Extended Hamiltonian Formalism of the Pure Space-Like Axial Gauge Schwinger Model

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We demonstrate that pure space-like axial gauge quantizations of gauge fields can be constructed in ways that are free from infrared divergences. To do so, we must extend the Hamiltonian formalism to include residual gauge fields. We construct an operator solution and an extended Hamiltonian of the pure space-like axial gauge Schwinger model. We begin by constructing an axial gauge formulation in auxiliary coordinates, \( x^\mu = (x^+, x^-) \), where \( x^+ = x^0 \sin \theta + x^1 \cos \theta, \ x^- = x^0 \cos \theta - x^1 \sin \theta \), and we take \( A_- = A^0 \cos \theta + A^1 \sin \theta = 0 \) as the gauge fixing condition. In the region \( 0 \leq \theta < \frac{\pi}{4} \), we can take \( x^- \) as the evolution parameter and construct a traditional canonical formulation of the temporal gauge Schwinger model in which residual gauge fields dependent only on \( x^+ \) are static canonical variables. Then we extrapolate the temporal gauge operator solution into the axial region, \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \), where \( x^+ \) is taken as the evolution parameter. In the axial region we find that we have to take the representation of the residual gauge fields realizing the Mandelstam-Leibbrandt prescription in order for the infrared divergences resulting from \( (\partial_-)^{-1} \) to be canceled by corresponding ones resulting from the inverse of the hyperbolic Laplace operator. We overcome the difficulty of constructing the Hamiltonian for the residual gauge fields by employing McCartor and Robertson’s method, which gives us a term integrated over \( x^- = \text{constant} \). Finally, by taking the limit \( \theta \to \frac{\pi}{2} - 0 \), we obtain an operator solution and the Hamiltonian of the axial gauge (Coulomb gauge) Schwinger model in ordinary coordinates. That solution includes auxiliary fields, and the representation space is of indefinite metric, providing further evidence that “physical” gauges are no more physical than “unphysical” gauges.

§1. Introduction

Axial gauges, \( n^\mu A_\mu = 0 \), specified by a constant vector \( n^\mu \), have been used recently in spite of their lack of manifest Lorentz covariance. One reason is that the Faddeev-Popov ghosts decouple from the theory in the axial gauge formulations.\(^1\) The case \( n^2 = 0 \), the light-cone gauge, has been extensively used in light-front field theory (LFFT) in attempts to find nonperturbative solutions of QCD. In LFFT, one usually makes the change of variables defined by \( x^+_l = \frac{x^0 + x^3}{\sqrt{2}}, \ x^-_l = \frac{x^0 - x^3}{\sqrt{2}} \) and specifies quantization conditions on the hyperplane \( x^+_l = 0 \). The advantages doing so are that the vacuum state contains only particles with nonnegative longitudinal momentum and that there exist relativistic bound-state equations of the Schrödinger type. (For a good overview of LFFT, see Ref. 3.)

It has been known for some time that axial gauge formulations are not ghost free, contrary to what was originally expected, and is still sometimes claimed. It was first
pointed out by Nakanishi\(^4\) that there exists an intrinsic difficulty in the axial gauge formulations so that an indefinite metric is indispensable even in QED. It was also noticed that in order to bring perturbative calculations done in the light-cone gauge into agreement with calculations done in covariant gauges, spurious singularities of the free gauge field propagator have to be regularized not as principal values, but according to the Mandelstam-Leibbrandt (ML) prescription.\(^5\) Shortly afterwards, Bassetto et al.\(^6\) obtained the ML form of the propagator in a canonical formalism. They quantized at equal time in light-cone gauge, which requires the introduction of a Lagrange multiplier field and its conjugate. These may be viewed as residual gauge degrees of freedom. Furthermore, Morara and Soldati\(^7\) found recently that the same is true in the light-cone temporal gauge formulation, in which \(x^+ + \sqrt{2}l\) is again taken as the evolution parameter, but \(A_0 + A_3\sqrt{2} = 0\) is taken as the gauge fixing condition. McCartor and Robertson\(^8\) showed that in the light-cone axial gauge formulation, where \(A_0 - A_3\sqrt{2} = 0\) is taken as the gauge fixing condition, the translational generator \(P_+\) consists of physical degrees of freedom integrated over the hyperplane \(x^+_l = \) constant and residual gauge degrees of freedom integrated over the hyperplane \(x^-_l = \) constant.

In McCartor and Robertson’s work the residual degrees of freedom were viewed as integration constants necessary to completely specify the solution to the constraint equations which relate constrained degrees of freedom to the independent degrees of freedom. This is the point of view we adopt in the present paper. Such equations often occur in quantum field theory, and little attention has been given to the question of what boundary conditions are appropriate to completely specify their solution. In the present paper we address this question. An example of the type of equation we have in mind is the equation for \(A_+\) in the light-cone gauge

\[
\partial_+^2 A_+ + \partial_- \partial_i A^i = J_+.
\]  

This equation admits a solution of the form

\[
C(x^+, x^\perp) + x^- B(x^+, x^\perp) + F(x),
\]

where \(F(x)\) is any solution to the equation. McCartor and Robertson showed that if \(F(x)\) is taken to be the usual solution, which results from the replacement \(\partial_\mu \rightarrow ik_\mu\) in the mode expansion, then neither \(C(x^+, x^\perp)\) nor \(B(x^+, x^\perp)\) is zero in the case of QED. On the other hand, in an earlier work,\(^9\) we showed that for the Schwinger model in the light-cone gauge, \(B(x^+, x^\perp)\) is zero while \(C(x^+, x^\perp)\) is not. It will be seen below that it is this last case that is peculiar: For all the solutions we give, both \(B(x^+, x^\perp)\) and \(C(x^+, x^\perp)\) are nonzero, except in the special case of the light-cone gauge, where \(B(x^+, x^\perp)\) happens to vanish.

Because the axial gauges can be viewed as continuous deformations of the light-cone gauge, extensions of the ML prescription outside the light-cone gauge formulation have also been studied. Lazzizzera,\(^10\) and Landshoff and Nieuwenhuizen,\(^11\) constructed canonical formulations for the non-pure space-like case \((n^0 \neq 0, n^2 < 0)\) in ordinary coordinates. However, in spite of many attempts, no one has yet succeeded
in constructing a consistent, pure space-like axial gauge \((n^0 = 0, n^2 < 0)\) formulation. One problem is that when the residual gauge fields are introduced as integration constants that occur when one integrates the axial gauge constraints, one cannot obtain their quantization conditions from the Dirac quantization procedure.\(^{12}\) Another problem is that the residual gauge fields are independent of \(x^3\) in the gauge \(A_3 = 0\), so that the Hamiltonian cannot be obtained by integrating densities made of those residual gauge fields over the three dimensional hyperplane \(x^0 = \) constant.

This motivates us to study the problem of introducing the residual gauge fields into pure space-like axial gauge formulations. To construct the Hamiltonians and to determine quantization conditions, we construct the pure space-like gauge as a continuous deformation of the light-cone gauge. In a previous, preliminary work\(^{13}\) we constructed an axial gauge formulation of noninteracting abelian gauge fields in the auxiliary coordinates \(x^\mu = (x^+, x^-, x^1, x^2)\), where

\[
x^+ = x^0 \sin \theta + x^3 \cos \theta, \quad x^- = x^0 \cos \theta - x^3 \sin \theta,
\]

and \(A_- = A^0 \cos \theta + A^3 \sin \theta = 0\) is taken as the gauge fixing condition. The same framework was used previously by others to analyze two-dimensional models.\(^{14}\) In the region \(0 \leq \theta < \frac{\pi}{4}\), we take \(x^-\) as the evolution parameter and construct the canonical temporal gauge formulation; in that case, the residual gauge fields are static canonical variables. By continuation, we can obtain operators satisfying the field equations in the axial region \(\frac{\pi}{4} < \theta < \frac{\pi}{2}\), where \(x^+\) is taken as the evolution parameter. In the axial region, we cannot use the conventional way of constructing the Hamiltonian for the residual gauge fields. Instead, we obtain it by integrating the divergence equation of the energy-momentum tensor over a suitable closed surface. Because the residual gauge fields do not depend on \(x^-\), their boundary surface \((x^- \rightarrow \pm \infty)\) contributions have to be kept. As a consequence, we obtain a Hamiltonian which includes a part from integrating the density involving the residual gauge fields over \(x^- = \) constant. Then, by taking the limit \(\theta \to \frac{\pi}{2} - 0\), we obtain the Hamiltonian in the axial gauge \(A_3 = 0\) formulation in ordinary coordinates.

