DISCUSSION

After some calculation, we obtain
\[ \int_0^{2\pi} \left[ \psi_1 \frac{\partial \psi_1}{\partial \theta} - \kappa \psi_1^2 \cos \theta \right] d\theta = \int_0^{2\pi} \int_0^{\pi} \left( R \left( \frac{\partial \psi_1}{\partial R} \right)^2 + \frac{1}{R} \left( \frac{\partial \psi_1}{\partial \theta} \right)^2 \right) dR d\theta \] (96)

Substituting the foregoing equation into equation (93), the following relation is obtained:
\[ \int_0^{2\pi} d\theta \left[ \frac{1}{c} (\psi_1)_{\theta=\theta_0} + \int_0^{\pi} \left( R \left( \frac{\partial \psi_1}{\partial R} \right)^2 \right. \right. \\
\left. \left. + \frac{1}{R} \left( \frac{\partial \psi_1}{\partial \theta} \right)^2 \right) dR \right] = 0 \] (97)

Similarly,
\[ \int_0^{2\pi} d\theta \left[ \frac{1}{c} (\psi_2)_{\theta=\theta_0} + \int_0^{\pi} \left( R \left( \frac{\partial \psi_2}{\partial R} \right)^2 \right. \right. \\
\left. \left. + \frac{1}{R} \left( \frac{\partial \psi_2}{\partial \theta} \right)^2 \right) dR \right] = 0 \] (98)

Since \( \psi_1 \) and \( \psi_2 \) are continuously differentiable with respect to \( R \) and \( \theta \) in \( \Omega \), the following relations should be held:
\[
\begin{align*}
\frac{1}{c} (\psi_n)_{\theta=\theta_0} &= 0, \quad (0 \leq \theta \leq 2\pi) \\
R \left( \frac{\partial \psi_n}{\partial R} \right) &= 0, \quad (0 \leq \theta \leq 2\pi, R \leq 1) \\
\frac{1}{R} \left( \frac{\partial \psi_n}{\partial \theta} \right) &= 0, \quad (0 \leq \theta \leq 2\pi, R \leq 1)
\end{align*}
\] (99)

where \( n = 1, 2 \). From the foregoing relations, we obtain
\[ \psi_\nu = 0 \quad \text{in} \quad \Omega, \quad \text{or} \quad c = \left( 1/c = 0 \right) \quad \text{and} \quad \psi_\nu = \text{const} \quad \text{in} \quad \Omega. \] (100)

Corollary 1

If there exists a limit \( \phi_\nu^* \) \((\nu = 1, 2)\) as mentioned before, then \( \phi_\nu^* \) is uniquely determined.

The Limiting Case, \( c = \infty \)

For the limiting case where \( c \) tends to infinity, the boundary condition (25) becomes as follows:
\[ \text{At} \quad R = 1 \]
\[ \frac{\partial}{\partial R} (\phi_1 - \phi_2) = -P \cos \theta / \kappa \] (102)

Theorem 2

The solution \( \phi_\nu \) \((\nu = 1, 2)\) of the equations (11) and (12) is uniquely determined except a constant under the boundary conditions (24) and (102).

Proof

Similarly as the proof of Theorem 1, \( \psi_\nu \) defined by equation (86) should satisfy the following relation:
\[ \int_0^{2\pi} d\theta \int_0^{\pi} \left( R \left( \frac{\partial \psi_\nu}{\partial R} \right)^2 \right) dR = 0 \] (103)

where \( \nu = 1, 2 \). Since \( \psi_\nu \) is continuously differentiable with respect to \( R \) and \( \theta \) in \( \Omega \), we obtain
\[
\left( \frac{\partial \psi_\nu}{\partial R} \right)^2 = 0 \quad \text{and} \quad \left( \frac{\partial \psi_\nu}{\partial \theta} \right)^2 = 0 \quad \text{in} \quad \Omega. \] (104)

From equation (104),
\[ \psi_\nu = \text{const} \quad \text{in} \quad \Omega. \] (105)

By using the first relation of equations (88),
\[ \psi_1 = -\psi_2 = A = \text{const} \] (106)

Namely, \( \phi_\nu \) is uniquely determined except a constant \( A \).

