Relativistic Bound State Solutions of a Two-Dimensional
Fermion Model with Color SU(3) Symmetry

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(Received January 25, 2001)

We implement color SU(3) symmetry on a relativistic two-dimensional fermion model proposed by Glöckle, Nogami and Fukui (GNF). We find that it is possible to construct solutions for several multiplets of two and three particles by using the solutions for the GNF model. Though our model is based on the single-particle theory (i.e., the model does not contain any anti-particles), we regard a particle belonging to $\mathbf{3}$ as an anti-particle while the particles belong to $\mathbf{3}$ representation of the SU(3) group. There are conditions for the coupling constant that must be satisfied in order that a solution is acceptable as a physical bound state. We construct bound state solutions of $\mathbf{3}$ for a particle-anti-particle pair, $\mathbf{1}$ for a particle-anti-particle pair and $\mathbf{1}$ for a three-particle system. We find that the singlet states form the tightest bound states among these solutions. We believe that our result provides some useful information for understanding quark confinement.

§1. Introduction

Although hadron dynamics is now believed to be described fundamentally by quantum chromodynamics (QCD), it is still difficult to understand completely how a hadron is constructed from quarks and gluons. To reach such an understanding, we must solve the bound state problem, or “quark confinement”. We may understand such a problem more transparently by finding an exact solution of a model which simulates QCD, even if the system under consideration is rather idealistic.

Several years ago Glöckle, Nogami and Fukui (GNF)1) proposed a quantum mechanical model of two Dirac particles interacting with each other through a $\delta$-function potential in 1+1 dimensional spacetime, and they found exact bound state solutions. Using these solutions, they studied the relativistic effect on the electric form factor of the hadron. The Lorentz invariance of this model is guaranteed by the fact that a Poincaré algebra is formed by the Hamiltonian, total momentum and boost operator. This model is a quantum mechanical version of the Bukhvostov-Lipatov model in field theory.2) The extension of the GNF model to a system of more than two particles has been discussed,3) and the present author found an $n$-particle bound state solution.4), 5)

In this article we introduce an intrinsic SU(3) symmetry, which we call “color symmetry”, to the above mentioned model and find two- and three-particle bound state solutions. If we regard the particles as quarks, then these solutions appear to correspond to mesons and baryon, respectively. As is shown in the next section we can construct bound state solutions for several SU(3) multiplets by using the known solutions of the GNF model mentioned above, and we find that the solutions for

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the color singlet states satisfy the boundary condition and have the largest binding energy. However, this does not imply quark confinement, because a particle can be isolated from the bound state by a finite energy, and in addition to the singlet states, the $3$ state for a particle-particle pair can also form a bound state. We believe, however, that our result provides an interesting piece of information for understanding quark confinement.

§2. Model and solutions

First we review the GNF model and its bound state solutions. We consider the Breit-type equation

$$H\Psi = E\Psi, \quad (2.1)$$

where the Hamiltonian $H$ is given by

$$H = H_0 + H_I, \quad (2.2)$$

$$H_0 = \sum_i \left\{ -i\alpha_i \frac{\partial}{\partial x_i} + m\beta_i \right\}, \quad (2.3)$$

$$H_I = -\frac{g}{2} \sum_{i \neq j} (1 - \alpha_i \alpha_j) \delta(x_i - x_j). \quad (2.4)$$

The suffixes $i$ and $j$ here denote the fermion species and $\alpha_i = \sigma_x$ and $\beta_i = \sigma_z$ are the $2 \times 2$ Dirac matrices, which operate on the particle $i$, where $\sigma_x$ and $\sigma_z$ are the Pauli matrices. Because the Hamiltonian $H$ and the total momentum $P = -i \sum \frac{\partial}{\partial x_i}$ commute each other, we can separate the coordinates $X = \sum x_i/n$ of the center of mass and the relative coordinates $x_i - x_j$, and we rewrite the wave function as

$$\Psi_p = e^{ipX} \Phi_p(x_i - x_j), \quad (2.5)$$

where $p$ is the eigenvalue of the total momentum of the particles. In the following it is sufficient to consider the case $p = 0$. Then we rewrite (2.1) as

$$H\Phi_0(x_i - x_j) = E_0\Phi_0(x_i - x_j). \quad (2.6)$$

The bound state solution of the above equation is given by

$$\Phi_0 = \exp \left\{ -h(x_i - x_j) + \frac{ig}{4} \sum_{i=1}^n \alpha_i g_i \right\} (\uparrow \cdots \uparrow), \quad (2.7)$$