In this paper we proceed to the case of interacting gauge fields. As a first step in finding ways to introduce the residual gauge fields into interacting axial gauge theories, we consider the exactly solvable Schwinger model. Previous attempts to construct the axial gauge \((A_1 = 0)\) operator solution to the Schwinger model have used a representation space isomorphic to (copies of) that of a positive metric free massive scalar field \(\tilde{\Sigma}\). In other words, they have assumed that \((A_1 = 0)\) is a physical gauge in that the entire representation space is physical. All such attempts encounter severe infrared difficulties.\(^{15}\) We expect that the difficulties can be overcome by introducing residual gauge fields depending only on \(x^0\). To study this possibility, we consider constructing an operator solution of the pure space-like axial gauge Schwinger model as a continuation from the light-cone gauge. We begin by constructing an operator solution in the region \(0 \leq \theta < \frac{\pi}{4}\). This is straightforward, because we can use the axial anomaly and the temporal gauge canonical quantization conditions as guiding principles in constructing the operator solution. We continue the operator solution into the axial region \(\frac{\pi}{4} < \theta < \frac{\pi}{2}\). In the axial region, the operator \((\partial_-)^{-1}\) becomes singular and gives rise to infrared divergences.
Furthermore, in the axial region, the Laplace operator, \( m^2 - n_- \partial_+^2 \) (where \( m^2 = \frac{e^2}{\pi} \) and \( n_- = \cos 2\theta \)), becomes hyperbolic, so that divergences result from its inversion. We find that the former divergences are canceled by the latter ones if we choose the representation of the residual gauge fields in such a way as to realize the ML prescription. As a consequence, we recover well-defined fermion field operators. Bilinear products of these give rise to bosonized expressions for the vector current and the energy-momentum tensor through the equal \( x^+ \)-time point-splitting procedure. We apply McCartor and Robertson’s method\(^8\) and obtain the Hamiltonian for the residual gauge fields. It consists of a part from the physical operators integrated over \( x^+ \) = constant and a part from the residual gauge operators integrated over \( x^- \) = constant. This is the way we extend the traditional Hamiltonian formulation to obtain the Hamiltonian for the residual gauge fields. Now, it is straightforward to obtain an operator solution and the Hamiltonian in the gauge \( A_1 = 0 \) (here, the Coulomb gauge) in ordinary coordinates. All one has to do is to take the limit \( \theta \to \frac{\pi}{2} - 0 \). While this limit exists, we cannot simply set \( \theta \) equal to \( \frac{\pi}{2} \). For this reason, one might say that we have not really quantized the Schwinger model at equal time in the Coulomb gauge. On the other hand, one might regard \( \theta \) as a regulating parameter. Then, we see that, as is the case in the light-cone gauge,\(^9\),\(^{16}\) proper regulation of the theory necessarily involves information off the initial value surface. In any event, the solution contains auxiliary fields, and the representation space is of indefinite metric. This provides further indication that “physical” gauges are no more physical than “unphysical” gauges, in that the representation space contains unphysical states and is of indefinite metric in all cases.

The paper is organized as follows. In §2, we use the axial anomaly and the temporal gauge quantization conditions as guiding principles to construct the temporal gauge operator solution of the Schwinger model in the auxiliary coordinates. In §3, we continue the solution into the axial region and show in detail that the infrared divergences inherent in the axial gauge quantizations are canceled if we choose the representation of the residual gauge fields that realizes the ML prescription. We also discuss properties of the axial gauge solution. Section 4 is devoted to concluding remarks.

We use the following conventions:

\[
g_{--} = \cos 2\theta, \quad g_{-+} = g_{+-} = \sin 2\theta, \quad g_{++} = -\cos 2\theta, \\
g^{--} = \cos 2\theta, \quad g^{-+} = g^{+-} = \sin 2\theta, \quad g^{++} = -\cos 2\theta, \\
n^\mu = (n^+, n^-) = (0, 1), \quad n_\mu = (n^+, n^-) = (\sin 2\theta, \cos 2\theta), \\
\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma^5 = -\sigma_3, \\
\gamma^+ = \gamma^0 \sin \theta + \gamma^1 \cos \theta = \begin{pmatrix} 0 & \sqrt{1 + n_+} \\ -\varepsilon(n_-)\sqrt{1 - n_+} & 0 \end{pmatrix}, \\
\gamma^- = \gamma^0 \cos \theta - \gamma^1 \sin \theta = \begin{pmatrix} 0 & \varepsilon(n_-)\sqrt{1 - n_+} \\ \sqrt{1 + n_+} & 0 \end{pmatrix}.
\]
§2. Temporal gauge formulation in the auxiliary coordinates

In this section we confine ourselves to the region $0 \leq \theta < \frac{\pi}{4}$ and choose $x^-$ as the evolution parameter. The temporal gauge Schwinger model in the auxiliary coordinates is defined by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - B(n \cdot A) + i \bar{\psi} \gamma^\mu (\partial_\mu + ieA_\mu) \psi,$$  

(2.1)

where $B$ is the Lagrange multiplier field, that is, the Nakanishi-Lautrup field in noncovariant formulations. From the Lagrangian we derive the field equations

$$\partial_\mu F^{\mu\nu} = n^\nu B + J^\nu, \quad J^\nu = e \bar{\psi} \gamma^\nu \psi,$$  

(2.2)

$$i \gamma^\mu (\partial_\mu + ieA_\mu) \psi = 0,$$  

(2.3)

and the gauge fixing condition

$$A_- = 0.$$  

(2.4)

The field equation of $B$,

$$\partial_- B = 0,$$  

(2.5)

is obtained by operating on (2.2) with $\partial_\nu$. Canonically conjugate momenta are defined by

$$\pi^- = \frac{\delta L}{\delta \partial_- A_-} = 0, \quad \pi^+ = \frac{\delta L}{\delta \partial_+ A_+} = F_{++}, \quad \pi_B = \frac{\delta L}{\delta \partial_- B} = 0, \quad \pi_\psi = \frac{\delta L}{\delta \partial_- \psi} = i \bar{\psi} \gamma^-.$$

(2.6)

Thus we impose the following equal $x^-$-time canonical quantization conditions:

$$[A_+(x), A_+(y)] = [\pi^+(x), \pi^+(y)] = 0, \quad [A_+(x), \pi^+(y)] = i \delta(x^+ - y^+),$$  

(2.7)

$$[A_+(x), \psi(y)] = [\pi^+(x), \psi(y)] = 0,$$  

(2.8)

$$\{\psi_\alpha(x), \psi_\beta(y)\} = 0, \quad \{\psi_\alpha(x), \psi_\beta^*(y)\} = ((\gamma^0 \gamma^-)^{-1})_{\alpha\beta} \delta(x^+ - y^+).$$  

(2.9)

Since in the present paper our aim is not to solve the Schwinger model in these gauges from scratch, but rather to write down such solutions and study their gauge dependent properties — particularly the existence and role of the auxiliary fields — we make use of gauge-independent properties known from previous solutions in other gauges, in particular, the axial anomaly. As is seen below, the axial anomaly is necessarily included in the solution and is required by the gauge-invariant point-splitting regulator. We therefore begin by regularizing the vector current $J^\mu$ using the gauge invariant point-splitting procedure and obtain the $J^\mu$ given by

$$J^\mu = j^\mu - m^2 A^\mu,$$  

(2.10)

where $j^\mu$ is the part given by the bilinear product of the $\psi$. It is easily seen that, making use of the relation $J^5_\mu = \varepsilon_{\mu\nu} j_\nu$, which holds in two spacetime dimensions, and assuming that $j^\mu$ satisfies $\varepsilon^{\mu\nu} \partial_\mu j_\nu = 0$, we get

$$\partial^\mu J^5_\mu = \varepsilon^{\mu\nu} \partial_\mu j_\nu = -m^2 \varepsilon^{\mu\nu} \partial_\mu A_\nu,$$  

(2.11)
where $\varepsilon^{+-} = -\varepsilon^{-+} = 1$, $\varepsilon^{++} = \varepsilon^{--} = 0$. Below, we show that the latter assumption holds.

