An imperfect spinless bose system is studied with use of the mixed state representation, which was originally developed to study highly excited nuclear states. With this representation, the degrees of freedom are classified into two types: One describes the thermal behaviour of the non-condensed boson and the other mainly describes the dynamical aspects of the condensed boson. The entropy of the system is specified by only the former type of degrees of freedom. Comparison is made with the conventional results for a dilute bose gas with an interaction of the hard sphere type.

§1. Introduction

Bose-Einstein condensation has been studied for the purpose of obtaining a microscopic description of superfluidity of $^4$He and superconductivity. Recently interest in it has been renewed through studies of dilute bose systems.\(^1\) The classical description of Bose-Einstein condensation is based on the idea that the great majority of bosons are condensed into the lowest state (i.e. the state $p = 0$) at low temperature.\(^2\)\(^-\)\(^5\) Further, by following this idea bosons condensed into the zero-momentum state $p = 0$ are considered to give no contribution to the entropy. In an imperfect bose system, however, there occurs a fragmentation of the condensed bosons due to the interaction between bosons; that is, considerable fractions of the condensed bosons in the state with $p = 0$ are excited into states with $p \neq 0$.

In related works, we have developed an approach called the mixed state representation, for the purpose of describing highly excited nuclear states.\(^6\),\(^7\) This approach has been successfully applied to the description of the thermal equilibrium state of the Lipkin model.\(^8\),\(^9\) We have shown that this approach of the mixed state representation provides a mean field approximation to the grand partition function. In this approach, two types of degrees of freedom are introduced. One describes the thermal aspects of the fermion system and, in the thermal equilibrium state, is represented by the distribution of the Fermi type. The other type mainly describes the dynamical aspects of the fermion system. The most conspicuous difference between these two types of degrees of freedom is that the entropy of a fermion system is specified by only the former one.
In this paper, we apply the approach of the mixed state representation to an imperfect spinless boson system and classify the degrees of freedom of the boson system into two types, one representing the degrees of freedom describing the thermal aspects of the non-condensed boson and the other mainly describing the dynamical aspects of the condensed bosons. Further, we show the following: The entropy is specified by only the former type of degrees of freedom. Therefore, even when there occurs fragmentation of the condensed bosons due to the interaction between bosons, the condensed bosons give no contribution to the entropy, except for the indirect effect on the entropy through the change in the boson excitation energy. At zero temperature, all the degrees of freedom are squeezed into the latter type, and thus the entropy vanishes.

In §2 we develop a mixed state representation of an imperfect spinless boson system. In this development, the formulation is very restricted in order to make clear the basic idea of the present approach and to make a comparison between the present result and the conventional results for a dilute boson system with an interaction of the hard sphere type. Through the definition of the mixed state, we classify the degrees of freedom into two types in the classical version. One is associated with a conserved quantity and can be freely specified as an initial condition. We take advantage of this fact to introduce the thermal aspects of the system, independently of the dynamics of the system. The other describes the dynamical aspects of the system. In §3 we define the density operator and the entropy corresponding to only the first type of degrees of freedom. With the use of this definition, we define the thermal equilibrium state of the imperfect boson system through the variational principle. Considering that the mixed state representation gives rise to the mean field approximation to the grand partition function of the fermion system, we perform this variation with respect to the free energy of grand canonical ensemble. In §4 we study the solution of equations derived with the variational principle. Section 5 is devoted to making a comparison between the present result and the conventional results for a dilute boson system with an interaction of the hard sphere type. We show that, under the approximation adopted in the conventional approach, our result appears to be similar to the conventional results. However, the difference between the present result and the conventional ones becomes conspicuous at many points. Concluding remarks are given in §6.

§2. A spinless boson system and the mixed state representation

The Hamiltonian with a two-body interaction among spinless bosons is given by

\[ H = \sum_p \epsilon_p \hat{a}_p^\dagger \hat{a}_p + \frac{1}{2} \sum_{p_1, p_2, p_3, p_4} V(p_1 - p_3) \delta(p_1 + p_2, p_3 + p_4) \hat{a}_p^\dagger \hat{a}_{p_2} \hat{a}_p \hat{a}_{p_3}, \]

(2.1a)

\[ \epsilon_p = \frac{1}{2m} p^2, \]

(2.1b)

where \( \hat{a}_p \) and \( \hat{a}_p^\dagger \) denote, respectively, boson creation and boson annihilation operators in the state specified by momentum \( p \), and \( m \) is the mass of a boson. In
what follows, we specify the state with an integer \( i \), corresponding to the periodic boundary conditions, i.e.,

\[
p_i = \frac{2\pi \hbar}{V^{1/3}} n_i,
\]

(2.2)

where \( n_i \) is a vector whose components are integers and \( V \) is the volume of the system under consideration. When the volume can be regarded to be sufficiently large, the discrete set of states will be transformed into a continuous one as follows:

\[
i \to p,
\]

(2.3a)

\[
\epsilon_i \to \epsilon p,
\]

(2.3b)

\[
\sum_i \to \frac{V}{(2\pi \hbar)^3} \int d^3 p.
\]

(2.3c)

With the discrete label, the state \( i = 0 \) corresponds to the state \( p = 0 \), and the state specified by \( \tilde{i} \) represents the time-reversed state of \( i \), i.e., the state \(-p_i\).

2.1. Mixed state

Here we define the mixed state for the present bose system in the following two steps: As the first step, we define the vacuum of a mixed state,

\[
|0\rangle = V|0_a\rangle|0_b\rangle,
\]

(2.4a)

\[
V = \exp \left[ \sum \theta_i \hat{a}_i^\dagger \hat{b}_i^\dagger - \theta_i^* \hat{b}_i \hat{a}_i \right],
\]

(2.4b)

where \( \hat{b}_i^\dagger \) denotes the creation operator of the auxiliary boson field in the time-reversed state of \( i \), and \( |0_a\rangle \) and \( |0_b\rangle \) represent, respectively, the vacuum of \( \hat{a}_i^\dagger \) and \( \hat{b}_i^\dagger \), i.e., \( \hat{a}_i|0_a\rangle = \hat{b}_i|0_b\rangle = 0 \). Then, the expectation values of the bilinear operators for the vacuum of mixed state are given as

\[
\langle \langle 0|\hat{a}_i^\dagger \hat{a}_j|0\rangle \rangle = \langle \langle 0|\hat{b}_i^\dagger \hat{b}_j|0\rangle \rangle = \delta_{ij} \sinh^2 |\theta_i|,
\]

\[
\langle \langle 0|\hat{a}_i^\dagger \hat{a}_j^\dagger|0\rangle \rangle = \langle \langle 0|\hat{a}_j \hat{a}_i|0\rangle \rangle = 0, \quad \langle \langle 0|\hat{b}_i^\dagger \hat{b}_j^\dagger|0\rangle \rangle = \langle \langle 0|\hat{b}_j \hat{b}_i|0\rangle \rangle = 0.
\]

(2.5)

The expectation value \( \sinh^2 |\theta_i| \) represents the occupation number of bosons in the state \( i \) for the vacuum \( |0\rangle \).