where function $g_i$ is defined by

$$g_i \equiv \epsilon(x_i - x_1) + \cdots + \epsilon(x_i - x_{i-1}) + \epsilon(x_i - x_{i+1}) + \cdots + \epsilon(x_i - x_n), \quad (2.8)$$

and

$$h(x_i - x_j) = m \sum_i x_i I_i^{\text{sin}} \left(x, \frac{g}{2} \right), \quad (2.9)$$
where $I_i^{\text{sin}}$ is defined by
\[ \exp\{i\theta_i g_i\} \equiv I_i^{\cos}(x; \theta) + i\alpha_i I_i^{\text{sin}}(x; \theta). \] (2.10)

The explicit forms of $I_i^{\text{sin}}$ and $I_i^{\cos}$ are given in Ref. 4). The abbreviated expression $(\uparrow \uparrow \cdots)$ denotes a direct product of eigenspinors of the matrices $\beta_i$ with eigenvalue $+1$. The energy eigenvalue $E_0$ (which is the mass of the bound state because the total momentum is set to zero) is given by
\[ E_0 = \frac{m \sin \frac{\pi g}{2}}{\sin \frac{g}{2}}. \] (2.11)

Below we give explicit expressions of the solutions for $n = 2$ and $3$.

1. $n = 2$
\[ \Phi_0 = \exp\left\{-m|x_1 - x_2| \sin \frac{g}{2} + \frac{i g}{4} \sum_{i=1}^{2} \alpha_i g_i\right\} (\uparrow \uparrow), \] (2.12)
\[ E_0 = \frac{m \sin g}{\sin \frac{g}{2}} = 2m \cos \frac{g}{2}. \] (2.13)

2. $n = 3$
\[ \Phi_0 = \exp\left\{-\frac{m}{2} \sum_{i=1}^{3} x_i g_i \sin g + \frac{i g}{4} \sum_{i=1}^{3} \alpha_i g_i\right\} (\uparrow \uparrow \uparrow), \] (2.14)
\[ E_0 = \frac{m \sin \frac{3g}{2}}{\sin \frac{g}{2}} = m \left(4 \cos^2 \frac{g}{2} - 1\right). \] (2.15)

As $h \to |x_i| \sin\{(n - 1)g/2\}$ for $|x_i| \to \infty$, it is necessary that
\[ \sin\left\{(n - 1)\frac{g}{2}\right\} > 0 \] (2.16)

or
\[ 0 < g < \frac{2\pi}{n - 1}, \] (2.17)

in order for the solution (2.7) to satisfy its boundary conditions. It is also necessary that
\[ 0 \leq E_0 = \frac{m \sin \frac{ng}{2}}{\sin \frac{g}{2}} \leq nm \] (2.18)

so that the solution may be acceptable as a physical bound state solution. Here, we can verify that $E_0 \leq nm$ is always satisfied.4) We assume that $g$ does not take a large value and do not take into account the peculiar periodic structure of the above solutions with respect to $g$, since such a structure originates from the pathological nature of the function $\epsilon(x_i - x_j)$ at $x_i = x_j$.5) 6)

Now, we implement color $SU(3)$ symmetry on the model above. First we suppose that each fermion belongs to the basic representation $3$ of the $SU(3)$ group. The interaction part of the Hamiltonian is then given by
\[ H_I = \frac{g}{2} \sum_{i \neq j}^{n} (1 - \alpha_i \alpha_j) \lambda_i \cdot \lambda_j \delta(x_i - x_j), \] (2.19)
where $\lambda_i = (\lambda_1, \lambda_2, \cdots, \lambda_8)_i$ denote the Gell-Mann $3 \times 3$ matrices of the adjoint representation of the $SU(3)$ algebra, which operate on the particle $i$. Let us write the wave function as

$$\Phi_{\alpha \beta \cdots} = q_\alpha^1 q_\beta^2 \cdots \Phi_0,$$  (2.20)

where $q_\alpha^i$ represents the $SU(3)$ spinor of the particle $i$ belonging to representation $3$. In the following we show that we can construct bound state solutions for multiplets of $n = 2$ and $3$ using the solutions (2.12) and (2.14). We see that we should require $g > 0$ so that the color singlet states can form bound states.