From (2.2), $J_-$ and $J_+$ can be expressed in terms of the bosonic operators as

$$J_- = g_- + J_+$$

$$J_+ = -\partial^+ F_- - n_- B,$$

$$J_- = \partial^- F_+ - n_+ B.$$  \hspace{1cm} (2.12)

Substituting these expressions into (2.11) yields

$$(\partial_\mu \partial^\mu + m^2)F_- = (n_- \partial_-^2 + 2n_+ \partial_+ \partial_- - n_- \partial_+^2)F_- = -n_- \partial_+ B.$$ \hspace{1cm} (2.13)

Because $\partial_- B = 0$, a particular solution to the equation is

$$-n_- \frac{m^2}{m^2 - n_- \partial_+^2} \partial_+ B.$$ \hspace{1cm} (2.14)

In the temporal gauge formulation, $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma}(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \frac{dp}{p^-} \{a(p_+) e^{-ip^- x} + a^*(p_+) e^{ip^- x}\},$$ \hspace{1cm} (2.15)

where $p^- = \sqrt{p^+^2 + m_0^2}$ with $m_0^2 = m^2$ and

$$[a(p_+), a(q_+)] = 0, \quad [a(p_+), a^*(q_+)] = \delta(p_+ - q_+).$$ \hspace{1cm} (2.16)

By integrating (2.14) with respect to $x^-$, we obtain the general solution for $A_+$ as

$$A_+ = -\frac{m}{m^2 - n_- \partial_+^2} \tilde{\Sigma} - \frac{n_- x^- \partial_+ - n_- \partial_+^2}{m^2 - n_- \partial_+^2} B + \text{integration constant},$$ \hspace{1cm} (2.17)

where $(\partial_-)^{-1}$ is defined by

$$(\partial_-)^{-1} f(x^-) = \frac{1}{2} \int_{-\infty}^{\infty} dy^- \varepsilon(x^- - y^-) f(y^-).$$ \hspace{1cm} (2.18)

This imposes, in effect, the principal value regularization. We determine the integration constant in the following way. Taking account of the fact that $\tilde{\Sigma}$ satisfies $(n_- \partial_-^2 + 2n_+ \partial_+ \partial_- - n_- \partial_+^2)\tilde{\Sigma} = 0$, we can rewrite the first term of (2.17) as

$$\frac{1}{\partial_-} \tilde{\Sigma} = -\frac{n_+ \partial_+ + \partial_-}{m^2 - n_- \partial_+^2} \tilde{\Sigma}.$$ \hspace{1cm} (2.19)

In the range of $\theta$ we are considering, we see that the operator $(\partial_-)^{-1}$ does not give rise to any infrared divergences. Now, the first term in (2.17) has the property
\[
\left[ \frac{\partial}{\partial x^-} \tilde{\Sigma}(x^+), \frac{\partial}{\partial y^-} \tilde{\Sigma}(y^+) \right] \neq 0 \quad \text{at equal } x^-\text{-time, because } \tilde{\Sigma} \text{ satisfies the following equal } x^-\text{-time commutation relations:}
\]
\[
[\tilde{\Sigma}(x), \tilde{\Sigma}(y)] = [\partial^- \tilde{\Sigma}(x), \partial^- \tilde{\Sigma}(y)] = 0, \quad [\tilde{\Sigma}(x), \partial^- \tilde{\Sigma}(y)] = i \delta(x^+ - y^+) \quad (2.20)
\]

Therefore, to preserve the canonical commutation relations (2.7), we must cancel the nonvanishing terms with contributions from another field, \( C \), the integration constant, which depends only on \( x^+ \). We find that if \( B \) and \( C \) satisfy the commutation relations
\[
[B(x), B(y)] = [C(x), C(y)] = 0, \quad [B(x), C(y)] = i(m^2 - n_- \partial_+^2) \delta(x^+ - y^+),
\]
\[
[B(x), \tilde{\Sigma}(y)] = [C(x), \tilde{\Sigma}(y)] = 0, \quad (2.21)
\]
then
\[
A_+ = -m \frac{\partial}{\partial_-} \tilde{\Sigma} + \frac{\partial_+}{m^2 - n_- \partial_+^2} (C - n_- x^- B) + \frac{n_+ m^2}{(m^2 - n_- \partial_+^2)^2} B, \quad (\partial_- C = 0) \quad (2.22)
\]
satisfies the quantization conditions in (2.7). In this way we can determine the integration constant by taking account of the canonical quantization conditions in the temporal gauge formulation.

Now that we have \( A_+ \), we proceed to solving the fermion field equation (2.3). A formal solution is given by
\[
\psi_\alpha(x) = \frac{Z_\alpha}{\sqrt{(\gamma^0 \gamma^-)_{\alpha\alpha}}} \exp[-i \sqrt{\pi} A_\alpha(x)], \quad (2.23)
\]
where \( Z_\alpha \) is a normalization constant and
\[
A_\alpha(x) = X(x) + (-1)^\alpha \lambda(x), \quad (2.24)
\]
with
\[
X = \frac{\partial_+}{\partial_-} \tilde{\Sigma} + \frac{m}{m^2 - n_- \partial_+^2} \left( C - n_- x^- B + \frac{n_+ n_-}{m^2 - n_- \partial_+^2} \partial_+ B \right),
\]
\[
\lambda = \tilde{\Sigma} + \frac{m}{m^2 - n_- \partial_+^2} \partial_+^{-1} B. \quad (2.25)
\]
Here the operator \( \partial_+^{-1} \) is defined in the same manner as in (2.18). To further examine the properties of the fermion field operators, such as their vacuum expectation values, we need explicit representations of \( B \) and \( C \). To realize the ML prescription, which shifts spurious singularities of the gauge field propagator at \( k_- = 0 \) above or below the real axis, depending on the sign of \( k_+ \), we choose the following representation for \( B \) and \( C \):
\[
B(x) = \frac{m}{\sqrt{2\pi}} \int_0^\infty dk_+ \sqrt{k_+} \{ B(k_+) e^{-ik_+ x^+} + B^*(k_+) e^{ik_+ x^+} \},
\]
\[
\frac{m}{m^2 - n_- \partial_+^2} C(x) = \frac{i}{\sqrt{2\pi}} \int_0^\infty \frac{dk_+}{\sqrt{k_+}} \{ C(k_+) e^{-ik_+ x^+} - C^*(k_+) e^{ik_+ x^+} \}. \quad (2.26)
\]
Here

\[ [B(k_+), C^*(q_+)] = [C(k_+), B^*(q_+)] = -\delta(k_+ - q_+), \quad (2.27) \]

and all other commutators are zero. As a matter of fact we obtain the \( x^- \)-ordered gauge field propagator

\[
D_{\mu\nu}(x - y) = \langle 0| \{ \theta(x^- - y^-) A_\mu(x) A_\nu(y) + \theta(y^- - x^-) A_\nu(y) A_\mu(x) \}|0\rangle
= \frac{1}{(2\pi)^2} \int d^2q D_{\mu\nu}(q) e^{-iq(x-y)}, \quad (2.28)
\]

where

\[
D_{\mu\nu}(q) = \frac{i}{q^2 - m^2 + i\epsilon} \left( -g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{q_- + i\epsilon \text{sgn}(q_+)} - n^2 \frac{q_\mu q_\nu}{(q_- + i\epsilon \text{sgn}(q_+))^2} \right). \quad (2.29)
\]

We define the physical subspace, \( V \), by

\[
V = \{ |\text{phys}\rangle | B(k_+)|\text{phys}\rangle = 0 \}. \quad (2.30)
\]

Then, we define the infrared parts, \( \Lambda_{\alpha}^{(0)} \), of \( \Lambda_\alpha \) to be

\[
\Lambda_{\alpha}^{(0)} = \frac{i}{\sqrt{2\pi}} \int_0^\kappa \frac{dk_+}{\sqrt{k_+}} \{ C(k_+) - C^*(k_+) + (-1)^{\alpha} (B(k_+) - B^*(k_+)) \}, \quad (2.31)
\]

where \( \kappa \) is a small positive constant. We subtract the infrared parts and do not rewrite the exponential function of them into normal ordered form. We refer to the part of the fermion field operator that is not rewritten into normal ordered form as a ‘spurion operator’.\(^{20}\) Then, the fermion field operators are defined as

\[
\psi_\alpha(x) = \frac{Z_\alpha}{\sqrt{(\gamma^0 \gamma^-)_{\alpha\alpha}}} \exp[-i\sqrt{\pi} \Lambda_{\alpha}^{(-)}(x)] \sigma_\alpha \exp[-i\sqrt{\pi} \Lambda_{\alpha}^{(+)}(x)], \quad (2.32)
\]

where \( \Lambda_{\alpha}^{(-)} \) and \( \Lambda_{\alpha}^{(+)} \) are the creation and annihilation operator parts of \( \Lambda_{\alpha} \equiv \Lambda_\alpha - \Lambda_{\alpha}^{(0)} \) and

\[
\sigma_\alpha = \exp \left[-i\sqrt{\pi} \left( \Lambda_{\alpha}^{(0)} - (-1)^{\alpha} Q \frac{Q}{2m} \right) \right]. \quad (2.33)
\]

Here, \( Q = - \int_{-\infty}^\infty dx^+ B(x) \); note that \( Q \) in \( \sigma_\alpha \) constitutes a Klein transformation.

We enumerate properties of the \( \psi_\alpha \) to show that the \( \psi_\alpha \) constitute the operator solution of the temporal gauge Schwinger model in the auxiliary coordinates.