Then the mixed state can be introduced by

\[
|m\rangle = U|0\rangle,
\]

(2.6a)

\[
U = \exp \left[ \sum \frac{1}{2} \delta_{i,0} \left( \Gamma_{i} \hat{T}_{i+} - \Gamma_{i}^{*} \hat{T}_{i-} \right) \right], \quad \left( \Gamma_{\tilde{i}} = \Gamma_{i} \right)
\]

(2.6b)

where we have used the generators of the \( su(1,1) \) algebra defined by

\[
\hat{T}_{i+} = \frac{1}{1 + \delta_{i,0}} \hat{a}_i^\dagger \hat{a}_i^\dagger, \quad \hat{T}_{i-} = \frac{1}{1 + \delta_{i,0}} \hat{a}_i \hat{a}_i, \quad \hat{T}_{i0} = \frac{1}{2(1 + \delta_{i,0})} \left( \hat{a}_i^\dagger \hat{a}_i + \hat{a}_i \hat{a}_i^\dagger + 1 \right).
\]

(2.7)
which satisfy the following commutation relations:
\[
[\hat{T}_{i+}, \hat{T}_{i-}] = -2\hat{T}_{i0}, \quad [\hat{T}_{i0}, \hat{T}_{i\pm}] = \pm \hat{T}_{i\pm}.
\] (2.8)

The expectation values for the mixed state are given by
\[
\langle m | \hat{a}^*_i \hat{a}_j | m \rangle = \delta_{i,j} \left\{ \sinh^2 |\theta_i| \cosh^2 |\Gamma_i| + (\sinh^2 |\theta_i| + 1) \sinh^2 |\Gamma_i| \right\},
\]
\[
\langle m | \hat{a}^*_i \hat{a}^*_j \hat{a}_j \hat{a}_i | m \rangle = \delta_{i,j} \frac{\Gamma_i^*}{2|\Gamma_i|} \sinh 2|\Gamma_i| (\sinh^2 |\theta_i| + \sinh^2 |\theta_i| + 1),
\]
\[
\langle m | \hat{\gamma}_i \hat{a}_i | m \rangle = \delta_{i,j} \frac{\Gamma_i}{2|\Gamma_i|} \sinh 2|\Gamma_i| (\sinh^2 |\theta_i| + \sinh^2 |\theta_i| + 1),
\] (2.9a)
\[
\langle m | \hat{a}^*_i \hat{a}^*_j \hat{a}_j \hat{a}_i | m \rangle = \langle m | \hat{a}^*_i \hat{a}^*_j \hat{a}_j \hat{a}_i | m \rangle \langle m | \hat{a}^*_k \hat{a}_k | m \rangle + \langle m | \hat{a}^*_i \hat{a}_i | m \rangle \langle m | \hat{a}^*_j \hat{a}_j | m \rangle + \langle m | \hat{a}^*_i \hat{a}_i | m \rangle \langle m | \hat{a}^*_j \hat{a}_j | m \rangle.
\] (2.9b)

2.2. Canonical variables

Here we introduce a new set of variables \{\(n_i, \phi_i\), \(N_i, \Phi_i\)\}, which are called canonical variables and defined by the relations
\[
n_i = \sinh^2 |\theta_i|, \quad e^{i\phi_i/\hbar} = \frac{\theta_i}{|\theta_i|},
\] (2.10)
\[
N_i = \frac{n_i + n^{\gamma} + 1}{1 + \delta_{i,0}} \sinh^2 |\Gamma_i|, \quad e^{i\Phi_i/\hbar} = \frac{\Gamma^*_i}{|\Gamma_i|},
\] (2.11a)
\[
N^{\gamma}_i = N_i, \quad \Phi_i = \Phi_i.
\] (2.11b)

These variables satisfy the canonicity conditions\(^{10}\)
\[
\langle m | i\hbar \partial_{\phi_i} | m \rangle = n_i, \quad \langle m | -i\hbar \partial_{n_i} | m \rangle = 0,
\] (2.12a)
\[
\langle m | i\hbar \partial_{N_i} | m \rangle = N_i, \quad \langle m | -i\hbar \partial_{\Phi_i} | m \rangle = 0.
\] (2.12b)

The derivation of the above variables is given in Appendix A.

In terms of the above variables, the expectation values (2.9) can be expressed as
\[
\langle m | \hat{a}^*_i \hat{a}_j | m \rangle = \delta_{i,j} \left\{ n_i + (1 + \delta_{i,0}) N_i \right\},
\]
\[
\langle m | \hat{a}^*_i \hat{a}^*_j \hat{a}_j \hat{a}_i | m \rangle = \delta_{i,j} (1 + \delta_{i,0}) e^{i\phi_i/\hbar} \sqrt{N_i \left( \frac{n_i + n^{\gamma} + 1}{1 + \delta_{i,0}} + N_i \right)},
\]
\[
\langle m | \hat{\gamma}_i \hat{a}_i | m \rangle = \delta_{i,j} (1 + \delta_{i,0}) e^{-i\phi_i/\hbar} \sqrt{N_i \left( \frac{n_i + n^{\gamma} + 1}{1 + \delta_{i,0}} + N_i \right)},
\] (2.13a)
\[
\langle m | \hat{a}^*_i \hat{a}^*_j \hat{a}_j \hat{a}_i | m \rangle = \delta_{i,j} \delta_{j,k} (1 + \delta_{i,0}) (1 + \delta_{k,0}) e^{i(\phi_i - \Phi_k)/\hbar}
\times \left[ N_i N_k \left( \frac{n_i + n^{\gamma} + 1}{1 + \delta_{i,0}} + N_i \right) \left( \frac{n_k + n^{\gamma} + 1}{1 + \delta_{k,0}} + N_k \right) + \delta_{i,k} \delta_{j,i} \{ n_j + (1 + \delta_{i,0}) N_i \} \{ n_j + (1 + \delta_{j,0}) N_j \} + \delta_{i,j} \delta_{j,k} \{ n_i + (1 + \delta_{i,0}) N_i \} \{ n_j + (1 + \delta_{j,0}) N_j \}. \right.
\] (2.13b)
Then, the expectation value of the generators of the \(su(1,1)\) algebra can be expressed in the following form:

\[
T_{i+} = \langle \langle m | \hat{T}_{i+} | m \rangle \rangle = e^{i\phi_i/\hbar} \sqrt{N_i \left( \frac{n_i + n_{\tilde{i}} + 1}{1 + \delta_{i,0}} + N_i \right)},
\]

\[
T_{i-} = \langle \langle m | \hat{T}_{i-} | m \rangle \rangle = e^{-i\phi_i/\hbar} \sqrt{N_i \left( \frac{n_i + n_{\tilde{i}} + 1}{1 + \delta_{i,0}} + N_i \right)},
\]

\[
T_{i0} = \langle \langle m | \hat{T}_{i0} | m \rangle \rangle = \frac{1}{2} \left( \frac{n_i + n_{\tilde{i}} + 1}{1 + \delta_{i,0}} + 2N_i \right).
\] (2.14)

