \times \mathbf{A}_0),
\]

where we have introduced a scalar potential (\( \nabla \cdot \mathbf{A}_0 \)) and a vector potential (\( \nabla \times \mathbf{A}_0 \)) (e.g., Pilant 1979). We also have the identity

\[
\frac{\delta(r)}{\rho 2\pi r} = \nabla \cdot \left( \frac{\mathbf{u} \ln(r)}{2\pi \rho} \right) - \nabla \times \left( \frac{\mathbf{u} \ln(r)}{2\pi \rho} \right).
\]
Thus from (21), (22) and (23), we obtain
\[ (\nabla^2 + k_p^2)A_p = -\mathbf{a} \frac{\ln(r)}{2\pi \rho \alpha^2}, \]
\[ (\nabla^2 + k_s^2)A_s = -\mathbf{a} \frac{\ln(r)}{2\pi \rho \beta^2}, \]
where
\[ k_p^2 = \frac{\omega^2}{\alpha^2}, \quad k_s^2 = \frac{\omega^2}{\beta^2}. \]
Notice that, as \( \mathbf{a} \) is a constant vector, equations (24) and (25) show that \( A_p \) and \( A_s \) are parallel to \( \mathbf{a} \). Dotting \( \mathbf{a} \) into both sides of (24) and (25) and letting \( A_p = \mathbf{a} \cdot A_p \) and \( A_s = \mathbf{a} \cdot A_s \), we obtain
\[ (\nabla^2 + \omega^2)A_p + k_p^2 A_p = -\frac{\ln(r)}{2\pi \rho \alpha^2}, \]
\[ (\nabla^2 + \omega^2)A_s + k_s^2 A_s = -\frac{\ln(r)}{2\pi \rho \beta^2}, \]
which are the scalar wave equations for the \( P \) and \( S \) potentials. In order to solve (26) and (27) by transform methods we first determine the Fourier transform of \( \ln(r)/2\pi \), which approaches zero as \( r \) goes to infinity. Taking a Fourier transform in \( x \) yields
\[ (\nabla_x^2 + \omega^2)x = \delta(x) \delta(z), \]
which approaches zero as \( r \) goes to infinity. Taking a Fourier transform in \( x \) yields
\[ (\nabla_x^2 - K_x^2)\psi = \delta(z), \]
which gives the \( x \)-transform of \( \ln(r)/2\pi \) as
\[ \psi = \exp(-|K_x| |z|) = -\frac{\exp(-|K_x| |z|)}{2|K_x|}, \]
where \( |\cdot| \) represents the modulus. As the procedures for solving (26) and (27) are the same we consider equation (26) only. We take a Fourier transform in \( x \) of (26) and substitute for the Fourier transform of \( \ln(r)/2\pi \) the expression given in the right-hand side of equation (28). This gives
\[ (\nabla_x^2 + \gamma^2)\tilde{\phi} = \frac{1}{\rho \alpha^2} \frac{\exp(-|K_x| |z|)}{(-2|K_x|)}, \]
where
\[ \gamma^2 = K_x^2 - K_s^2. \]
We know that the solution of \( (\nabla_x^2 + \gamma^2)\phi = \delta(z) \) is given by \( \exp(\gamma |z|)/(2\gamma) \). Therefore, the solution of equation (29) is given by the following convolution:
\[ \tilde{A}_p = \frac{1}{\rho \alpha^2} \int_{-\infty}^{\infty} d\zeta \frac{\exp(-|K_x| |z| - \zeta)}{(-2|K_x|)} \frac{\exp(i\gamma |\zeta|)}{(2\gamma)}. \]
Taking the inverse Fourier transform of this equation, we obtain
\[ A_p(x, z) = -\frac{1}{8\pi \rho \alpha^2 i} \int_{-\infty}^{\infty} dK_x \exp(iK_x x) \times \int_{-\infty}^{\infty} d\zeta \exp(-|K_x| (|z| - \zeta) + i\gamma |\zeta|). \]
Now we set \( z = 0 \) and \( x = r \) to obtain
\[ A_p(r) = -\frac{1}{4\pi \rho \alpha^2 i} \int_{-\infty}^{\infty} dK_x \exp(iK_x r) \times \int_{0}^{\infty} d\zeta \exp(-|K_x| \zeta - i\gamma \zeta), \]
or
\[ A_p(r) = -\frac{1}{4\pi \rho \alpha^2 i} \int_{-\infty}^{\infty} dK_x \frac{\exp(iK_x r)}{|K_x|(|K_x| - i\gamma)}. \]
Substituting \( K_x = \omega p \) in equation (31), we express \( A_p \) as an integral over ray-parameter \( p \). Thus equation (31) represents the scalar \( P \)-wave potential due to a point-force in a 2-D elastic medium.