1. The Dirac equation is satisfied:

\[
i\gamma^\mu (\partial_\mu + ieA_\mu)\psi = 0. \quad (2.34)
\]

To show this, we use the fact that \( [A_{\alpha}^{(+)}(x), \psi_\alpha(x)] = 0 \), where \( A_{\alpha}^{(+)} \) is the annihilation operator part of \( A_\alpha \).

2. The canonical commutation relations (2.8) and anticommutation relations (2.9) are satisfied.

3. The vacuum expectation value of \( \psi_1 \) vanishes, whereas that of \( \psi_2 \) diverges, as it does in the light-cone gauge solution.\(^9\)
(4) By applying the gauge invariant point-splitting procedure to $e\bar{\psi}\gamma^\mu\psi$, we obtain the vector current $J^\mu = m\partial^\mu X - m^2 A^\mu = m\varepsilon^{\mu\nu}\partial_\nu\lambda$. This verifies that $j^\mu$ is described as $j^\mu = m\partial^\mu X$ so that it satisfies $\varepsilon^{\mu\nu}\partial_\mu j_\nu = 0$. The charge operator, $Q$, is given by

$$Q = \int_{-\infty}^{\infty} dx^+ J^- (x) = -\int_{-\infty}^{\infty} dx^+ B(x), \quad (2.35)$$

where the derivative terms integrate to zero.

(5) The conserved chiral current $J^{5\mu} = m\varepsilon^{\mu\nu}\partial_\nu X$ is similarly obtained from $e\bar{\psi}\gamma^\mu\gamma^5\psi$. The chiral charge operator, $Q_5$, is given by

$$Q_5 = -\int_{-\infty}^{\infty} dx^+ J^{-5} (x) = \int_{-\infty}^{\infty} dx^+ \partial_+ C (x), \quad (2.36)$$

where the derivative terms, except $\partial_+ C$, integrate to zero. The term $\partial_+ C$ does not integrate to zero, because $\psi$ contains the singular operator $\partial_+^{-1} B$.

(6) Applying the gauge invariant point-splitting procedure to the fermi products in the symmetric energy-momentum tensor and subtracting a diverging c-number (we refer to this procedure\(^2\)) as $R$, we obtain

$$\Theta_+^- = \frac{i}{2} R(\bar{\psi}\gamma^- \partial_+ \psi - \partial_+ \bar{\psi}\gamma^- \psi) - A_+ J^- - \frac{1}{2} \{A_+, B\}$$

$$= \frac{1}{2} \{\partial_+ \lambda, \partial^- \lambda\} - \frac{1}{2} \{A_+, B\}, \quad (2.37)$$

$$\Theta_+^+ = \frac{i}{2} R(\bar{\psi}\gamma^+ \partial_+ \psi - \partial_+ \bar{\psi}\gamma^+ \psi) - A_+ J^+ + \frac{1}{2} (F_+)^2$$

$$= -\frac{n_+}{2} \{(\partial_+ \lambda)^2 + (\partial^- \lambda)^2\} + \frac{1}{2} (F_+)^2,$$  \hspace{1cm} (2.38)

$$\Theta_-^- = -\frac{i}{2} R(\bar{\psi}\gamma^- \partial_+ \psi - \partial_+ \bar{\psi}\gamma^- \psi) + A_+ J^- + \frac{1}{2} (F_+)^2$$

$$= \frac{n_-}{2} \{(\partial_+ \lambda)^2 + (\partial^- \lambda)^2\} + \frac{1}{2} (F_+)^2,$$  \hspace{1cm} (2.39)

$$\Theta_-^+ = -\frac{i n_+}{2n_-} R(\bar{\psi}\gamma^+ \partial_+ \psi - \partial_+ \bar{\psi}\gamma^+ \psi) - \frac{i}{2n_-} R(\bar{\psi}\gamma^+ \gamma^5 \partial_+ \psi - \partial_+ \bar{\psi}\gamma^+ \gamma^5 \psi)$$

$$+ \frac{1}{n_-} J_- A_+ + \frac{n_+}{n_-} J^+ A_+ = \frac{1}{2} \{\partial_+ \lambda, \partial^- \lambda\}. \quad (2.40)$$

Here, and hereafter, we write the symmetrization of any two operators $A$ and $B$ as $AB + BA = \{A, B\}$.

(7) Translational generators consist of those of the constituent fields:

$$P_- = \int_{-\infty}^{\infty} dx^+ : \Theta_-^- :$$

$$= \int_{-\infty}^{\infty} dx^+ \frac{1}{2} : \left\{ n_-(\partial_+ \hat{\Sigma})^2 + n_-(\partial_- \hat{\Sigma})^2 + m^2 \hat{\Sigma}^2 + B \frac{n_+}{m^2 - n_+ \partial_+^2} B \right\} :,$$

$$P_+ = \int_{-\infty}^{\infty} dx^+ : \Theta_-^+ : = \int_{-\infty}^{\infty} dx^+ \left\{ \partial_+ \hat{\Sigma} \partial^- \hat{\Sigma} - B \frac{1}{m^2 - n_+ \partial_+^2} \partial_+ C \right\}. \quad (2.41)$$
(8) The Heisenberg equation for \( C \) does not hold, as is seen from
\[
[P_-, C(x)] = in^2B(x), \tag{2.42}
\]
but those for the canonical variables do hold.

\section*{§3. Cancellation of infrared divergences resulting from \( \partial_-^{-1} \)}

In this section we continue \( \theta \) into the region \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \) and take \( x^+ \) as the evolution parameter. In accordance with the change of the evolution parameter, we change the Fourier expansion of \( \tilde{\Sigma} \) so as to be convenient for calculating equal \( x^+ \)-time commutation relations. By making use of \( \frac{dp_+}{p} = -\frac{dp_-}{p^+} \), we change integration variables from \( p_+ \) to \( p_- \) and rewrite \( \tilde{\Sigma} \) as
\[
\tilde{\Sigma}(x) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \notag \{ a(p_-)e^{-ip_-x} + a^*(p_-)e^{ip_-x} \}, \tag{3.1}
\]
where \( p^+ = \sqrt{p_-^2 + m_0^2} \) with \( m_0^2 = |n_-|m^2 \) and
\[
[a(p_-), a(q_-)] = 0, \quad [a(p_-), a^*(q_-)] = \delta(p_- - q_-). \tag{3.2}
\]
\( \tilde{\Sigma} \) satisfies the following equal \( x^+ \)-time commutation relations:
\[
[\tilde{\Sigma}(x), \tilde{\Sigma}(y)] = [\partial^+ \tilde{\Sigma}(x), \partial^+ \tilde{\Sigma}(y)] = 0, \quad [\tilde{\Sigma}(x), \partial^+ \tilde{\Sigma}(y)] = i\delta(x - y). \tag{3.3}
\]
Furthermore, in accordance with the change of the evolution parameter, the conjugate momentum of \( \psi \) changes to
\[
\pi_{\psi}^{(a)} = \frac{\delta L}{\delta \partial_+ \psi} = i\tilde{\psi}\gamma^+. \tag{3.4}
\]
Therefore, we have to modify the normalization of \( \psi \) in (2.32) accordingly:
\[
\psi_{\alpha}^{(a)}(x) = \frac{Z_{\alpha}^{(a)}}{\sqrt{(\gamma^0\gamma^+)^{\alpha\alpha}}} \exp[-i\sqrt{\pi}A_{\alpha r}^-(x)]\sigma_{\alpha}\exp[-i\sqrt{\pi}A_{\alpha r}^+(x)]. \tag{3.5}
\]
Here we have appended the superscript “(a)” to \( \psi \) and to the normalization constant to indicate that the normalization is altered.

