Off-Shell Formulation of Supergravity on an Orbifold

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An off-shell formulation is given for the supersymmetry in singular spaces that has recently been developed in an on-shell formalism by Bergshoeff, Kallosh and Van Proeyen using a supersymmetry singlet ‘coupling constant’ field and a 4-form multiplier field in five-dimensional space-time. We present this formulation for a general supergravity-Yang-Mills-hypermultiplet coupled system compactified on an orbifold $S^1/Z_2$. Relations between the bulk cosmological constant and brane tensions of the boundary planes are discussed.

§1. Introduction

Concerning the supersymmetrization of the Randall-Sundrum scenario1) on the orbifold $S^1/Z_2$, there have appeared two distinct approaches, one due to Altendorfer, Bagger and Nemeschansky,2) and the other due to Gherghetta and Pomarol3) and Falkowski, Lalak and Pokorsky.4) The difference between these approaches is that the signs of the $U(1)_R$ gauge coupling constant $g$ and the gravitino ‘mass term’ change across the branes in the latter approach, while they do not in the former approach. (Hence the brane in the latter approach resembles the thin limit of the domain wall.) Only in the latter approach can the supersymmetry requirement give relations between the cosmological constant in the bulk space and the brane tensions of the two boundary planes. These relations are identical to the relations that are required for the existence of the Randall-Sundrum warp solution.1) Moreover, it is believed that the latter case appears from the heterotic M-theory on $S^1/Z_2$5) after the reduction to five dimensions by compactifying on a Calabi-Yau 3-fold.6)

In a paper entitled “Supersymmetry in Singular Spaces,” Bergshoeff, Kallosh and Van Proeyen7) (BKVP) gave an interesting formulation for realizing ‘dynamically’ this situation in which the sign of the coupling constant changes across a brane. Specifically, they considered the Maxwell/Einstein gauged supergravity system in five dimensions8) and lifted the gauge coupling constant $g$ of $U(1)_R$ to a supersymmetry singlet field $G(x)$. Then, introducing a 4-form gauge field $H_{\mu\nu\rho\sigma}$ also, they succeeded in constructing a supersymmetric action for the system on an $S^1/Z_2$ orbifold and realized a coupling ‘constant’ whose sign changes, $G(y) = g\epsilon(y)$, as the solution of the equation of motion.

However, their construction is heuristic and is presented only in an on-shell formulation for the pure Maxwell/Einstein gauged supergravity system. It is thus unclear how it changes when the system is varied. The purpose of this paper is,

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therefore, to give an off-shell generalization of the BKVP formulation. In our formulation, the coupling field $G(x)$ appears as the ratio of the scalar component fields of two vector multiplets, and the 4-form gauge field $H_{\mu\nu\rho\sigma}$ is included essentially as a dual of the scalar component field of a linear multiplet.

This paper is organized as follows. In §2 we briefly explain the form of the invariant off-shell action for the general supergravity-Yang-Mills-hypermultiplet system in 5D,\(^9\) which we consider in this paper. Next, in §3, we present a new form of the linear multiplet in which the constrained vector and auxiliary scalar components are rewritten in terms of 3-form and 4-form gauge fields, respectively. Such a linear multiplet was briefly considered in Ref. 10), but we present the details here for the first time. Using this new form of the linear multiplet as a supermultiplet containing BKVP’s 4-form gauge field $H_{\mu\nu\rho\sigma}$, we give an off-shell version of the 4-form gauge field in a 5D bulk. In §4 we discuss the compactification of the system on the orbifold $S^1/Z_2$ and construct a brane action that again gives an off-shell generalization of BKVP’s brane action. In §5, we discuss the relation between the cosmological constant in the bulk and the brane tensions of the boundary planes, based on the obtained action. We show that various results obtained by previous authors are obtained from our general results by reducing it to simpler systems. Section 6 is devoted to discussion. Some technical points are treated in the Appendix concerning the parametrizations of the target manifold $U(2,q)/U(2) \times U(q)$ of the hypermultiplet scalar fields and the non-linear Lagrangian.

§2. Supergravity action in a five-dimensional bulk

The invariant action for a general system of Yang-Mills and hypermultiplet matter coupled to supergravity in an off-shell formulation was first obtained in Ref. 9) (which we refer to as ‘I’ henceforth) on the basis of the super Poincaré tensor calculus given in Ref. 11). However, the calculation given there is very tedious because there is no conformal $S$-supersymmetry. Weyl multiplets in 5D conformal supergravity were constructed very recently by Bergshoeff et al.,\(^{12}\) and the full superconformal tensor calculus was presented by Fujita and Ohashi in Ref. 10), which we refer to as II, where it was explained how easily the result of I can be rederived using superconformal tensor calculus. We here, therefore, follow the technique developed in II.

We here consider a system of $n + 1$ vector multiplets $V^I$ ($I = 0, 1, 2, \ldots, n$) of some gauge group $G$ and $r$ hypermultiplets $H^\alpha$ ($\alpha = 1, 2, \ldots, 2r$), which give a certain representation of $G$ with representation matrix $(gt_I)^\alpha_\beta$. The field contents of the Weyl multiplet, vector multiplet and hypermultiplet are listed in Table I.

In the superconformal framework, the action obtained in I results if we fix the extraneous gauge freedoms of dilatation $D$, conformal supersymmetry $S$ and special conformal-boost $K$ symmetries by the conditions\(^{10}\)

$$D: \mathcal{N} = 1, \quad S: \Omega^I I_N I = 0, \quad K: \hat{D}_\alpha \mathcal{N} = 0, \quad (2.1)$$

where $\mathcal{N}(M)$ is the homogeneous cubic function of the scalar component fields $M^I$ of the vector multiplets $V^I$, which uniquely characterizes the vector part action of
Table I. Field contents of the multiplets.