5 REFLECTION FROM AN IRREGULAR INTERFACE

In order to evaluate reflections from a curved interface (Fig. 3), we follow the procedure outlined in Frazer & Sen (1985). The surface of material discontinuity \( \Sigma \) is made coincident with a part of the surface of integration such that the volume \( V \) contains both points \( x_1 \) and \( x_2 \). We will use equation (4) and the point \( x_1 \) and \( x_2 \) used in equation (4) will be replaced by \( x_1 \) and \( x_2 \), respectively. Now, we replace \( \phi_1 \) by its equivalent scattered field appearing to emanate from points outside volume \( V \). Although the physical source point is inside \( V \), the virtual source region is not inside \( V \), and so none of the assumptions used in the derivation above is violated. Thus we have
\[ \phi_4(x_3) = \int_{\Sigma} \mathbf{n} \cdot (\phi_1 \nabla \phi_1 - \phi \nabla \phi_4) dA, \]
where the superscript ‘r’ stands for the reflected field at the interface and the subscripts ‘r’ and ‘s’ for the receiver and source wavefields, respectively. The symbol \( \phi_4(x_3) \) represents the field reflected from \( \Sigma \), measured at the point \( x_3 \), due to a source at \( x_3 \).

In our earlier approach we assumed that the wavefronts and the interface are locally planar and used the plane wave reflection coefficient to evaluate the scattered field. Here, we will make assumptions that are less severe. Both the fields \( \phi_1 \) and \( \phi_2 \) will be expressed as integrals over ray parameters and then we will use the plane wave reflection coefficient to express the reflected field at the interface.

[Figure 3. PSPI for reflection from a curved interface.]
Thus, the wavefronts are actually planar although the interfaces are assumed to be locally plane ignoring the interaction of the wavefield between neighbouring points of the interfaces. At the interface, the incident scalar field due to an explosion source (equation 20) \( \phi_s \) can be written as

\[
\phi_s(x_i) = \frac{i}{2\pi \rho \alpha_1} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(p_x - p_s) \cdot (x_1 - x_s)\right]}{(2q_2)},
\]

where \( \mathbf{P} \) is the slowness vector, \( q_s \) is the vertical slowness such that \( \mathbf{P} = x_p + zq_s \), and \( x_i \) is a point on the interface. Now \( \mathbf{P} = (1/\alpha)\mathbf{t}_p \), where \( \mathbf{t}_p \) is the unit vector normal to the plane wavefront. Therefore,

\[
\phi_s(x_i) = \frac{i}{2\pi \rho \alpha_1^2} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(\mathbf{t}_p \cdot (x_1 - x_s))\right]}{(2q_2)}.
\]

In order to calculate the reflected field at point \( x_s \), we apply Snell’s law at that point. For each plane wave appearing in the above equation we calculate the angle of incidence with respect to the normal \( \mathbf{n} \) at \( x_i \) and then apply Snell’s law to obtain the reflected field. Let \( \mathbf{t}_p \) be the direction of the reflected wavefield; then following Frazer & Sen (1985), we have

\[
\mathbf{t}_p = \mathbf{t}_p^r \cdot (1 - mn)
\]

where \( \mathbf{l} \) is the unit dyadic. Thus, the reflected field on the upper surface of \( \Sigma \) will be given as

\[
\phi_r(x_i) = \frac{i}{2\pi \rho \alpha_1} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(\mathbf{t}_p^r \cdot (x_1 - x_s))\right]}{(2q_2)},
\]

where \( R_{pp} \) is the plane wave reflection coefficient (for potentials) computed for the local slowness at the interface. Also, since the medium is homogeneous above the interface, \( \alpha_1 = \alpha_2 \). The incident wavefield at \( x_i \) due to the receiver can be written as

\[
\phi_r(x_i) = \frac{i}{2\pi \rho \alpha_1} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(\mathbf{t}_p \cdot (x_1 - x_s))\right]}{(2q_2)}.
\]