We begin by pointing out that because \( n_- = \cos 2\theta < 0 \) in the region \( \frac{\pi}{4} < \theta < \frac{\pi}{2} \), the Laplace operator \( m^2 - n_-\partial_+^2 \), which operates on the residual gauge fields, becomes hyperbolic, so that its inverse gives rise to divergences. Therefore, we regularize them using the principal value prescription. We then recover well-defined fermion field operators, because linear infrared divergences resulting from \( \partial_+^{-1} \tilde{\Sigma} \) are canceled by those resulting from the square of the inverse of the hyperbolic Laplace operator. To show that this actually happens, we rewrite \( \psi_{\alpha}^{(a)*}(x)\psi_{\alpha}^{(a)}(y) \) into normal ordered form. It is tedious but straightforward to obtain
\[
\psi_{\alpha}^{(a)*}(x)\psi_{\alpha}^{(a)}(y) = \frac{Z_{\alpha}^{(a)}}{(\gamma^0\gamma^+)^{\alpha\alpha}} \exp[M_{\alpha\alpha}(x - y)]\exp[-i\sqrt{\pi}(A_{\alpha r}^-(x) - A_{\alpha r}^-(y))] \notag \times \exp[-i\sqrt{\pi}(A_{\alpha r}^+(x) - A_{\alpha r}^+(y))], \tag{3.6}
\]
where

\[ M_{\alpha\alpha}(x) = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \left( 1 + \left( \frac{p^+}{p_-} \right)^2 + 2(-1)^{\alpha} \frac{p^+}{p_-} \right) e^{-ip^-x} + \frac{1}{2} \int_{0}^{\infty} dk_+ \left( - \frac{im^2 n_- x^-}{m^2 + n_- k_+^2} + \frac{2m^2 n_- n_+ k_+}{(m^2 + n_- k_+^2)^2} \right) e^{-ik_+x^+} \]

\[ + (-1)^{\alpha} \int_{0}^{\infty} dk_+ \left( \frac{n_- k_+}{m^2 + n_- k_+^2} e^{-ik_+x^+} - \frac{e^{-ik_+x^+} - \theta(\kappa - k_-)}{k_+} \right). \]

(3.7)

We must show that \( M_{\alpha\alpha}(x) \) does not diverge when \( x^+ = 0 \), because in what follows we make use of the values of \( M_{\alpha\alpha}(x - y) \) at \( x^+ = y^+ \) to calculate the equal \( x^+ \)-time anticommutation relations and to calculate the dynamical operators.

Note that the terms on the first line of (3.7) are given only by the physical operators \( \tilde{\Sigma} \) and \( \frac{\partial}{\partial x^-} \tilde{\Sigma} \), whereas the terms on the second and third lines are given only by the residual gauge fields \( B \) and \( C \). Therefore, for convenience, we call the terms on the first line “physical contributions” and the rest “residual gauge contributions”.

Since \((p^+)^2 = p_-^2 + m_0^2\), only the term \( \frac{m_0^2}{p_-^2} \) among the physical contributions gives rise to a linear infrared divergence:

\[ \frac{1}{4} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \frac{m_0^2}{p_-^2} e^{-ip^-x^-} = \frac{1}{4} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \frac{m_0^2}{p_-^2} - \frac{m_0^2}{2} \int_{0}^{\infty} \frac{dp_-}{p^+} \frac{1 - \cos p_-x^-}{p_-^2}. \]

(3.8)

A logarithmic divergence appears in the third term of the physical contributions, but it is regularized by the principal value prescription. To verify that the linear divergence in (3.8) is canceled by that resulting from the residual gauge field’s contribution, we change integration variables from \( p_- \) to \( p_+ \), which is given in terms of \( p_- \) by \( p_+ = \sqrt{p_-^2 + m_0^2 - n_- p_-} \). Note that \( p_- \) is conjugate to the spatial variable in the axial gauge formulation, whereas \( p_+ \) is conjugate to the spatial variable in the temporal gauge formulation. Note, furthermore, that the residual gauge field’s contributions are given in (3.7) as integrals with respect to the momentum \( k_+ \), which is conjugate to the spatial variable in the temporal gauge formulation. Consequently, we obtain

\[ \frac{1}{4} \int_{-\infty}^{\infty} \frac{dp_-}{p^+} \frac{m_0^2}{p_-^2} = \frac{1}{4} \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p^2 - m_0^2}} \frac{n_-^2 m_0^2}{(n_+ p_+ - \sqrt{p^2 - m_0^2})^2} + \frac{1}{4} \int_{m_0}^{\infty} \frac{dp_+}{\sqrt{p^2 - m_0^2}} \frac{n_-^2 m_0^2}{(n_+ p_+ + \sqrt{p^2 - m_0^2})^2}. \]

(3.9)

We see that the integrals are well-defined at their lower limits. We also see that the first term is singular at \( p_+ = \frac{m_-}{n_-} (\equiv a) \) and gives rise to linear divergence, whereas the second term is well-defined. In the neighborhood of \( p_+ = a \), the integrand of the first term behaves as

\[ \frac{1}{4} \frac{1}{\sqrt{p^2 - m_0^2}} \frac{n_-^2 m_0^2}{(n_+ p_+ - \sqrt{p^2 - m_0^2})^2} \approx \frac{n_- a}{4} \frac{1}{(p_+ - a)^2}. \]

(3.10)
Those divergences in the residual gauge field’s contribution that can be regularized by the principal value prescription are so regularized. Then we find that when \(x^+ = 0\), the first term on the second line of (3.7) is equal to zero:

\[
\int_0^\infty dk_+ \frac{n_- m^2}{m^2 + n_- k_+^2} = \int_0^\infty \frac{m^2}{k_+^2 - a^2} = 0. 
\]  
(3.11)

However, we cannot regularize a linear divergence resulting from a double pole using the principal value prescription. In fact, when \(x^+ = 0\), the second term on the second line of (3.7) gives rise to a linear divergence as follows

\[
\int_0^\infty \frac{n_+ m_2 k_+}{(m^2 + n_+ k_+^2)^2} = - \frac{n_+ a^2}{4} \int_0^\infty \frac{1}{(k_+^2 - a^2)} - \frac{1}{(k_+^2 + a^2)}. 
\]  
(3.12)

Thus we see that the linear divergence originating from (3.10) is canceled by that resulting from the residual gauge field’s contributions. It turns out that when \(x^+ = 0\), the terms on the third line of (3.7) gives rise to a finite value, \(\log \frac{\kappa}{\sqrt{|k^2 - a^2|}}\), so that by summing all the finite contributions, we obtain

\[
M_{\alpha\alpha}(x)|_{x^+ = 0} = K_0(m_0|x^-|) + i(-1)^{\alpha-1} \frac{\pi}{2} \varepsilon(x^-) 
- \frac{m_0^2}{2} \int_0^\infty \frac{dp_-}{p^+} - \cos \frac{p_- x^-}{p^2} + \text{constant} + (-1)^{\alpha} \log \frac{\kappa}{\sqrt{|k^2 - a^2|}} 
\]

\[
= - \left( \log \frac{m_0 \gamma}{2} + \log(x^- + i\varepsilon(-1)^{\alpha-1}) \right) + i(-1)^{\alpha-1} \frac{\pi}{2} + F_\alpha(|x^-|), 
\]  
(3.13)

where \(K_0(m_0|x^-|)\) is the modified Bessel function of order 0, which possesses the logarithmic term \(-\log \frac{m_0 |x^-|}{2} + \gamma\). We have gathered the remainder of \(K_0\) and the other terms to make a continuous function \(F_\alpha(|x^-|)\).

Now that we have finished showing that no divergences appear from (3.7), we proceed to study other properties of the fermion field operators.

(1) The equal \(x^+\)-time anticommutation relations hold. For example, when \(x^+ = y^+\), we find that

\[
\exp[M_{\alpha\alpha}(x - y)] = \frac{2}{m_0 \gamma} \frac{i(-1)^{\alpha-1}}{x^- - y^- + i\varepsilon(-1)^{\alpha-1}} e^{F_\alpha(|x^- - y^-|)}, 
\]  
(3.14)

so that if we take \(Z^{(a)}_\alpha\) to be

\[
(Z^{(a)}_\alpha)^2 = \frac{m_0 \gamma}{4\pi} e^{-F_\alpha(0)}, 
\]  
(3.15)

then we obtain

\[
\{\psi_\alpha^+(x), \psi_\alpha^-(y)\}|_{x^+ = y^+} = \exp[F_\alpha(|x^- - y^-|) - F_\alpha(0)] 
\frac{2\pi(\gamma^0 \gamma^+)^{\alpha\alpha}}{\gamma^0 \gamma^+} \left( \frac{i(-1)^{\alpha-1}}{x^- - y^- + i\varepsilon(-1)^{\alpha-1}} + \frac{i(-1)^{\alpha-1}}{y^- - x^- + i\varepsilon(-1)^{\alpha-1}} \right) 
\times \exp[-i\sqrt{\pi}(A_\alpha^-(x) - A_\alpha^-(y))] \exp[-i\sqrt{\pi}(A_\alpha^+(x) - A_\alpha^+(y))] 
\times \frac{1}{(\gamma^0 \gamma^+)^{\alpha\alpha}} \delta(x^- - y^-). 
\]  
(3.16)
When $\alpha \neq \beta, \{\psi^*_\alpha(x), \psi_\beta(y)\}|_{x^+ = y^+}$ vanishes, due to the Klein transformation factors.

(2) The vector current, $J^\mu = me^{\mu\nu}\partial_\nu \lambda$, is obtained this time by applying the equal $x^+$-time point-splitting procedure to $e\bar{\psi}\gamma^\mu \psi$.