The expectation values of the Hamiltonian and the number operator can be expressed as

\[
H = \langle \langle m | \hat{H} | m \rangle \rangle = \sum_i \epsilon_i N_i + \frac{1}{2} V(0) N^2 + \frac{1}{2} \sum_{i,j} V(i - j) N_i N_j
\]

\[
+ \frac{1}{2} \sum_{i,j} V(i - j) (1 + \delta_{i,0}) (1 + \delta_{j,0}) e^{i(\Phi_i - \Phi_j)/\hbar}
\]

\[
\times \sqrt{N_i N_j \left( \frac{n_i + n_{\tilde{i}} + 1}{1 + \delta_{i,0}} + N_i \right) \left( \frac{n_j + n_{\tilde{j}} + 1}{1 + \delta_{j,0}} + N_j \right)},
\] (2.15)

\[
N = \langle \langle m | \sum_i \hat{N}_i | m \rangle \rangle = \sum_i N_i,
\] (2.16a)

\[
N_i = \langle \langle m | \hat{a}^*_i \hat{a}_i | m \rangle \rangle = n_i + (1 + \delta_{i,0}) N_i.
\] (2.16b)

The dependence of the mixed state on time is determined by the variational principle

\[
\delta \int L dt = 0,
\] (2.17)

where we have used the definition

\[
L = \langle \langle m | i\hbar \frac{d}{dt} - \hat{H} | m \rangle \rangle
\]

\[
= \sum_{i \geq 0} \hat{N}_i \frac{d\Phi_i}{dt} + \sum_i n_i \frac{d\phi_i}{dt} - H.
\] (2.18)

Then, we have the following equations of motion:

\[
\frac{d\Phi_i}{dt} = \frac{\partial H}{\partial N_i}, \quad \frac{dN_i}{dt} = -\frac{\partial H}{\partial \Phi_i},
\] (2.19a)

\[
\frac{d\phi_i}{dt} = \frac{\partial H}{\partial n_i}, \quad \frac{dn_i}{dt} = -\frac{\partial H}{\partial \phi_i}.
\] (2.19b)
With the use of the Poisson brackets defined by
\[
\{A, B\} = \sum_i (\partial_{\phi_i} A \cdot \partial_{n_i} B - \partial_{n_i} A \cdot \partial_{\phi_i} B) + \sum_{i \geq 0} (\partial_{\Phi_i} A \cdot \partial_{N_i} B - \partial_{N_i} A \cdot \partial_{\Phi_i} B),
\]
the above equations of motion can be expressed in the form
\[
\begin{align*}
\frac{dN_i}{dt} &= \{N_i, H\}, & \frac{d\Phi_i}{dt} &= \{\Phi_i, H\}, \\
\frac{dn_i}{dt} &= \{n_i, H\}, & \frac{d\phi_i}{dt} &= \{\phi_i, H\}.
\end{align*}
\]
(2.21a)
(2.21b)

It is easily ascertained that the quantities \(N\) and \(n_i\) are conserved, i.e.,
\[
\frac{dN}{dt} = 0, \quad \frac{dn_i}{dt} = 0.
\]
(2.22)

The first equation represents conservation of the total boson number. The physical meaning of \(n_i\) can be recognized from the following discussion: The \(su(1,1)\) algebra can be reproduced in the classical form
\[
\hbar \{T_{i,0}, T_{i,\pm}\} = \pm T_{i,\pm}.
\]
(2.23)

Hence, the classical image of the Casimir operator of the \(su(1,1)\) algebra can be given by
\[
T_i^2 = T_{i,0}^2 - T_{i,\pm} = \frac{1}{4} \left( \frac{n_i + n_{\tilde{i}} + 1}{\delta_{i,0}} \right)^2.
\]
(2.24)

The invariance of \(n_i\) guarantees the invariance of \(T_i\). From the conservation of these quantities, we have another conserved quantity,
\[
\frac{dN}{dt} = 0; \quad N = \sum_i (1 + \delta_{i,0})N_i.
\]
(2.25)

\section{3. Thermal equilibrium state}

In the previous section, we showed that the quantities \(n_i\), representing the occupation number of bosons in the vacuum of the mixed state, are conserved. We can rather arbitrarily specify these quantities through initial conditions. Hence we can describe the dynamical behaviour of the present system within the space defined by the set of quantities \(n_i\). In this section, we specify these quantities with relation to a static state of thermal equilibrium.

\subsection{3.1. Density and entropy operators}

We define a thermal equilibrium state in the following way. The first step is to define the density operator
\[
\hat{\rho} = \prod_i A_i \exp[B_i \hat{a}_i^* \hat{a}_i],
\]
(3.1a)
\[ A_i = \frac{1}{\cosh^2 |\theta_i|}, \quad B_i = \ln \tanh^2 |\theta_i|. \quad (3.1b) \]

In the above definition, we have used the notation
\[ \tilde{a}_i^* = U \hat{a}_i^* U^\dagger, \quad \hat{a}_i = U \hat{a}_i U^\dagger. \quad (3.2) \]

The basis of states is provided by
\[ \left\{ \prod_i \frac{1}{\sqrt{n_i!}} (\tilde{a}_i^*)^{n_i} |\tilde{0}\rangle \right\}, \quad (3.3a) \]
\[ |\tilde{0}\rangle = U |0\rangle. \quad (3.3b) \]

Then these states satisfy the relation
\[ \tilde{\rho} \prod_i \frac{1}{\sqrt{n_i!}} (\tilde{a}_i^*)^{n_i} |\tilde{0}\rangle = \prod_i A_i e^{n_i B_i} \frac{1}{\sqrt{n_i!}} (\hat{a}_i^*)^{n_i} |\tilde{0}\rangle, \quad (3.4) \]
and the following property of the density operator can be easily ascertained:
\[ \text{Tr} [\tilde{\rho}] = 1. \quad (3.5) \]

With the aid of the density operator, we can give the expectation value of an operator \( \hat{O} \) by
\[ \langle \hat{O} \rangle = \text{Tr} [\tilde{\rho} \hat{O}]. \quad (3.6) \]

Then we have the relation
\[ \langle \hat{O} \rangle = \langle \langle m | \hat{O} |m \rangle \rangle, \quad (3.7) \]
which can be easily ascertained with the use of (3.4) and the relations
\[ \hat{a}_i^* = \tilde{a}_i^* \cosh |\Gamma_i| + \tilde{a}_i^* \frac{\Gamma_i}{|\Gamma_i|} \sinh |\Gamma_i|, \]
\[ \hat{a}_i = \tilde{a}_i^* \frac{\Gamma_i}{|\Gamma_i|} \sinh |\Gamma_i| + \tilde{a}_i \cosh |\Gamma_i|. \quad (3.8) \]

Therefore we have shown that the mixed state defined in the previous section corresponds to a thermal state described by the density operator \( \tilde{\rho} \). The expectation value (3.6) can be easily evaluated with the use of the mixed state.

