Circumferential Stresses in Curved Beams

Robert Schmidt. The common formula for the circumferential normal stress \( \sigma_0 \) (author’s Eq. (1)) usually is not obtained by superposing bending stress and direct stress, as stated by the author, but is a direct consequence of Winkler’s hypothesis. This hypothesis that plane normal cross-sections of the curved beam before deformation remain plane, normal, and inextensional during deformation leads to the following expression for the extensional strain \( \epsilon_0 \) (Schmidt, 1979; Huddleston, 1968; Schmidt and DaDeppo, 1970):

\[
\epsilon_0 = \frac{1}{r} \left[ R\epsilon_c - (R-r)\beta' \right],
\]

where \( \epsilon_c \) and \( \beta' \) are the strain and angle of rotation, respectively, of an element \( Rd\theta \) of the centroidal curve, and \( \beta' = d\delta/d\theta \). With the Hooke law \( \sigma_0 = E\epsilon_0 \), the first integral in the author’s Eq. (3) yields the force

\[
N = \frac{EA(Re + \beta')}{R - e}.
\]

The bending moment \( M \) is given by

\[
M = \int (R-r)\sigma dA = \frac{EAR(\epsilon_c - \beta')}{R - e}.
\]

Solving the last two equations for \( \epsilon_c \) and \( \beta' \), we obtain

\[
\epsilon_c = \frac{RN - M}{EAR}, \quad \beta' = \frac{eN - M}{Ee},
\]

two structurally similar expressions. Substituting this result into our expression for \( \sigma_0 \) and employing Hooke’s law, \( \sigma_0 = E\epsilon_0 \), we obtain the author’s Eq. (1).

Now we may say that the author’s numerical examples make it clear that the Winkler hypothesis of rigid normal sections, although very dependable in the case of pure flexure, should be used with great caution in other cases of applied loads. Needless to say, we should appreciate the author’s revealing calculations.

That the author’s semi-empirically modified formulas have yielded more accurate results at some points is not surprising, since the modification was contrived with that intent. It also should be kept in mind that, in view of the Saint-Venant principle, Winkler’s theory was not intended for extremely thick curved beams.

In the case \( P > 0, M_b = -PR, a/b = 2, \) for example, at \( \theta = 90 \text{ deg} \) and \( b, s_0 = (a/P)a_0 = 25.766 \) as calculated by the plane-stress theory (Timoshenko and Goodier, 1970) for \( P > 0, M_b = 0; \) and \( s_0 = -23.265 = -1.5 \times 2 \times 7.755 \) from page 74 of Timoshenko and Goodier (1970) for \( P = 0, M_b = -PR; \) so that \( s_0 = s_b = 25.766 - 23.265 = 2.501, \) a small difference. Author’s formula (1) analogously yields 25.176 - 23.176 = 2.000, so that the ratio 2.00/2.501 = 1.071. The approximate value of the dimensionless stress at \( r = a, \theta = 90 \text{ deg} \) can seemingly be improved by employing a stress concentration factor of circa 1/4 found in most elementary textbooks on mechanics of materials in the case of straight rods; thus, 2 \times 1/4 = 25. The calculations in this paragraph demonstrate that small differences result when large stresses are superposed in the author’s example, and this fact is a partial explanation of the inaccuracy of the rationally derived Formula (1).

Additional References


Author’s Closure

The author would like only to add that all these assessments of accuracy are based on rectangular cross-sections because exact solutions are available only for plane-stress problems. The practical situation typically involves a nonrectangular cross-section and is fully three dimensional. In this case, the standard of comparison must be experimental results or finite elements results, and very few are available.

Quite recently I was able to compare curved-beam formulas with results from a detailed three-dimensional finite element analysis of a crane hook whose cross-section was the typical one, i.e., a slender trapezoid with rounded ends and larger part toward the center of curvature. The hook was loaded by a force directed through its center of curvature so as to create both bending and tension on the cross-section analyzed. At the inside, Eqs. (1) and (4) of the Note gave circumferential stress values that were, respectively, 14 and 8 percent lower than the finite element stress.

I suspect that most of the discrepancy comes from displacements not being independent of coordinate \( z \), as is assumed in the analysis of curved beams. This effect appeared in another finite element study (Cook, 1989) where in one test case of a