(a) case $n = 2$:

The interaction part of the Hamiltonian for $n = 2$ is written

$$H_I = g(1 - \alpha_1 \alpha_2) \lambda_1 \cdot \lambda_2 \delta(x_1 - x_2).$$  (2.21)

The tensor product of $3 \times 3$ is reduced to the irreducible representations $3 + 6$. The wave function of the $3$ multiplet is given by

$$\Phi_{(3)}^\alpha = \epsilon^{\alpha \beta \gamma} q_\beta^1 q_\gamma^2 \Phi_0.$$  (2.22)

We find that operation of $H_I$ in (2.21) on the wave function above gives

$$(H_I \Phi_{(3)})^\alpha = g \epsilon^{\alpha \beta \gamma}(1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) \sum_{i=1}^{8} (\lambda_i q_\beta^1)(\lambda_i q_\gamma^2) \Phi_0$$

$$= -\frac{8}{3} g \epsilon^{\alpha \beta \gamma}(1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) q_\beta^1 q_\gamma^2 \Phi_0$$

$$= -\frac{8}{3} g (1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) \Phi_{(3)}^\alpha.$$  (2.23)

This shows that we can obtain the solution for the $3$ state from (2.12) by substituting $\frac{8}{3} g$ for $g$. The boundary condition (2.16) is now written

$$\sin \frac{4}{3} g > 0,$$  (2.24)

and the mass of this bound state is given by

$$E_0 = m_{2-3} = 2m \cos \frac{4}{3} g > 0.$$  (2.25)

These are satisfied by a small positive $g$.

The wave function of the $6$ multiplet state is given by

$$\Phi_{(6)} = \{q_\alpha^1 q_\beta^2 + q_\beta^1 q_\alpha^2\} \Phi_0.$$  (2.26)

We then have

$$(H_I \Phi_{(6)})_{\alpha \beta} = \frac{4}{3} g (1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) \Phi_{(6)}_{\alpha \beta},$$  (2.27)

which shows that the solution is obtained by substitution of $-\frac{4}{3} g$ for $g$, as before. The boundary condition is now

$$-\sin \frac{2}{3} g > 0,$$  (2.28)
which is not satisfied by a small positive $g$. We thus find that only the $\mathbf{3}$ state can form a bound state for the particle-particle pair if $g > 0$.

Next, let us consider a particle-anti-particle pair. First, however, we should note that because our model is based on the single-particle theory, we do not have an anti-particle in our model. For this reason, we regard a particle which belongs to the $\mathbf{3}$ representation as an anti-particle, and we assume that the anti-particle is coupled to the $\mathbf{3}$ particles through the potential with $g$ taking the opposite sign ($g \rightarrow -g$). The interaction part (2.21) of the Hamiltonian is thus rewritten as

$$H_I = -g(1 - \alpha_1 \alpha_2) \lambda_1 \cdot \lambda_2^* \delta(x_1 - x_2),$$

where we suppose that the particle 2 is the anti-particle and $\lambda_2^*$ denotes the complex conjugate of $\lambda_2$. We now construct solutions for $\mathbf{1}$ and $\mathbf{8}$ multiplets, because $\mathbf{3} \times \mathbf{3} = \mathbf{1} + \mathbf{8}$. The wave function of the $\mathbf{1}$ state is written

$$\Phi_{(1)} = q_1^{\alpha} q_2^{\alpha} \Phi_0.$$  

(2.30)

The operation of $H_I$ of (2.29) on this function gives

$$H_I \Phi_{(1)} = -\frac{16}{3} g(1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) \Phi_{(1)}.$$  

(2.31)

Then the boundary condition becomes

$$\sin \frac{8}{3} g > 0,$$

(2.32)

and the mass is given by

$$m_{2-1} = 2m \cos \frac{8}{3} g.$$  

(2.33)

The wave function of the $\mathbf{8}$ state is given by

$$\Phi_{(8)} = \left\{ q_1^{1-2\alpha} - \frac{1}{3} q_2^{1-2\alpha} \delta_{\alpha \beta} q_1^{1-2\gamma} \right\} \Phi_0,$$

(2.34)

and we find

$$(H_I \Phi_{(8)})_{\alpha} = \frac{2}{3} g(1 - \alpha_1 \alpha_2) \delta(x_1 - x_2) \Phi_{(8)}.$$  

(2.35)

This gives the boundary condition

$$-\sin \frac{1}{3} g > 0,$$

(2.36)

which is not satisfied by positive $g$. Comparing Eqs. (2.24), (2.25), (2.32) and (2.33), we find that the $\mathbf{1}$ state of the particle-anti-particle pair can form the tightest bound state among the two-particle states.