As the second step, we introduce the entropy operator by
\[ \hat{\eta} = -k \ln \tilde{\rho}, \quad (3.9) \]
where \( k \) is the Boltzmann constant. Then the entropy is defined as the expectation value of the entropy operator:
\[ S = -k \langle \ln \tilde{\rho} \rangle = k \sum_i \left\{ (n_i + 1) \ln (n_i + 1) - n_i \ln n_i \right\}, \quad (3.10) \]
where the quantities \( n_i \) are the occupation numbers in the vacuum of the mixed state \( |0\rangle \). It is noteworthy that the entropy \( S \) is specified by only \( n_i \) and does not depend on \( N_i \) or \( \Phi_i \). Hence we can say that the variables \( n_i \) and \( \phi_i \) mainly describe the thermal behaviour of the system and the variables \( N_i \) and \( \Phi_i \) describe the dynamical aspects.
3.2. Static state of thermal equilibrium

As the final step, we define the Helmholtz free energy. Considering that the mixed state representation has provided a mean field approximation to the grand partition function of the fermion system, we introduce the free energy of the grand canonical ensemble to specify the quantities $n_i$ and $(N_i, \Phi_i)$ as follows:

\begin{equation}
\mathcal{F}(T, V, \mu) = E - TS - \mu N,
\end{equation}

\begin{equation}
E = H = \langle \langle m | \hat{H} | m \rangle \rangle,
\end{equation}

\begin{equation}
N = \sum_i N_i,
\end{equation}

where $T$ and $\mu$ denote the temperature and chemical potential, respectively. Then the static state of thermal equilibrium can be specified with the aid of the variational principle:

\begin{equation}
\delta \mathcal{F}(T, V, \mu) = 0.
\end{equation}

The above variation should be performed with respect to the variables $(n_i, \phi_i)$ and $(N_i, \Phi_i)$.

From this variation, we have formally two types of equations:

\begin{equation}
\frac{\partial \mathcal{F}}{\partial n_i} = -kT \ln \frac{n_i + 1}{n_i} + \epsilon_i - \mu + V(0)N + \sum_j V(i - j)N_j
+ \frac{1}{2} \left( e^{\phi_i/h} \Delta_i + e^{-\phi_i/h} \Delta_i^* \right) \frac{N_i}{\sqrt{N_i (N_i + 2T_i)}},
\end{equation}

\begin{equation}
\frac{\partial \mathcal{F}}{\partial \phi_i} = 0,
\end{equation}

\begin{equation}
\frac{\partial \mathcal{F}}{\partial N_i} = \epsilon_i + \epsilon_i^* - 2\mu + 2V(0)N + 2 \sum_j V(i - j)N_j
+ \left( e^{\phi_i/h} \Delta_i + e^{-\phi_i/h} \Delta_i^* \right) \frac{N_i + T_i}{\sqrt{N_i (N_i + 2T_i)}},
\end{equation}

\begin{equation}
\frac{\partial \mathcal{F}}{\partial \Phi_i} = \frac{i}{\hbar} \left( e^{\phi_i/h} \Delta_i - e^{-\phi_i/h} \Delta_i^* \right) \frac{N_i (N_i + 2T_i)}{\sqrt{N_i (N_i + 2T_i)}}.
\end{equation}

Here we have used the definitions

\begin{equation}
\Delta_i = \sum_j V(i - j)(1 + \delta_{j,0}) e^{-\phi_j/h} \sqrt{N_j (N_j + 2T_j)},
\end{equation}

\begin{equation}
T_i = \frac{n_i + n_i^* + 1}{2(1 + \delta_{i,0})},
\end{equation}

and made the assumption

\begin{equation}
V(i) = V(|i|).
\end{equation}

In the succeeding sections, we investigate solutions of the above equations.
§4. Imperfect bose gas with repulsive interaction

In this section, we consider a bose gas with repulsive interaction and investigate the solution of Eqs. (3·13) and (3·14). In order to obtain a basic understanding of the physical ingredients involved in the variables $N_i$, we consider the case of an ideal bose gas, that is, the case of $V(i) = 0$. Next, we consider the case of a repulsive interaction.

4.1. Ideal bose gas

In this case, the Helmholtz free energy is written as

$$\mathcal{F} = E - TS - \mu N,$$

$$E = \sum_i \epsilon_i N_i,$$

$$S = k \sum_i \{ (n_i + 1) \ln(n_i + 1) - n_i \ln n_i \},$$

$$N_i = n_i + (1 + \delta_{i0}) N_i.$$  \hspace{1cm} (4.1d)

Since the internal energy $E$ and the entropy $S$ are increasing functions of the variables $n_i$, the free energy $\mathcal{F}$ takes its extremum at the variables $n_i$ satisfying (3·13), i.e.,

$$\frac{\partial \mathcal{F}}{\partial n_i} = \epsilon_i - \mu - kT \ln \frac{n_i + 1}{n_i} = 0.$$

Then we have the following expression for $n_i$:

$$n_i = \frac{1}{\exp \left[ \frac{1}{kT} (\epsilon_i - \mu) \right] - 1},$$  \hspace{1cm} (4.2)

where the chemical potential $\mu$ should be negative. On the other hand, the free energy $\mathcal{F}$ is monotonically increasing with respect to the variables $N_i$. Hence, they should take as small values as possible to give the minimum value of $\mathcal{F}$, that is

$$N_i = 0 \quad \text{for} \quad i \neq 0.$$  \hspace{1cm} (4.3)

The value of $N_0$ remains arbitrary, as long as we do not fix the total boson number $N$ on average. When $N$ is fixed, there occur two cases. In one case, we have the relation

$$N = n = \sum_i n_i,$$

for a proper negative $\mu$, and all the variables $N_i$ vanish; that is,

$$n_i = \exp \left[ \frac{(\epsilon_i - \mu)}{kT} \right] - 1,$$

$$N_i = 0 \quad \text{for all} \quad i,$$  \hspace{1cm} (4.5b)
where $\mu$ should be determined by Eq. (4.4) with fixed $N$. The other is the case in which Eq. (4.4) cannot hold for any negative $\mu$. Then $N_0$ cannot vanish; that is, we have

$$n_i = \frac{1}{\exp[(\epsilon_i - 0_+) / kT] - 1},$$

$$N_0 = \frac{1}{2} \left( N - \sum_i n_i \right), \quad N_i = 0 \quad \text{for} \quad i \neq 0. \quad (4.6a)$$

In the limit $V \to \infty$, (4.5) can be expressed as follows:

$$n_p = \frac{1}{\exp \left( \frac{1}{2 \pi m \mathbf{p}^2 - \mu} / kT \right) - 1},$$

$$N_p = 0 \quad \text{for all} \quad p, \quad (4.7a)$$

and the quantity $\mu$ should be determined through the relation

$$N = \frac{V}{(2\pi \hbar)^3} \int n_p d^3 p.$$  

$$N_\mathbf{p} = 0 \quad \text{for all} \quad \mathbf{p}, \quad (4.7b)$$

The expression (4.7) is the conventional one for an ideal bose gas in a non-condensed phase, and (4.8) corresponds to that in the condensed phase. The quantity $N_0$ denotes the number of bosons condensed into the state $\mathbf{p} = 0$.