(3) The conserved chiral current $J^{\mu 5} = me^{\mu\nu}\partial_\nu X$ is similarly obtained from $e\bar{\psi}\gamma^\mu \gamma^5 \psi$.

(4) As a consequence of our choice of the representation of the residual gauge fields, we obtain the ML form of the $x^+$-ordered gauge field propagator

$$D_{\mu\nu}(x - y) = \langle 0 | \{\theta(x^+ - y^+)A_\mu(x)A_\nu(y) + \theta(y^+ - x^+)A_\nu(y)A_\mu(x)\}|0\rangle$$

$$= \frac{1}{(2\pi)^2} \int d^2 q D_{\mu\nu}(q) e^{-iq(x-y)}, \quad (3.17)$$

where

$$D_{\mu\nu}(q) = \frac{i}{q^2 - m^2 + i\epsilon} \left( -g_{\mu\nu} + \frac{n_{\mu} q_{\nu} + n_{\nu} q_{\mu}}{q_+ + i\epsilon \text{sgn}(q_+)} - n^2 \frac{q_{\mu} q_{\nu}}{(q_+ + i\epsilon \text{sgn}(q_+))^2} \right)$$

$$- \delta_{\mu+} \delta_{\nu+} \frac{i}{2} \left( \frac{1}{(q_+ + i\epsilon)^2} + \frac{1}{(q_- + i\epsilon)^2} \right). \quad (3.18)$$

We give a detailed derivation of this in the Appendix.

(5) Applying the gauge invariant point-splitting procedure to the fermi products in the symmetric energy-momentum tensor and subtracting a divergent $c$-number (we refer to this procedure as $R$), we obtain

$$\Theta^+_- = \frac{in_+}{2n_-} R(\bar{\psi}\gamma^- \partial_- \psi - \partial_- \bar{\psi}\gamma^- \psi) - \frac{i}{2n_-} R(\bar{\psi}\gamma^- \gamma^5 \partial_- \psi - \partial_- \bar{\psi}\gamma^- \gamma^5 \psi)$$

$$- \frac{1}{2} \{A_+, B\} = \frac{1}{2} \{\partial_+ \lambda, \partial^\lambda \} - \frac{1}{2} \{A_+, B\}, \quad (3.19)$$

$$\Theta^+_- = -\frac{i}{2} R(\bar{\psi}\gamma^- \partial_- \psi - \partial_- \bar{\psi}\gamma^- \psi) + \frac{1}{2} (F_-)^2$$

$$= -\frac{n_-}{2} \{(\partial_+ \lambda)^2 + (\lambda)^2\} + \frac{1}{2} (F_-)^2, \quad (3.20)$$

$$\Theta^+_- = \frac{1}{2} (F_-)^2 + \frac{i}{4} R(\bar{\psi}\gamma^- \partial_- \psi - \partial_- \bar{\psi}\gamma^- \psi)$$

$$- \frac{n_+}{4} R(\bar{\psi}\gamma^- \gamma^5 \partial_- \psi - \partial_- \bar{\psi}\gamma^- \gamma^5 \psi) + \frac{n_+}{4} R(\bar{\psi}\gamma^+ \gamma^5 \partial_- \psi - \partial_- \bar{\psi}\gamma^+ \gamma^5 \psi)$$

$$= \frac{n_-}{2} \{(\partial_+ \lambda)^2 + (\lambda)^2\} + \frac{1}{2} (F_-)^2, \quad (3.21)$$

$$\Theta^+_- = \frac{i}{2} R(\bar{\psi}\gamma^+ \partial_- \psi - \partial_- \bar{\psi}\gamma^+ \psi) = \frac{1}{2} \{\partial_+ \lambda, \partial^+ \lambda\}. \quad (3.22)$$

We must take special care in the axial gauge formulation when we derive the conserved translational generators from the divergence equation

$$\partial_\nu \Theta^\nu_{\mu} = 0. \quad (3.23)$$

The problem is that, although $x^-$ is a space coordinate, $\Theta^{--}$ does not vanish in the limits $x^- \to \pm \infty$. This is because the residual gauge fields do not depend on $x^-$. 

and because $A_\perp$ depends explicitly on $x^-$. Therefore we have to retain the value $\lim_{x^- \to \pm\infty} \Theta_{\mu}^-$. To take this fact into account, we integrate the divergence equation $\partial_{\nu}\Theta_{\mu}^{\nu} = 0$ over the closed surface shown in Fig. 1, whose bounds $T$ and $L$ are taken to $\infty$ after all calculations are finished. We remark that we can use this surface even in the limit $\theta \to \frac{\pi}{2} - 0$, in contrast to that used in Refs. 8] and 22).

It is straightforward to obtain

$$
0 = \left( \int_{-L}^{L} dx^- \left[ \Theta_{\mu}^+ (x) \right]_{x^+ = T}^{x^+ = -T} + \int_{-T}^{T} dx^+ \left[ \Theta_{\mu}^- (x) \right]_{x^- = L}^{x^- = -L} \right), \tag{3.24}
$$

where

$$
\left[ \Theta_{\mu}^+ (x) \right]_{x^+ = T}^{x^+ = -T} = \Theta_{\mu}^+ (x) |_{x^+ = T} - \Theta_{\mu}^+ (x) |_{x^+ = -T},
$$

$$
\left[ \Theta_{\mu}^- (x) \right]_{x^- = L}^{x^- = -L} = \Theta_{\mu}^- (x) |_{x^- = L} - \Theta_{\mu}^- (x) |_{x^- = -L}. \tag{3.25}
$$

In what follows we refer to operators composed of $\tilde{\Sigma}$ as ‘physical’ operators and operators composed of $B$ and $C$ as ‘residual gauge operators’.

It can be shown that products of physical operators and residual gauge operators vanish, so that the physical operator parts and the residual gauge operator parts decouple in (3.24). This result reflects the fact that the physical operator parts are conserved by themselves. It can also be shown that the residual gauge operators that are not well-defined in the limit $L \to \infty$ cancel among themselves. After obtaining the decoupled and well-defined expressions, we take the limit $L \to \infty$. This limit enables us to discard the physical operator parts $\theta_{\mu}^- (x)$ of $\Theta_{\mu}^- (x)$, as is usually done when one calculates the conserved physical generators in the traditional axial gauge formulations. Finally, we take the limit $T \to \infty$. At this stage we assume that we can integrate the remaining residual gauge operators in $\Theta_{\mu}^- (x)$ by parts in the $x^+$ direction. This assumption is justified, because the residual gauge operator parts are conserved by themselves. As a consequence, we obtain

$$
0 = \int_{-\infty}^{\infty} dx^- \left[ \theta_{+}^+ \right]_{x^+ = -\infty}^{x^+ = \infty} - \int_{-\infty}^{\infty} dx^+ \left[ B \frac{1}{m^2 - n_- \partial_+^2} \partial_+ C \right]_{x^- = -\infty}^{x^- = \infty}, \tag{3.26}
$$

$$
0 = \int_{-\infty}^{\infty} dx^- \left[ \theta_{+}^+ \right]_{x^+ = \infty}^{x^+ = -\infty} + \frac{1}{2} \int_{-\infty}^{\infty} dx^+ \left[ B (x) \frac{n_-}{m^2 - n_- \partial_+^2} B (x) \right]_{x^- = -\infty}^{x^- = \infty}, \tag{3.27}
$$

where

$$
\theta_{+}^+ = \frac{n_+}{n_-} : \partial_- \tilde{\Sigma} \partial_+ \tilde{\Sigma} : - \frac{1}{2n_-} : \{ \left( \partial^+ \tilde{\Sigma} \right)^2 + \left( \partial_- \tilde{\Sigma} \right)^2 + m_0^2 (\tilde{\Sigma})^2 \} :,
$$

$$
\theta_{-}^+ = : \partial_- \tilde{\Sigma} \partial_+ \tilde{\Sigma} :, \tag{3.28}
$$

Fig. 1.
We see that the physical and residual gauge terms are conserved independently in (3.26) and (3.27). In fact, the second terms vanish trivially, because they are independent of \( x^- \), so that the first physical terms also vanish. It follows that the values \( \int_{-\infty}^{\infty} dx^- \theta_+^{\mu} \) are conserved. Furthermore, the integrands of the second terms of (3.26) and (3.27) are independent of \( x^- \), so that integrals of them with respect to \( x^+ \) over the whole interval \((-\infty, \infty)\) become constant and thus are conserved. Thus it is natural to take

\[
P_+ = \int_{-\infty}^{\infty} dx^- \theta_+^{\mu}(x) - \int_{-\infty}^{\infty} dx^+ B(x) \frac{1}{m^2 - n_- \partial_+^2} \partial_+ C(x), \tag{3.29}
\]

\[
P_- = \int_{-\infty}^{\infty} dx^- \theta_+^{\mu}(x) + \frac{1}{2} \int_{-\infty}^{\infty} dx^+ B(x) \frac{n_-}{m^2 - n_- \partial_+^2} B(x) \tag{3.30}
\]
as the conserved translational generators in the axial gauge formulation. It is easily shown that they are the correct ones in the axial gauge formulation. These consist of the physical operator part integrated over \( x^+ = \text{constant} \) and the residual gauge operator part integrated over \( x^- = \text{constant} \). By making use of the commutation relations (2.21) and (3.3), it is easy to show that the Heisenberg equations for \( A_+ \) and \( \psi^{(a)} \) hold. This shows that we have successfully constructed the extended Hamiltonian formulation of the axial gauge Schwinger model.