Therefore, using (32), (35), and (36), we have

\[
\phi_r(x_i) = \frac{-i\omega}{(2\pi \rho \alpha_1^2)(2\pi \rho \alpha_1^2)} \int dp_x \int dx_1 \int dp_y \frac{R}{(2q_2)(2q_2)} \times \exp\left[i\omega \frac{R_{pp}}{\alpha_1} (\mathbf{t}_p^r \cdot (x_1 - x_s) + \mathbf{t}_p \cdot (x_1 - x_s))\right].
\]

where

\[
R = R_{pp}(n \cdot t_p - n \cdot t_p^r)/\alpha_1.
\]

Equation (37) gives the reflected wavefield at the receiver point \( x_i \) in terms of a three fold integral. Replacing the integrals over \( p_y \) and \( p_r \), with their stationary phase values, yields the CSPI result. Note that we used equation (20) to evaluate the source wavefield. Therefore, the resulting equation (37) represents the PP (displacement potential) measured at the receiver point due to an explosion source. The integrals for \( P-S, S-P \) and \( S-S \) reflections can be derived in the same manner and need not be given here. The expression for calculating the pressure response due to an explosion source in an acoustic medium will be very similar to equation (37). If, however, we need to calculate the PP reflection at the receiver point due to a unidirectional point force at \( x_s \), we need to substitute (31) for \( \phi_s \) in (32) and follow the same procedure outlined above. In order to derive the equation for the transmitted field at the receiver point in a situation where \( x_i \) and \( x_s \) are separated by a material discontinuity, we simply replace the reflection coefficient term in (37) with its associated transmission coefficient term (Frazer & Sen 1985).

### 6 Multilayered Medium

We now consider a medium consisting of three layers as shown in Fig. 4. We wish to calculate the reflected wavefield at the receiver at point \( x_r \) for the generalized ray path shown by the dashed line in the figure. Different parameters and symbols used in the following derivation are shown in Fig. 4. The derivation is carried out in the following steps. First we write the incident wave field at \( x_i^- \) (the upper part of the interface \( S_i \)) as

\[
\phi_s(x_i^-) = \frac{i}{2\pi \rho \alpha_1^2} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(\mathbf{t}_p \cdot (x_1 - x_s))\right]}{(2q_2)}.
\]

Next, to calculate the field at \( x_i^+ \) (the lower side of \( S_i \)), we transmit each plane wave given by the above equation using Snell’s law and plane wave transmission coefficients. We define (Frazer & Sen 1985):

\[
l_i = 1 - mn, \quad \sigma = \frac{1}{\alpha_1}, \quad \mathbf{t}_p^r = \alpha_2 \sigma \mathbf{n} + \sqrt{(1 - \sigma^2)} \mathbf{n}.
\]

Thus, we have

\[
\phi_r(x_i) = \frac{i}{2\pi \rho \alpha_1^2} \int dp_x \frac{\exp\left[i(\omega/\alpha_1)(\mathbf{t}_p^r \cdot (x_1 - x_s))\right]}{(2q_2)}.
\]

where \( PP_{12} \) is the downward looking transmission coefficient for the potential. Next we use a KH surface integral over the surface \( S_i \) and an integral over ray-parameter \( \rho \) (Fig. 4) to compute the field at \( x_s \). The result is

\[
\phi_s(x_i) = \frac{(i)^{\frac{i\omega}{2\pi \rho \alpha_1^2}}}{(2\pi \rho \alpha_1^2)(2\pi \rho \alpha_2^2)} \times \int dp_x \int dx_1 \int dp_y T_{12} \times \exp\left[i\omega \frac{R_{pp}}{\alpha_1} (\mathbf{t}_p^r \cdot (x_1 - x_s) + \mathbf{t}_p \cdot (x_1 - x_s))\right].
\]