The entropy can be expressed in the same form for both phases:

$$S = \frac{V}{(2\pi \hbar)^3} \int d^3 p \left\{ \ln (n_p + 1) - n_p \ln n_p \right\}. \quad (4.9)$$

Therefore the bosons condensed into the state $\mathbf{p} = 0$ give no contribution to the entropy.

Since the free energy $\mathcal{F}$ does not depend on the variables $\phi_i$ and $\Phi_i$, these variables are completely arbitrary, even for the condensed phase.

4.2. **Bose gas with repulsive interaction**

In this case, the free energy $\mathcal{F}$ is given by

$$\mathcal{F} = E - TS - \mu N,$$

$$E = \sum_i \epsilon_i N_i + \frac{1}{2} V(0) N^2 + \frac{1}{2} \sum_{i,j} V(i - j) N_i N_j$$

$$+ \frac{1}{2} \sum_{i,j} V(i - j)(1 + \delta_{i,0})(1 + \delta_{j,0}) e^{(\phi_i - \Phi_j) / \hbar}$$

$$\times \sqrt{N_i (N_i + 2T_i)} N_j (N_j + 2T_j). \quad (4.10a)$$

$$N_0 = \frac{1}{2} \left( N - \sum_i n_i \right), \quad N_i = 0 \quad \text{for} \quad i \neq 0. \quad (4.6b)$$

In addition, Eq. (4.6) can be written as

$$n_p = \frac{1}{\exp \left( \frac{1}{2 \pi m \mathbf{p}^2 - \mu} / kT \right) - 1},$$

$$N_\mathbf{p} = 0 \quad \text{for} \quad \mathbf{p} \neq 0. \quad (4.8a)$$

$$N_\mathbf{p} = 0 \quad \text{for all} \quad \mathbf{p}, \quad (4.8b)$$

The expression (4.7) is the conventional one for an ideal bose gas in a non-condensed phase, and (4.8) corresponds to that in the condensed phase. The quantity $N_0$ denotes the number of bosons condensed into the state $\mathbf{p} = 0$.

The entropy can be expressed in the same form for both phases:

$$S = \frac{V}{(2\pi \hbar)^3} \int d^3 p \left\{ \ln (n_p + 1) - n_p \ln n_p \right\}. \quad (4.9)$$

Therefore the bosons condensed into the state $\mathbf{p} = 0$ give no contribution to the entropy.

Since the free energy $\mathcal{F}$ does not depend on the variables $\phi_i$ and $\Phi_i$, these variables are completely arbitrary, even for the condensed phase.
Imperfect Bose System and Its Mixed State Representation

As a typical case, we first consider that in which
\[ N_i = 0 \quad \text{for all} \quad i. \quad (4.11) \]

Then \( \mathcal{F} \) takes its extremum for the variables \( n_i \) determined by the equation
\[
\frac{\partial \mathcal{F}}{\partial n_i} = \epsilon_i - \mu + V(0)N + \sum_j V(i - j)n_j - kT \ln \frac{n_i + 1}{n_i} = 0, \quad (4.12)
\]
which gives rise to the following expression for \( n_i \):
\[
n_i = \frac{1}{\exp \left[ \frac{\epsilon_i - \mu + V(0)N + \sum_j V(i - j)n_j}{kT} \right] - 1}. \quad (4.13)
\]

Here the chemical potential \( \mu \) must satisfy the relation
\[
\mu < \epsilon_i + V(0)N + \sum_j V(i - j)n_j \quad \text{for all} \quad i. \quad (4.14)
\]

This solution describes a bose gas in the non-condensed phase.

When \( N \) is fixed to a certain value on average, the chemical potential should be determined by the equation
\[
N = n = \sum_i n_i = \sum_i \frac{1}{\exp \left[ \frac{\epsilon_i - \mu + V(0)N + \sum_j V(i - j)n_j}{kT} \right] - 1}. \quad (4.15)
\]

As the temperature becomes lower, it eventually becomes impossible to satisfy Eq. (4.15). Then, the overswelled bosons are condensed in the state \( p = 0 \), and \( N_0 \) does not vanish. Further, this situation gives rise to a competition between the increase of the kinetic energy and the decrease of the correlation energy depending on the relative phase
\[
e^{i(\Phi_i - \Phi_0)/\hbar}
\]
and thus to non-vanishing \( N_i (i \neq 0) \). The extremum of the free energy \( \mathcal{F} \) with respect to the variables \( n_i, \Phi_i \) and \( \{N_i; i \neq 0\} \) is determined by
\[
\frac{\partial \mathcal{F}}{\partial n_i} = \epsilon_i - \mu + V(0)N + \sum_j V(i - j)n_j
\]
\[
+ \frac{1}{2} \left( e^{i\Phi_i/h} \Delta_i + e^{-i\Phi_i/h} \Delta_i^* \right) \frac{N_i}{\sqrt{N_i} (N_i + 2T_i)}
\]
\[
- kT \ln \frac{n_i + 1}{n_i} = 0, \quad (4.16)
\]
\[
\frac{\partial F}{\partial \Phi_i} = \frac{i}{\hbar} \left( e^{i\Phi_i/\hbar} \Delta_i - e^{-i\Phi_i/\hbar} \Delta_i^* \right) \sqrt{N_i (N_i + 2T_i)} \]
\[
= 0, \quad (4.17a)
\]
\[
\frac{\partial F}{\partial N_i} = \epsilon_i + \epsilon_i^* - 2\mu + 2V(0)N + 2 \sum_j V(i-j)N_j 
+ \left( e^{i\Phi_i/\hbar} \Delta_i + e^{-i\Phi_i/\hbar} \Delta_i^* \right) \frac{N_i + T_i}{\sqrt{N_i (N_i + 2T_i)}} 
= 0 \quad \text{for } i \neq 0. \quad (4.17b)
\]

From Eq. (4.17a), we have the relation
\[
\Phi_i - \Psi_i = l_i \pi \hbar; \quad l_i = \text{integer}, \quad (4.18)
\]
where we have used the quantity \(\Psi_i\), defined by
\[
e^{i\Psi_i/\hbar} = \frac{\Delta_i^*}{|\Delta_i|}. \quad (4.19)
\]

In order to make the correlation energy as small as possible, we can set
\[
\Phi_0 = 0, \quad \Phi_i - \Phi_0 = \pi \hbar, \quad (i \neq 0) \quad (4.20)
\]
without loss of generality. With this choice of phase angles \(\Phi_i\), we have
\[
\Delta_i^* = e^{i\Phi_0/\hbar} \sum_j V(i-j)(1+\delta_{j,0})e^{i(\Phi_j-\Phi_0)/\hbar} \sqrt{N_j(N_j + 2T_j)} 
= 2V(i) \sqrt{N_0(N_0 + 2T_0)} - \sum_{j \neq 0} V(i-j) \sqrt{N_j(N_j + 2T_j)},
\]
\[
= \Delta_i^{(0)} - \Delta_i^{(ex)}, \quad (4.21a)
\]
\[
e^{-i\Phi_i/\hbar} \Delta_i^* = (-1)^{1+\delta_{i,0}} \left( \Delta_i^{(0)} - \Delta_i^{(ex)} \right). \quad (4.21b)
\]