Now, let \( \phi_\nu = \kappa \phi_\nu \). Substituting \( \phi_\nu \) instead of \( \phi_\nu \), into equations (11), (12), (24), and (25), making \( c \) become infinite and keeping \( \kappa \) as a constant, we obtain the same equations for (11), (12), and (24), while we obtain
\[ \frac{\partial}{\partial R} \phi_\nu = -P \cos \theta \] (107)

for equation (25).

Corollary 2

For the limiting case when \( \kappa = 0 \) and \( \kappa \) is uniquely determined except a constant under degenerated boundary conditions.

The Limiting Case, \( c = 0, \kappa = \text{const} \)

Dividing both sides of equations (11) and (12) by \( \kappa \) and making \( \kappa \) become infinite, we then have
\[ \frac{\partial \phi_\nu}{\partial \chi} = 0 \] (108)

under appropriate condition.

On the other hand, the boundary conditions (24) and (25) become,
\[ \text{At} \quad R = 1 \quad \text{At} \quad R = 1 \]
\[ \phi_1 + \phi_2 = 0 \quad \text{or} \quad \phi_1 = 0 \]
\[ \phi_1 - \phi_2 = 0 \quad \text{or} \quad \phi_2 = 0 \] (109)

where \( \kappa \) tends to infinity while \( \kappa \) is constant. General solution of equation (108) is
\[ \phi_\nu = f_\nu(Y), \quad (\nu = 1, 2) \] (110)

where \( f_\nu(Y) \) is an arbitrary function of \( Y \) only. Substituting the result of equation (110) into equation (109), we obtain
\[ f_\nu (\sin \theta) = 0 \] (111)

for every value of \( \theta \) in \( 0 \leq \theta \leq 2\pi \). This conclusion implies that \( f_\nu(Y) \) is identically zero in \( \Omega \). From this fact, it is easily derived that there is no solution other than the solution such that
\[ \phi_\nu = 0 \quad \text{in} \quad \Omega. \] (112)

On the Stability of the Impact Damper

Authors' Closure

Dr. Sadek and Professor Mills should not have been disturbed by our assumption of two symmetric impacts per cycle since we have demonstrated the existence and stability of the corresponding solution over a wide range of system parameters. Furthermore, extensive experimental studies with a mechanical model and an analog computer, in addition to digital computer simulation of impact dampers (Ref. 13 of our paper) indicates that when the impact damper has periodic solutions, most of the time this motion has indeed two symmetric impacts per cycle.

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As for Professor Egle, his question regarding the procedure involved in determining the matrix $P$ indicates that he is not familiar with perturbation techniques. He is referred to any appropriate text on the subject.

His second comment deserves a more detailed answer since it offers an additional test of the validity of our stability analysis. Some of the experimental data (Ref. [6] of the Discussion) that, according to Professor Egle represent steady-state motion with two symmetric impacts per cycle, lie outside the stable region predicted by our stability analysis. This led Professor Egle to conclude that our stability analysis contradicts an "experimentally determined fact." In order to check these "facts," we simulated the impact damper on a digital computer, using the set of parameters used by Professor Egle. The experimental data that we obtained disagreed with what Professor Egle reports he observed in his experiments. We found the data points in question to be unstable (in the sense of two symmetric impacts per cycle); hence they do not contradict the predictions of our stability analysis. On the other hand, these data points are in a region where symmetric two impacts per cycle are stable, according to Professor Egle’s own stability analysis. More detailed discussion of Professor Egle’s stability analysis are presented in our discussion of his paper.