Figure 4. PSPI for a multilayered medium.
where

\[ T_{12} = -PP_{21} \left( \frac{n_1 \cdot t_2^p}{\alpha_2} + \frac{n_1 \cdot t_1^p}{\alpha_1} \right) \]  \tag{42}

In the next step, the reflected field at \( x_1^+ \) is given by

\[ \phi_0(x_1^+) = \frac{i^2 (\omega)^2}{(2\pi \rho_1 \alpha_1^2)(2\pi \rho_2 \alpha_2^2)} \times \left[ \int dx_1 \int dx_2 \int dp_1 \int dp_2 \, \frac{T_{12}R_{23}}{(2q_1)(2q_2)} \times \exp \{ i\omega [P_1 \cdot (x_1 - x_2) + P_2 \cdot (x_2 - x_1)] \} \right] \]  \tag{43}

where

\[ R_{23} = PP_{23} \left( \frac{n_2 \cdot t_1^p}{\alpha_2} - \frac{n_2 \cdot t_2^p}{\alpha_1} \right) \]  \tag{44}

and

\[ t_2^p = t_1^p \cdot (1 - 2n_2n_2). \]  \tag{45}

Continuing in this manner, the field at the receiver is given by

\[ \phi_0(x) = \frac{(i\omega)^2}{(2\pi \rho_1 \alpha_1^2)(2\pi \rho_2 \alpha_2^2)} \times \left[ \int dx_1 \int dx_2 \int dp_1 \int dp_2 \, \frac{T_{12}R_{23}T_{21}}{(2q_1)(2q_2)(2q_4)} \times \exp \{ i\omega [P_1 \cdot (x_1 - x_2) \right.

\[ + P_1 \cdot (x_2 - x_1) + P_2 \cdot (x_2 - x_1) + P_2 \cdot (x_1 - x_2)] \} \]  \tag{46}

where

\[ T_{21} = -(PP)_{21} \left( \frac{n_2 \cdot t_1^p}{\alpha_2} + \frac{n_2 \cdot t_2^p}{\alpha_1} \right). \]  \tag{47}

Thus, the response of the generalized ray shown in Fig. 4 is given by a sevenfold integral as in equation (46) above. The response of any other generalized ray with many branches and including converted phases can easily be written out simply by inspection. Such integrals will be referred to as \textit{multifold phase space path integrals} (PSPI). These integrals resemble the Feynman phase space path integrals used in quantum mechanics. In this formulation, the contribution of all possible paths in the phase space are summed to evaluate the response of a single generalized ray. As explained in Fig. 4, plane waves from the source and receiver are coupled at each point on the interface by means of a scalar KH integral. We note here that if a generalized ray touches \( n \) points, then the PSPI for that ray has fold \((2n + 1)\), whereas the CSP for that ray has fold \( n \). Note that the power of a full PSPI is seldom needed and that a reduced fold CSP/PSPI can be derived directly, without the use of the stationary phase method.

7 SYNTHETIC SEISMOGRAMS

It is likely that PSPI's can be computed more rapidly in the time domain than in the frequency domain. However, in the frequency domain, attenuation in the medium can be included properly by making the velocities frequency dependent and complex. We discuss here only the frequency-domain calculation of the single interface \( P-P \) reflection problem considered earlier.

The PSPI synthetic seismograms are calculated in two steps. The first step involves calculation of wavenumber (ray parameter) integrals and the second step involves the computation of surface integrals. Since the ray-parameter integrals are oscillatory, they can be evaluated rapidly by using a generalized Filon method (GFM) quadrature formula (Frazer & Gettrust 1984). Frequency-dependent sampling intervals are chosen to avoid spatial aliasing and the ray-parameter integrals are tapered at endpoints using Hanning windows to avoid truncation phases. The ray parameter \( p \) is converted to local slowness at the interface for the evaluation of the generalized reflection and transmission coefficients. Complex angles of incidence are allowed in order to include the evanescent regime. For example, referring to Fig. 5, we note that for a ray with ray parameter \( \rho \), we have

\[ p = \frac{\sin(\theta)}{\alpha}, \quad q = \frac{\cos(\theta)}{\alpha}, \]

\[ p_1 = \frac{\sin(\theta')}{\alpha}, \quad q_1 = \frac{\cos(\theta')}{\alpha}, \]

and

\[ \theta = \theta - \psi. \]

The above relations hold even in the evanescent regime and by substitution for \( \theta_1 \) in the expression for \( p_1 \) and \( q_1 \) we obtain

\[ p_1 = p_\alpha \cos(\psi_1) - q_\alpha \sin(\psi_1), \]

and

\[ q_1 = q_\alpha \cos(\psi_1) + p_\alpha \sin(\psi_1). \]

These equations can be used to calculate the local slowness at each point along the interface. Having computed the slowness integrals, the surface integrals can be carried out simply by the method explained in Sen & Frazer (1985).