Finally, we discuss the charges in the axial gauge formulation. The current, being gauge invariant, is the same as in the temporal gauge:

\[
J^- = - \partial_+ \left( m \hat{\Sigma} + \frac{m^2}{m^2 - n_- \partial_+^2} \partial_+^{-1} B \right),
\]

\[
J^+ = \partial_- \left( m \hat{\Sigma} + \frac{m^2}{m^2 - n_- \partial_+^2} \partial_+^{-1} B \right) = m \partial_- \hat{\Sigma}. \tag{3.31}
\]

Using the fact that the current has zero divergence and also using the relation

\[
\frac{m^2}{m^2 - n_- \partial_+^2} B = B + \frac{n_- \partial_+^2}{m^2 - n_- \partial_+^2} B, \tag{3.32}
\]

we find the charge to be

\[
Q = - \int_{-\infty}^{\infty} dx^+ B(x) + \int_{-\infty}^{\infty} dx^- m \partial_- \hat{\Sigma}(x). \tag{3.33}
\]

In the temporal gauge formulation (and also in the light-cone gauge\(^9\) and Landau gauge\(^20\) solutions) we set the second term equal to zero. The justification for that is that \( \hat{\Sigma}(x) \) is a massive field, and since it has no infrared singularities, the term involving the integral over it commutes with the \( \psi \) field. The question is somewhat more complicated in the axial gauge case, since the singularity coming from \( \partial_+^{-1} \) causes the commutator of the second term in (3.33) with fermi field to become nonzero. On the other hand, we cannot keep both terms in (3.33), since in that case the fermi field would carry twice the correct charge. We believe that the correct charge operator is obtained from (3.33) by letting the second term provide vanishing
topological charges. In previous attempts to formulate the Coulomb gauge Schwinger model (using a representation space which is entirely physical), the fermion field operators were represented as solitons constructed solely out of the $\tilde{\Sigma}$, so that the $\tilde{\Sigma}$ tended to nonvanishing values in the limits $x^{-}\to\pm\infty$. That led, inevitably, to infrared difficulties. In our formulation, the fermion field operators are defined in the same manner as those in the temporal gauge formulation so that it is reasonable for them not to carry any topological charges. To realize this, we define the operator $(\partial_-)^{-1}$ as a finite integral,

$$\frac{1}{2} \int_{-L}^{L} \varepsilon(x^--y^-)f(y^-)dy^-,$$

and take the limit $L\to\infty$ after all relevant calculations are finished. Note that taking the limit $L\to\infty$ before all relevant calculations give the definition in (2.18) and that changing the order is justified by the fact that the infrared divergences are all eliminated. We therefore take the charge operator to be

$$Q = -\int_{-\infty}^{\infty} dx^+ B(x).$$

Similarly, we take the chiral charge to be $Q_5 = \int_{-\infty}^{\infty} dx^+ \partial_+ C(x)$.

§4. Concluding remarks

In this paper we have shown that $x^-$-independent residual gauge fields can be introduced as regulator fields in the pure space-like axial gauge formulation of the Schwinger model by extending the Hamiltonian formalism à la McCartor and Robertson. Because we do not have consistent pure space-like axial gauge quantization conditions, we have extrapolated the solution in the temporal gauge formulation and checked that it is also a solution in the axial gauge formulation. One conclusion to be drawn from this work involves the representation of the residual gauge fields: We have found that in the axial gauge we must choose the representation that specifies the ML prescription in order for the linear infrared divergences resulting from the operator $(\partial_-)^{-2}$ to be canceled. For this cancellation to take place, it is necessary for us to change integration variables to ones proportional to $n_+ = \sin 2\theta$. It follows from this that a pure space-like axial gauge formulation in ordinary coordinates (this is the case of $\theta = \frac{\pi}{2}$, which gives us the Coulomb gauge) has to be defined as the limit $\theta \to \frac{\pi}{2} - 0$. Regulating the solution in this way involves information off the initial value surface ($x^0 = 0$), but in any event, the solution includes auxiliary fields, and the representation space is of indefinite metric.

In contrast, the case $\theta = \frac{\pi}{4}$ (the light-cone axial gauge formulation) can be defined by simply setting $\theta = \frac{\pi}{4}$ (but the theory still has to be regulated by splitting the fermi products off the initial value surface). This is because, by virtue of the relation $n^2 = n_- = \cos 2\theta = 0$, we do not have the linear divergences resulting from $(\partial_-)^{-2}$ and from the square of the inverse of the hyperbolic Laplace operator, except for the contact term in the most singular component of the gauge field propagator. We have also found that in order for the equal $x^+$-time anticommutation relations to
be satisfied in the axial gauges, we have to alter the representation of the $\tilde{\Sigma}$ from the temporal one to the axial one and make a modification of the overall normalization of the fermion field operators.

In contrast to the axial gauges, it seems that in the temporal gauges we can use either the representation of the residual gauge fields which specifies the ML prescription or the one which specifies the principal value prescription. If we use the latter, then both components of the fermion field have vanishing vacuum expectation value.

Now that we have the operator solution in the axial gauge Schwinger model, we can calculate commutation relations of the $A_\mu$. It turns out that

$$[A_\mu(x), A_\nu(y)] = i\{-g_\mu\nu \Delta(x-y; m^2) + (n_\mu \partial_\nu + n_\nu \partial_\mu) \partial_- E(x-y) - n^2 \partial_\mu \partial_\nu E(x-y)\},$$

(4.1)

where $\Delta(x; m^2)$ is the commutator function of the free field of mass $m$, and

$$E(x) = \frac{1}{\partial_-^2} \Delta(x; m^2) - \frac{x^-}{m^2 - n_- \partial_-^2} \delta(x^+) + \frac{2n_+ \partial_+}{(m^2 - n_- \partial_-^2)^2} \delta(x^+).$$

(4.2)

It remains to be determined whether we can use (4.1) to obtain consistent pure space-like axial gauge quantization conditions, which are needed to quantize interacting pure space-like axial gauge fields. We leave this task for subsequent studies.

**Appendix A**

*Derivation of (3.16)*

In this appendix we give a detailed derivation of the $x^+$-ordered gauge field propagator

$$D_{\mu\nu}(q) = \int d^2x \langle 0| \{ \theta(x^+) A_\mu(x) A_\nu(0) + \theta(-x^+) A_\nu(0) A_\mu(x) \}|0\rangle e^{iq\cdot x}. \quad (A.1)$$

It is straightforward to show that the contributions from the $\tilde{\Sigma}$ and from the residual gauge fields are given, respectively, by

$$D^p_{++}(q) = \frac{i}{q^2 - m^2 + i\varepsilon} \frac{m^2}{q^2 - m^2 + i\varepsilon} \left(-g_{++} + \frac{2n_+ q_+}{q_-} - n^2 \frac{q_+^2}{q_-^2}\right) - \frac{i}{q_-^2},$$

$$D^q_{++}(q) = \int_0^\infty dk_+ \left[ \delta'(q_-) \frac{n_- k_+^2}{m^2 + n_- k_+^2} \left( \frac{i}{k_+ - q_- - i\varepsilon} - \frac{i}{k_+ + q_- + i\varepsilon} \right) \right.$$  

$$\left. + \delta(q_-) \frac{2n_+ k_+ m^2}{(n_- k_+^2 + m^2)^2} \left( \frac{i}{k_+ - q_- - i\varepsilon} + \frac{i}{k_+ + q_- + i\varepsilon} \right) \right]. \quad (A.2)$$