Our reply to the rest of Professor Egle’s comments is the following:

1. We did state in the introduction of our paper that “it is assumed that, during a period of the sinusoidal excitation, two impacts occur at equal time intervals and at opposite sides of the container.” We note the existence of other types of steady-state motion in Ref. [13] of our paper.

2. The limitation that $|\alpha| \leq d/2$ is implied and satisfied automatically by equation (11).

3. Equations (18) and (19) are convenient ways of expressing the relationships among the variables of the impact damper.

4. The stability analysis presented in our paper can easily be extended to cover any periodic motion of the impact damper; we considered only the symmetric impact case because it is by far the simplest to analyze.

In Table 1, considering $J/a^4$, Professor Kennedy’s 9-term values for rhombuses, as given by him (column 4), are compared with the sets of accurate upper and lower bounds given by Argyris and Kelsey (columns 2 and 3) and by Rao and Hussainy (columns 6 and 7). One is forced to conclude that there is some serious error in Kennedy’s values (column 4). It would appear that the values are inflated by a factor sec $\alpha$. On correcting them by a multiplying factor cos $\alpha$, one obtains the data in column 5. Comparison of this corrected data with the other information in the table shows that the Galerkin procedure followed by Professor Kennedy yields underestimates for the stiffness (as it should be) and that convergence is rather slow, becoming worse with increasing skew angles.

In this context, the following comments are relevant:

1. Argyris and Kelsey’s variational solutions are highly accurate (for torsional stiffness and maximum stress) as they have taken the sectional symmetry properly into account and because their assumption of linear variation of shear across the thickness is valid in most of the cross section (substantial variations occur only in small regions at the obtuse corners). They obtain bounds for $J$ by applying the principles of virtual displacements and virtual forces; see columns 2 and 3.

2. Rao and Hussainy use a polar series satisfying the differential equation exactly at all stages of truncation and also exactly satisfying the boundary conditions along the sides adjoining an acute angle. The only remaining condition, viz., the symmetry condition on the short diagonal, is satisfied to a high degree of accuracy by a simple procedure that insures the nullification of the first few terms of the Taylor expansion of the error on that diagonal. Thus a high degree of accuracy is achieved also in the final solution with only a few terms in the series. Certain additional constraints are placed on the errors in the symmetry condition so as to change the nature of the approximating sequence for the solution. The two sequences provide close bounds for the true solution for all skew angles. The convergence is particularly good for large angles of skew, i.e., for very thin sections; see columns 6 and 7.

This method can be applied with nearly equal ease to the analysis of parallelogram sections—one satisfies the skew symmetry condition on the short diagonal instead of the symmetry condition. The method is highly versatile; it can effectively deal with and has already been applied to a range of boundary-value problems with arbitrary rectilinear boundaries; for example, references [6, 7, 8].

3. Sathyanarayana analyzed the parallelogram shape by using harmonic Fourier series in oblique coordinates and applying Fourier analysis along the edges. His numerical data for rhombuses are included also in Table 1. The algebra and computations in this method are much too heavy. Further, the convergence is more inferior to that by the simpler methods used by Rao and Hussainy.

4. Professor Kennedy’s stress function

$$\phi = (1 - \eta^2)(1 - \xi^2) \sum_{n=0,1,2,...} a_n(\eta\xi)^n$$

is well chosen. It is consistent with the symmetry condition in the problem, viz.,

$$\phi(\eta, \xi) = \phi(-\eta, -\xi)$$

and the first term accounts for the dominant linear parts in shear-stress distributions along constant $\eta$ and $\xi$ lines.

Information in Tables 1 and 2 of the original Note and Table 3 of this discussion shows that in the Galerkin solution with this stress function convergence suffers with increasing angle of skew or increasing side ratio.

In Professor Kennedy’s Note, accuracy of the solution is assessed by differences between successive approximations. This could be misleading. Small differences between successive approximations do not necessarily imply accuracy of the result.