From Eq. (4.17b), we have the following expression for \(i \neq 0\):
\[
N_i = \left\{ \frac{1}{2} \left( \epsilon_i + \epsilon_i^* \right) - \mu + V(0)N + \sum_j V(i-j)N_j \right\} T_i, \quad (4.22a)
\]
\[
E_i = \sqrt{\left\{ \frac{1}{2} \left( \epsilon_i + \epsilon_i^* \right) - \mu + V(0)N + \sum_j V(i-j)N_j \right\}^2 - |\Delta_i|^2}. \quad (4.22b)
\]

With the aid of this result, we have the following expression for \(n_i\) from Eq. (4.16):
\[
n_i = \frac{1}{\exp \left[ E_i/kT \right] - 1} \quad \text{for } i \neq 0, \quad (4.23a)
\]
\[
n_0 = \frac{1}{\exp \left[ \left( V(0)N + \sum_j V(j)N_j + \Delta_0 \frac{N_0}{\sqrt{N_0(N_0 + 2T_0)}} - \mu \right)/kT \right] - 1}. \quad (4.23b)
\]
In the derivation of (4.23a), we have used the relation
\[ \epsilon_i^\sim = \epsilon_i. \] (4.24)

In the above expressions \( N \) stands for the summation
\[ N = \sum_i \{ n_i + (1 + \delta_{i,0})N_i \}. \] (4.25)

For either phase, the entropy is given by
\[ S = k \sum_i \{ (n_i + 1) \ln (n_i + 1) - n_i \ln n_i \}. \] (4.26)

As the temperature becomes lower, the degrees of freedom are transposed from \((n_i, \phi_i)\) to \((N_i, \Phi_i)\), which do not contribute to the entropy. (That is, the latter describe the dynamical aspects of the boson system.) In the condensed phase, the phase given by \( \Phi_0 \) is incoherent with respect to the other phases specified by \( \Phi_i \) \((i \neq 0)\).

§5. Comparison with the approach of Bogoliubov

We now study the behaviour of the solution obtained in the previous section in more detail. For this purpose, we compare the present approach with that of Bogoliubov. Here we adopt a short-range hard-sphere-type repulsive interaction.

5.1. Short range approximation

First we adopt the following approximation:
\[ V(i) \rightarrow V(0) : \text{[Approximation I]} \]

Thus we have
\[ \sum_j V(i - j)N_j \rightarrow V(0) \sum_j N_j = V(0)N, \]
\[ \Delta_i \rightarrow 2V(0)\sqrt{N_0(N_0 + 2T_0)} - V(0) \sum_{j \neq 0} \sqrt{N_j(N_j + 2T_j)} \]
\[ = \Delta^{(0)} - \Delta^{(ex)} \]
\[ = \Delta. \] (5.1)

Then the solution for the condensed phase obtained in the preceding section can be rewritten as follows:
\[ n_i = \frac{1}{\exp \left( \frac{E_i}{kT} \right) - 1}, \quad N_i = T_i \left\{ \frac{\epsilon_i - \mu + 2V(0)N_i}{E_i} - 1 \right\}, \] (5.2a)
\[ E_i = \sqrt{\{\epsilon_i - \mu + 2V(0)N_i\}^2 - \Delta^2}; \quad \text{for } i \neq 0, \] (5.2b)
\[ n_0 = \frac{1}{\exp \left( \frac{2V(0)N + \Delta \sqrt{N_0(N_0 + 2T_0)} - \mu}{kT} \right) - 1}. \]

The quantity \( \Delta^{(\text{ex})} \) can be determined by solving the equation

\[ \Delta^{(\text{ex})} = V(0) \sum_{i \neq 0} \left( \frac{1}{2} + n_i \right) \frac{\Delta^{(0)} - \Delta^{(\text{ex})}}{E_i} \]

for given values of \( T, V, \mu \) and \( N_0 \). In the above expression, therefore, the independent variables are \( T, V, \mu \) and \( N_0 \). When the average value of \( N \) is fixed to a certain value, we can determine \( N_0 \) with the relation

\[ N_0 = \frac{1}{2} \left( N - \sum_i n_i - \sum_{i \neq 0} N_i \right). \]

The chemical potential \( \mu \) still remains undetermined, even for fixed \( N \). Therefore we need another principle to determine it. With this in mind, we demand that the entropy takes its maximum for fixed \( T, V \) and \( N_0 \). Then, because of the inequality

\[ \frac{\partial S}{\partial \mu}_{T,V,N_0} > 0, \]

the chemical potential in the condensed phase should take its possible maximum value,

\[ \mu_0 = 2V(0)N - \Delta. \]

For this value of the chemical potential, \( n(= \sum_i n_i) \) takes its maximum, and further, in the limit of large volume but finite density \( N/V \), \( n_0/N \) can be regarded as negligible. For the time being, however, we treat \( T, V, \mu \) and \( N_0 \) as independent variables.

We have the following expressions of the internal energy and the free energy for the thermal equilibrium state obtained above:

\[ \mathcal{E} = E - \mu N \]
\[ = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 \]
\[ + \Delta^{(0)} \sqrt{N_0(N_0 + 2T_0)} - \frac{1}{2} \Delta^{(\text{ex})} \sum_{i \neq 0} \sqrt{N_i(N_i + 2T_i)} \]
\[ + \frac{1}{2} \sum_{i \neq 0} \{E_i - \epsilon_i + \mu - 2V(0)N\} \]
\[ + \delta \sum_{i \neq 0} \delta_i n_i + \frac{1}{2} \sum_{i \neq 0} \left( E_i - \epsilon_i + \mu - 2V(0)N \right) \]

\[ \mathcal{F} = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 \]
\[ + \Delta^{(0)} \sqrt{N_0(N_0 + 2T_0)} - \frac{1}{2} \Delta^{(\text{ex})} \sum_{i \neq 0} \sqrt{N_i(N_i + 2T_i)} \]
\[ + \frac{1}{2} \sum_{i \neq 0} \{E_i - \epsilon_i + \mu - 2V(0)N\} \]
\[ + kT \sum_{i \neq 0} \ln \left[ 1 - \exp \left\{ - \frac{1}{kT} E_i \right\} \right] \]

\[ - \frac{n_0 N_0}{\sqrt{N_0(N_0 + 2T_0)}} + kT \ln \left[ 1 + \frac{1}{n_0} \right]. \tag{5.7} \]

5.2. Dilute gas approximation

Further, for a dilute bose gas, the assumption is made that

\[ \Delta \rightarrow \Delta^{(0)} : \quad \text{[Approximation II]} \]

Then we have

\[ n_i = \frac{1}{\exp \left[ \frac{E_i}{kT} \right] - 1}, \quad N_i = T_i \left\{ \frac{\epsilon_i - \mu + 2V(0)N}{E_i} - 1 \right\}, \tag{5.8a} \]

\[ E_i = \sqrt{\{\epsilon_i - \mu + 2V(0)N\}^2 - 4V(0)^2N_0(N_0 + 2T_0)}, \tag{5.8b} \]

\[ n_0 = \frac{1}{\exp \left[ \{2V(0)N + 2V(0)N_0 - \mu\} / kT \right] - 1}. \tag{5.8c} \]