(b) case $n = 3$:

The interaction part of the Hamiltonian for $n = 3$ is written

$$H_I = \frac{g}{2} \sum_{i \neq j}^3 (1 - \alpha_i \alpha_j) \lambda_i \cdot \lambda_j \delta(x_i - x_j)$$

$$= g \{(1 - \alpha_1 \alpha_2) \lambda_1 \cdot \lambda_2 \delta(x_1 - x_2) + (1 - \alpha_2 \alpha_3) \lambda_2 \cdot \lambda_3 \delta(x_2 - x_3)$$

$$+ (1 - \alpha_3 \alpha_1) \lambda_3 \cdot \lambda_1 \delta(x_3 - x_1)\}.$$  

(2.37)
We reduce the three particle state to the irreducible representations as $3 \times 3 \times 3 = 1 + 8 + 8 + 10$. Below we construct solutions of the 1 and 10 states.

The wave function of the 1 state is given by

$$
\Phi_{(1)} = \epsilon^{\alpha \beta \gamma} q_\alpha q_\beta q_\gamma \Phi_0,
$$

and the operation of $H_I$ (2.37) on this wave function gives

$$
H_I \Phi_{(1)} = -\frac{8}{3} g \sum_{i \neq j}^3 (1 - \alpha_i \alpha_j) \delta(x_i - x_j) \Phi_{(1)},
$$

and the boundary condition is given by

$$
\sin \frac{8}{3} g > 0.
$$

The mass of this state is

$$
m_{3-1} = m \left\{ 4 \cos^2 \frac{4}{3} g - 1 \right\}.
$$

The wave function of the 10 state is given by

$$
\Phi_{(10)\alpha \beta \gamma} = \sum_{\text{perm}(\alpha \beta \gamma)} q_\alpha^1 q_\beta^2 q_\gamma^3 \Phi_0,
$$

where the summation is taken over all the permutations of $(\alpha, \beta, \gamma)$, and we find

$$
(H_I \Phi_{(10)})_{\alpha \beta \gamma} = \frac{4}{3} g \sum_{i \neq j}^3 (1 - \alpha_i \alpha_j) \delta(x_i - x_j) \Phi_{(10)\alpha \beta \gamma}.
$$

This gives the boundary condition

$$
-\sin \frac{4}{3} g > 0,
$$

which is not satisfied by positive $g$.

We summarize our results in Table I. We note that the boundary conditions are the same for the singlet states of $n = 2$ and $n = 3$ and that the singlet states have the largest binding energies among the multiplets for each $n$.

§3. Summary

We implemented color $SU(3)$ symmetry on the GNF model for two- and three-particle states. We assume that the particles belong to the basic representation $3$ of the $SU(3)$ group. Though the GNF model is based on the single-particle theory (i.e., there is no anti-particle in the model), we regarded a particle belonging to the $3$ representation as an anti-particle, and assumed that the anti-particle interacts with the particles through the potential with $g$ taking the opposite sign. We were
able to construct bound state solutions for the particle-particle and the particle-
anti-particle pairs and the three-particle system by making use of the solutions of
the original GNF model and its extension to the three-body system. The results are
listed in Table I. We find that among these solutions, the color-singlet states form
the tightest bound states if \( g > 0 \). This does not, however, imply quark confinement,
because the \( \mathbf{3} \) state of the particle-particle pair also can form a bound state, though it
has smaller binding energy than that of the \( \mathbf{1} \) state of the particle-anti-particle pair,
and also because a particle can be extracted from the bound state by finite energy.
The author, however, believes that these results shed some light on the problem of
quark confinement.

It is an interesting task to fill the blank of Table I, i.e. to find a solution for the
\( \mathbf{8} \) state of the three-particle system. It is also interesting to construct solutions for
two-particle and two-anti-particle systems, though they seem to be more complicated
because we cannot utilize the known solutions of the GNF model directly in that
case.

**References**

   **83** (1990), 84.
   **83** (1990), 835.