Our first numerical example is a two-layered acoustic medium consisting of an acoustic player (\( P \)-wave velocity = \( 1.0 \text{ km s}^{-1} \), density = 1.0, thickness = \( 2.0 \text{ km} \) over an acoustic half-space (\( P \)-wave velocity = \( 2.0 \text{ km s}^{-1} \), density = 1.0). We calculate the primary reflection (pressure response

\[ \text{Figure 5. Conversion of ray-parameter values to the local slowness at the interface.} \]
Phase space path integrals

Next, we consider a two-layered acoustic medium as shown in Fig. 7(a). In this example, we calculate the primary reflection (pressure response) from a truncated reflector. The source-receiver geometry is shown in Fig. 7(a). The reflected ray path shows a shadow zone beyond a range of 4.0 km ($p_s = 0.707$ km$^{-1}$). A fairly common mistake is to calculate the response for this model as a single $p$-integral, truncated at the ray parameter corresponding to the reflection from the source to the corner; such a calculation is shown in Fig. 7(b). We also evaluate this using a twofold PSPI (Fig. 7c) and a fourth-order acoustic finite-difference algorithm (Fig. 7d). The free surface was treated as an absorbing boundary for the finite difference calculation and the line source synthetics were converted to approximate point-source solutions (Vidale & Helmberger 1987). The PSPI models the diffractions and reflections very well. The match of amplitude, phase and traveltime is excellent between the PSPI and finite difference seismograms. As expected, the synthetics shown in Fig. 7(b) computed by a single ray-parameter integral are in error. For example, we see a truncation phase arriving with a phase velocity of

from an explosion source) from the interface, recorded at the surface receivers. This can be easily evaluated by a single ray-parameter integral (Aki & Richards 1980). Evaluating this by means of a PSPI will require using a threefold integral similar to the one given by equation (37). Our numerical experiments (Sen & Frazer 1988) showed that we do not need to use a full threefold integral and that a twofold PSPI gives identical results. We, therefore, used a twofold integral obtained by replacing the receiver ray-parameter integral ($p_r$) by its equivalent geometrical acoustics Green's function. In Fig. 6(a), we show the synthetic seismograms computed by using a single $p$-integral for this model. Similarly, Fig. 6(b) shows the profile computed by using a twofold PSPI. Fig. 6(c) shows an expanded plot of single-$p$ and PSPI seismograms in the range 3.6-5.6 km for better comparison. The match between these two sets of synthetics is excellent. We notice that the post-critical reflection (both amplitude and phase) and head waves are modelled very well by PSPI. This confirms the accuracy of the PSPI formulation for the 1-D model.

Figure 6. Synthetic seismograms for a layer over half-space model computed (a) by a single ray-parameter integral, (b) by twofold PSPI (see text for details) and (c) an expanded plot of single-$p$ and PSPI seismograms in the range 3.6-5.6 km for comparison. The head wave is marked by a dashed line.
8 DISCUSSION AND CONCLUSIONS

In this paper we have demonstrated the usefulness of the PSPI method over other ray-based methods. The tangent plane approximation used in the PSPI is more accurate than the usual tangent plane approximation, since the wavefronts are actually planar, although the interfaces are still assumed to be locally plane. The use of ray-parameter integrals in the KH integrand enables us to include classical head waves, although the head waves from curved interfaces are only approximate and the whispering gallery head waves will not be included. The angles of incidence are allowed to be complex and thus the evanescent regime is also included. Although the full PSPI formulation requires evaluation of a series of integrals, a careful examination of the Snell rays often enables one to reduce the fold by replacing many of the integrals with this stationary phase point values.

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