Then the free energy and the internal energy are reduced to the following forms:

\[ \mathcal{E} = E - \mu N \]

\[ = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 + \Delta^{(0)} \sqrt{N_0(N_0 + 2T_0)} \]

\[ + \frac{1}{2} \sum_{i \neq 0} \{E_i - \epsilon_i + \mu - 2V(0)N\} + \sum_{i \neq 0} E_i n_i \]

\[ + (2V(0)N - \mu)n_0, \tag{5.9} \]

\[ \mathcal{F} = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 + \Delta^{(0)} \sqrt{N_0(N_0 + 2T_0)} \]

\[ + \frac{1}{2} \sum_{i \neq 0} \{E_i - \epsilon_i + \mu - 2V(0)N\} \]

\[ + kT \sum_{i \neq 0} \ln \left[ 1 - \exp \left\{ - \frac{1}{kT} E_i \right\} \right] \]

\[ - \frac{n_0 N_0}{\sqrt{N_0(N_0 + 2T_0)}} + kT \ln \left[ 1 + \frac{1}{n_0} \right]. \tag{5.10} \]

At sufficiently low temperature, where the values \( n_i \) become negligibly small, and therefore \( T_i \) can be approximated as \( 1/2(1 + \delta_{i,0}) \), the above expressions become similar to those obtained by Bogoliubov under the approximation

\[ N_0 \approx \frac{1}{2} N, \quad \text{i. e.,} \quad \Delta^{(0)} \approx 2V(0)N_0 \approx V(0)N. \quad \text{[Approximation III]} \]

At zero temperature, we have the following expression for \( \mathcal{E} \) under the approximations I, II and III:

\[ \mathcal{E}(T = 0) = \frac{1}{2} V(0)N^2 + \frac{1}{2} \sum_{i \neq 0} (E_i - \epsilon_i - V(0)N), \tag{5.11a} \]
In the above expressions, we have adopted the following form of $\mu$:

$$
\mu(T=0) = 2V(0)N - \Delta^{(0)} \longrightarrow V(0)N.
$$

The above result is equivalent to the result of the Bogoliubov approach at zero temperature.\(^4,11\) For the expression of $E$, we have a difference due to the Fock effect in the mean field approximation. Therefore our approach can be regarded as an extension of the Bogoliubov approach to finite temperature. The quantity $E_i$ corresponds to the quasi-boson energy at finite temperature. In the succeeding discussion, we do not adopt the approximation III, but rather determine $N_0$ self-consistently from the set of Eqs. (5·8).

In the limit of large $V$, with fixed $N/V$, the present result can be rewritten as follows under the approximations I and II:

Non-condensed phase:

$$
n_p = \frac{1}{\exp [(\epsilon_p + 2V(0)N - \mu)/kT] - 1}, \quad \mathcal{N}_p = 0,
$$

$$
N = \frac{V}{(2\pi\hbar)^3} \int n_p d^3p,
$$

$$
\mathcal{E} = -V(0)N^2 + \frac{V}{(2\pi\hbar)^3} \int (\epsilon_p + 2V(0)N - \mu)n_p d^3p,
$$

$$
\mathcal{F} = -V(0)N^2 + \frac{kT}{(2\pi\hbar)^3} \int \ln \left[1 - \exp \left\{1 - \frac{\epsilon_p + 2V(0)N - \mu}{kT}\right\}\right] d^3p.
$$

Condensed phase:

$$
n_p = \frac{1}{\exp [E_p/kT] - 1},
$$

$$
\mathcal{N}_p = \left(\frac{1}{2} + n_p\right) \left(\frac{\epsilon_p + 2V(0)N - \mu}{E_p} - 1\right) \quad \text{for} \quad p \neq 0,
$$

$$
E_p = \sqrt{(\epsilon_p + 2V(0)N - \mu)^2 - (2V(0)N_0)^2},
$$

$$
\mathcal{E} = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 + 2V(0)N_0^2 + \frac{V}{2(2\pi\hbar)^3} \int (E_p - \epsilon_p - 2V(0)N + \mu)d^3p
$$

$$
+ \frac{V}{(2\pi\hbar)^3} \int E_p n_p d^3p,
$$

$$
\mathcal{F} = -V(0)N^2 + 2\{2V(0)N - \mu\}N_0 + 2V(0)N_0^2 + \frac{1}{2} \frac{V}{(2\pi\hbar)^3} \int (E_p - \epsilon_p - 2V(0)N + \mu)d^3p
$$

$$
+ \frac{kT}{(2\pi\hbar)^3} \int \ln \left[1 - \exp \left(-\frac{1}{kT}E_p\right)\right] d^3p.
$$
where \( N \) is defined by
\[
N = 2N_0 + \frac{V}{(2\pi \hbar)^3} \int (n_p + N_p) d^3p. \tag{5.13f}
\]

In Eqs. (5.13), the independent variables are given by the set of variables
\[
\{T, V, \mu, N_0\}. \tag{5.13g}
\]

§6. Concluding remarks

We have applied the mixed state representation to an imperfect bose system in order to describe Bose-Einstein condensation. The thermal equilibrium state was defined by a variation of the Helmholtz free energy of the grand canonical ensemble. Then, the free energy can be expressed as a function of the variables \( T, V, \mu \) and \( N_0 \) for a boson system with an interaction of the hard sphere type.

Within the variational principle, the value of \( N_0 \) cannot be specified. Hence we have adopted the following prescription: First we specified the chemical potential \( \mu \) such that the normal component of the boson number becomes as large as possible, following the procedure adopted in the case of an ideal bose gas. Then, the independent variable \( N_0 \) was specified in terms of overswelled bosons with a fixed \( N \) on average, using Eq. (5.13f). The chemical potential specified in the above way is reduced to that obtained using the Thomas-Fermi approximation at zero temperature. The chemical potential specified in this way corresponds to that obtained with the first-order Popov approximation.\(^{12}\)

As a result of classifying the degrees of freedom into two types, the present approach specifies the thermal component of the boson density in the condensed phase in the form
\[
n_{\text{TH}} = n = \sum_{i \neq 0} \frac{1}{\exp[E_i/kT] - 1},
\]

where the term \( n_0 \) in the summation is dropped because it becomes negligible in the limit of large \( V \). On the other hand, we have the following expression of the boson number in excited states, found using the conventional grand partition function within the mean field approximation:\(^{13}\)
\[
n_{\text{ex}} = \sum_{i \neq 0} \left( \frac{|u_i|^2 + |v_i|^2}{\exp[E_i/kT] - 1} \right). \tag{6.2}
\]

Here \( u_i \) and \( v_i \) are so-called coefficients of the Bogoliubov transformation given by
\[
\begin{align*}
    u_i^2 &= \frac{1}{2} \left( \frac{\epsilon_i - \mu + 2V(0)N}{E_i} + 1 \right), \\
    v_i^2 &= \frac{1}{2} \left( \frac{\epsilon_i - \mu + 2V(0)N}{E_i} - 1 \right).
\end{align*} \tag{6.3}
\]