Here, we have made use of the fact that $q^2 = n_- q_-^2 + 2n_+ q_+ q_- - n_- q_+^2$ and that $n^2 = n_-$. Note that the explicit $x^-$ dependence gives rise to the factor $\delta'(q_-)$. Note also that there is no on-mass-shell condition for the residual gauge fields, so that there remains a $k_-$ integration. As a consequence, there arise singularities resulting from the inverse of the hyperbolic Laplace operator. It turns out that when we
regularize the singularities using principal value regularization, the integral on the second line of (A.2) is well-defined. In fact we can rewrite its integrand as a sum of simple poles:

\[
\frac{n_{-}k_{+}^2}{m^2 + n_{-}k_{+}^2} \left( \frac{i}{k_{+} - q_{+} - i\varepsilon} - \frac{i}{k_{+} + q_{+} - i\varepsilon} \right)
\]

\[
= \frac{n_{-}q_{+}^2}{m^2 + n_{-}q_{+}^2} \left( \frac{i}{k_{+} - q_{+} - i\varepsilon} - \frac{i}{k_{+} + q_{+} - i\varepsilon} \right)
\]

\[- \frac{n_{-}a}{m^2 + n_{-}q_{+}^2} \left( \frac{i}{k_{+} - a} - \frac{i}{k_{+} + a} \right), \quad (A.3)
\]

where \( a = \frac{m}{\sqrt{-n_{-}}} \). Direct calculation then gives

\[
\int_{0}^{\infty} dk_{+} \left( \frac{i}{k_{+} - q_{+} - i\varepsilon} - \frac{i}{k_{+} + q_{+} - i\varepsilon} \right) = -\pi \text{sgn}(q_{+}), \quad (A.4)
\]

\[
P \int_{0}^{\infty} dk_{+} \left( \frac{1}{k_{+} - a} - \frac{1}{k_{+} + a} \right) = 0, \quad (A.5)
\]

where \( \text{sgn}(q_{+}) \) is obtained because the \( k_{+} \) integration region is limited to \((0, \infty)\). On the other hand, the integral on the third line of (A.2) yields a linear divergence. We can see this by rewriting the integrand as a sum of simple and double poles:

\[
\frac{2n_{+}k_{+}m^2}{(m^2 + n_{-}k_{+}^2)^2} \left( \frac{i}{k_{+} - q_{+} - i\varepsilon} + \frac{i}{k_{+} + q_{+} - i\varepsilon} \right)
\]

\[
= \frac{2n_{+}q_{+}m^2}{(m^2 + n_{-}q_{+}^2)^2} \times \left( \frac{i}{k_{+} - q_{+} - i\varepsilon} - \frac{i}{k_{+} + q_{+} - i\varepsilon} \right)
\]

\[+ n_{+}a \frac{n_{-}q_{+}^2 - m^2}{(m^2 + n_{-}q_{+}^2)^2} \left( \frac{i}{k_{+} - a} - \frac{i}{k_{+} + a} \right)
\]

\[- \frac{n_{+}m^2}{n_{-}m^2 + n_{-}q_{+}^2} \left( \frac{i}{(k_{+} - a)^2} + \frac{i}{(k_{+} + a)^2} \right), \quad (A.6)
\]

We can evaluate the integrals of the first and second terms on the right-hand side with the help of (A.4) and (A.5). However, we cannot regularize the linear divergence resulting from the double pole by the principal value prescription. We show below that this linear divergence cancels that resulting from the factor \((q_{-}^2)^{-1}\) of \( D_{++}^{q_{+}} \) in (A.2). For later convenience we rewrite the linearly divergent integration in the form

\[
P \int_{0}^{\infty} dk_{+} \left( \frac{n_{+}}{(k_{+} - a)^2} + \frac{n_{+}}{(k_{+} + a)^2} \right) = P \int_{-\infty}^{\infty} dk_{+} \frac{n_{+}}{(k_{+} - a)^2} = -\int_{-\infty}^{\infty} dq_{-} \frac{n_{-}}{q_{-}^2} \delta(q_{-}), \quad (A.7)
\]

where we have changed the integration variable from \( k_{+} \) to \( q_{-} = \frac{-n_{-}^2}{n_{+}}(k_{+} - a) \). Substituting (A.4), (A.5) and (A.7) into (A.2) yields

\[
D_{++}^{q_{+}}(q) = \frac{i}{q^2 - m^2 + i\varepsilon} \left( -in_{+}^2q_{+}^2\pi\text{sgn}(q_{+})\delta'(q_{-}) - 2in_{+}q_{+}\delta(q_{-})\pi\text{sgn}(q_{+}) \right.
\]

\[-m^2\delta(q_{-}) \int_{-\infty}^{\infty} dq_{-} \frac{1}{q_{-}^2} \right), \quad (A.8)
\]
where we have made use of the identity
\[ \frac{1}{q^2 - m^2 + i\varepsilon} \delta'(q_-) = - \frac{1}{m^2 + n_q^+} \delta'(q_-) + \frac{2n_q^+}{(m^2 + n_q^+)^2} \delta(q_-). \] (A.9)

Thus, for the sum of $D^\mu_+(q)$ and (A.8), we obtain
\[ D_{++}(q) = \frac{i}{q^2 - m^2 + i\varepsilon} \left( - g_{++} + \frac{2n_q^+}{q_- + i\varepsilon \text{sgn}(q_+)} \frac{1}{m^2 + n_q^+} - in^2 q_+^2 \pi \text{sgn}(q_+) \delta'(q_-) \right. \\
- n^2 q_+^2 \left( \frac{1}{q_-^2} - \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \right) - i \left( \frac{1}{q_-^2} - \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \right), \] (A.10)

where we have made use of the identity
\[ \delta(q_-) \frac{q_-^2 - m^2 + i\varepsilon}{q^2 - m^2 + i\varepsilon} = \delta(q_-) \left( 1 + \frac{n^2 q_+^2}{q^2 - m^2 + i\varepsilon} \right). \] (A.11)

Now we show that the term $\frac{1}{q_-^2} - \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2}$ does not give rise to any divergences when we calculate $D_{++}(x)$ by substituting (A.10) into (A.1). To show this we consider the integral
\[ \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \delta(q_-) \left( \frac{1}{q_-^2} - \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \right) e^{-iq_x x^-} = \int_{-\infty}^{\infty} dq_- \frac{e^{-iq_x x^-}}{q_-^2 (q^2 - m^2 + i\varepsilon)} + \frac{1}{m^2 + n_q^+} \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2}. \] (A.12)

We can rewrite the second line further as
\[ \int_{-\infty}^{\infty} dq_- \frac{e^{-iq_x x^-} - 1}{q_-^2} \frac{1}{q^2 - m^2 + i\varepsilon} + \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} \left( \frac{1}{m^2 + n_q^+} + \frac{1}{q^2 - m^2 + i\varepsilon} \right) \right. \\
= \int_{-\infty}^{\infty} dq_- \left( \frac{e^{-iq_x x^-} - 1}{q_-^2} \frac{1}{q^2 - m^2 + i\varepsilon} + \frac{2n_q^+ + n_q^+}{q_- (m^2 + n_q^+)^2} \frac{1}{q^2 - m^2 + i\varepsilon} \right). \] (A.13)

We see that the last integral diverges at most logarithmically. However, because logarithmic divergences can be regularized by the principal value prescription, there arise no divergences in (A.13). This verifies that the identity
\[ \frac{1}{q_-^2} + i\pi \text{sgn}(q_+) \delta'(q_-) - \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} = Pf \frac{1}{q_-^2} + i\pi \text{sgn}(q_+) \delta'(q_-) = \frac{1}{(q_- + i\varepsilon \text{sgn}(q_+))^2} \] (A.14)

holds, where $Pf$ denotes Hadamard’s finite part. It should also be noted that
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dq_- \left( \delta(q_-) \int_{-\infty}^{\infty} dq_- \frac{1}{q_-^2} - \frac{1}{q_-^2} \right) e^{-q_x x^-} = \frac{1}{\pi} \int_{0}^{\infty} dq_- \frac{1 - \cos q_x x^-}{q^2} = \frac{|x^-|}{2}, \] (A.15)
where the Fourier transform of the last term is $-\frac{1}{2} \left( \frac{1}{(q_- + i\epsilon)^2} + \frac{1}{(q_- - i\epsilon)^2} \right)$. Thus we obtain

$$D_{++}(q) = \frac{i}{q^2 - m^2 + i\epsilon} \left( -g_{++} + \frac{2n_+q_+}{q_- + i\epsilon \text{sgn}(q_+)} - \frac{n_+^2q_+^2}{(q_- + i\epsilon \text{sgn}(q_+))^2} \right)$$

$$- \frac{1}{2} \left( \frac{1}{(q_- + i\epsilon)^2} + \frac{1}{(q_- - i\epsilon)^2} \right).$$

(A-16)

This completes the demonstration that, due to the residual gauge fields, the linear divergence resulting from $\partial_+^{-1}$ is eliminated from the $x^+$-ordered gauge field propagator.

References

2) L. Susskind, Phys. Rev. 165 (1968), 1535.