The above expression can be rewritten in the form
\[
n_{\text{ex}} = \sum_{i \neq 0} \left\{ \frac{1}{2} \left( \frac{\epsilon_i - \mu + 2V(0)N}{E_i} - 1 \right) + \frac{\epsilon_i - \mu + 2V(0)N}{E_i} \frac{1}{\exp[E_i/kT] - 1} \right\}. \]
\[ \sum_{i \neq 0} \left\{ \frac{\epsilon_i - \mu + 2V(0)N}{E_i} \left( \frac{1}{2} + \frac{1}{\exp[E_i/kT] - 1} \right) \right\} + \frac{1}{\exp[E_i/kT] - 1} \]

\[ \sum_{i \neq 0} N_i + \sum_{i \neq 0} n_i. \]  

This form coincides with that obtained in the present approach. In the present approach, however, the second term on the RHS of the last equality is regarded as the thermal component and the first term as the excited fraction of the condensed bosons due to the interaction between bosons. The second term on the RHS of Eq. (6·2) is called the “thermal depletion” \(^{14}\) or the “thermal density”. \(^{15}\) By stipulating that only the degrees of freedom related with \(n_{\text{TH}}\) contribute to the entropy, it is important to make a clear distinction between the thermal and the condensed component. This point will be discussed in detail in a succeeding paper.

In a succeeding paper, we make clear the relation to the conventional approach based on the grand partition function with the mean field approximation and discuss the behaviour of the present result in more detail using numerical analysis.

For simplicity, we have developed the present formulation for a homogeneous system of imperfect spinless boson. An extension of the present formulation that enables us to describe a trapped system is also a future project.

**Appendix A**

**Canonicity Condition and Canonical Variables**

In this appendix we derive a set of canonical variables satisfying the canonicity condition

\[ \langle \langle m | i \hbar \partial_Q | m \rangle \rangle = P, \quad \langle \langle m | -i \hbar \partial_P | m \rangle \rangle = 0. \]  

(A·1)

First we note the relation

\[ V^\dagger U^\dagger \partial_x U V = V^\dagger \partial_x V + V^\dagger (U^\dagger \partial_x U) V. \]  

(A·2)

The expectation value of the first term on the RHS can be written as

\[ \langle 0_b | \langle 0_a | V^\dagger \partial_x V | 0_a \rangle | 0_b \rangle \]

\[ = \int_0^1 \langle 0_b | \langle 0_a | \exp \left[ -\lambda \sum_i (\theta_i \hat{a}_i^\dagger \hat{b}_i^\dagger - \theta_i^* \hat{b}_i \hat{\tilde{a}}_i) \right] \sum_i \left( \theta_{i,x} \hat{a}_i^\dagger \hat{b}_i^\dagger - \theta_i^* \hat{b}_i \hat{\tilde{a}}_i \right) \right] | 0_a \rangle | 0_b \rangle d\lambda \]

\[ = \int_0^1 \sum_i (\theta_i \hat{a}_i^\dagger \hat{b}_i^\dagger - \theta_i^* \hat{b}_i \hat{\tilde{a}}_i) \frac{1}{|\theta_i|} \sinh \lambda |\theta_i| \cosh \lambda |\theta_i| |d\lambda| \]

\[ = \frac{1}{2} \sum_i (\theta_i \hat{a}_i^\dagger \hat{b}_i^\dagger - \theta_i^* \hat{b}_i \hat{\tilde{a}}_i) \frac{1}{|\theta_i|^2} \sinh^2 |\theta_i| , \]

where we have used the definition

\[ \theta_{i,x} = \partial_x \theta_i, \quad \theta_{i,x}^* = \partial_x \theta_i^*. \]  

(A·3)
With the replacement
\[ n_i = \sinh^2 |\theta_i|, \quad e^{i\phi_i/\hbar} = \frac{\theta_i^*}{|\theta_i|}, \]
we have the expression
\[ \langle 0_b | V^1 i\hbar \partial_x V | 0_a \rangle | 0_b \rangle = \sum_i n_i \partial_x \phi_i. \]  
(A.5)

In the same way, we can evaluate the expectation value of \( U^1 \partial_x U \) for the vacuum of the mixed state \( | 0 \rangle \). First we evaluate \( U^1 \partial_x U \):

\[
U^1 \partial_x U = \int_0^1 \exp \left[ -\lambda \sum_{i \geq 0} (\Gamma_i \dot{\bar{T}}_{i+} - \Gamma_i^* \dot{\bar{T}}_{i-}) \right] \sum_{j \geq 0} (\Gamma_{j,x} \dot{\bar{T}}_{j+} - \Gamma_{j,x}^* \dot{\bar{T}}_{j-}) \exp \left[ \lambda \sum_{i \geq 0} (\Gamma_i \dot{\bar{T}}_{i+} - \Gamma_i^* \dot{\bar{T}}_{i-}) \right] d\lambda
\]

\[
= \sum_{i \geq 0} \int_0^1 \left[ \Gamma_{i,x} \left( \cosh^2 \lambda |\Gamma_i| \dot{\bar{T}}_{i+} + \left( \frac{\Gamma_i^*}{|\Gamma_i|} \sinh^2 \lambda |\Gamma_i| \right)^2 \dot{\bar{T}}_{i-} + \frac{\Gamma_i^*}{|\Gamma_i|} \sinh 2\lambda |\Gamma_i| \dot{\bar{T}}_{0} \right) \right]
\]

Then its expectation value is given by
\[
\langle \langle m | \partial_x | m \rangle \rangle = \sum_{i \geq 0} \frac{n_i + n_i + 1}{2(1 + \delta_{i,0})} \frac{\Gamma_i^*}{|\Gamma_i|^2} \sinh^2 |\Gamma_i|. \]  
(A.6)

With the replacement
\[ N_i = \frac{n_i + n_i + 1}{1 + \delta_{i,0}} \sinh^2 |\Gamma_i|, \quad e^{i\phi_i/\hbar} = \frac{\theta_i^*}{|\theta_i|}, \]
we have the relation
\[ \langle \langle 0 | U^1 i\hbar \partial_x U | 0 \rangle \rangle = \sum_{i \geq 0} N_i \partial_x \Phi_i. \]  
(A.8)

With the replacements
\[ n_i = \sinh^2 |\theta_i|, \quad e^{i\phi_i/\hbar} = \frac{\theta_i^*}{|\theta_i|}, \]
we finally have the relations
\[ \langle \langle m | i\hbar \partial_x \phi_i | m \rangle \rangle = n_i, \quad \langle \langle m | - i\hbar \partial_{\lambda_i} | m \rangle \rangle = 0, \]  
(A.10a)
\[ \langle \langle m | i\hbar \partial_{\Phi_i} | m \rangle \rangle = N_i, \quad \langle \langle m | - i\hbar \partial_{\lambda_i} | m \rangle \rangle = 0. \]  
(A.10b)
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References