<table>
<thead>
<tr>
<th>field</th>
<th>type</th>
<th>restrictions</th>
<th>SU(2)</th>
<th>Weyl-weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{\mu}^{a}$</td>
<td>boson</td>
<td>fünfbein</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\psi_{\mu}$</td>
<td>fermion</td>
<td>$SU(2)$-Majorana</td>
<td>2</td>
<td>$-\frac{1}{2}$</td>
</tr>
<tr>
<td>$b_{\mu}$</td>
<td>boson</td>
<td>real</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$V_{\mu}^{ij}$</td>
<td>boson</td>
<td>$V_{\mu}^{ij} = (V_{\mu}^{ji})^{*}$</td>
<td>3</td>
<td>0</td>
</tr>
<tr>
<td>$v_{ab}$</td>
<td>boson</td>
<td>real, antisymmetric</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\chi^{i}$</td>
<td>fermion</td>
<td>$SU(2)$-Majorana</td>
<td>2</td>
<td>$\frac{3}{2}$</td>
</tr>
<tr>
<td>$D$</td>
<td>boson</td>
<td>real</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

The system. We use notation like $N_{I} \equiv \partial N / \partial M^I$, $N_{IJ} \equiv \partial N / \partial M^I \partial M^J$, etc., and $\hat{D}_\mu$ represents the full superconformal covariant derivative. The $Q$ supersymmetry transformation that preserves these gauge conditions is given by the combination of $Q$, $S$ and $K$ as

$$
\tilde{\delta}_Q(\varepsilon) = \delta_Q(\varepsilon) + \delta_S(\eta^{I}(\varepsilon)) + \delta_K(\xi^\alpha(\varepsilon)),
$$

$$
\eta^{I}(\varepsilon) = \frac{N_I}{12\pi} \gamma^i \cdot \hat{F}^{I}(W) \varepsilon^i + \frac{N_{IJ}}{3N} Y^{IJ} j^j + \frac{N_{I}}{3N} \Omega^{Ij} (2i \bar{\varepsilon} \Omega^I),
$$

omitting the unimportant parameter $\xi^\alpha_K(\varepsilon)$. The resultant $Q$ transformation laws of the Weyl multiplet, the vector multiplet and the hypermultiplet are completely the same as (I 6-8), (I 6-9) and (I 6-10), respectively, given in I, provided that the following translation rules are used (with the present notation, which is the same as that in II, on the LHS and that of I on the RHS):

$$
V_{\mu}^{ij} \leftrightarrow \tilde{V}_{\mu}^{ij}, \quad v_{ab} \leftrightarrow \tilde{v}_{ab}, \quad \frac{N_{I}}{3N} Y^{Iij} \leftrightarrow -\tilde{r}^{ij},
$$

$$
\chi^{i} \leftrightarrow 16 \bar{\chi}^{i} + 3\gamma^{i} \cdot \hat{R}^{i}(Q), \quad D \leftrightarrow 8\bar{C} - \frac{3}{2} \hat{R}(M) + 2\bar{v}^{2},
$$

$$
\Omega \leftrightarrow \lambda, \quad \zeta^\alpha \leftrightarrow \xi^\alpha, \quad Y^{Iij} \leftrightarrow \tilde{Y}^{Iij} - M^{Ij} \tilde{t}^{ij}.
$$

We therefore do not give these $Q$ transformation rules here, but simply cite the explicit form of the action for the present supergravity-Yang-Mills-hypermultiplet system:

$$
\mathcal{L}_0 = \mathcal{L}_{\text{hyper}} + \mathcal{L}_{\text{vector}} + \mathcal{L}_{\text{C-S}} + \mathcal{L}_{\text{aux}},
$$

$$
e^{-1} \mathcal{L}_{\text{hyper}} = \nabla^{a} A_{a}^{\alpha} \nabla_{\alpha} A_{a}^{i} - 2i \bar{\zeta}^{\alpha} (\bar{\nabla} + gM) \zeta_{\alpha}.
$$
\[
+ \mathcal{A}_{i}^{\alpha}(gM)^{2}a^{\beta}A_{\beta}^{i} - 4i\bar{\psi}_{a}^{i}b^{a}c^{a}\zeta_{a}\nabla_{b}A_{i}^{\alpha} - 2i\bar{\psi}_{a}^{(i\gamma \alpha \beta \gamma)}A_{j}^{\beta}A_{\alpha i}^{\gamma}bA_{a i}
\]
\[
+ \mathcal{A}_{i}^{\alpha}(8ig\bar{\Omega}_{\alpha i}^{\beta}b^{\beta} - 4ig\bar{\psi}_{a}^{i}c^{a}M_{\alpha \beta}b^{\beta})
\]
\[
+ 4ig\bar{\psi}_{a}^{(i\gamma \beta \gamma \Omega)}A_{j}^{\beta} - 2ig\bar{\psi}_{a}^{i\gamma \beta \gamma \Omega_{i}^{j}M_{\alpha \beta}A_{j}^{\beta}}
\]
\[
e^{-1}L_{\text{vector}} = -\frac{1}{2}R(\omega) - i\bar{\psi}_{a}^{i}b\mu \nu \rho \psi_{\rho} + (\bar{\psi}_{a}^{i}b)(\bar{\psi}_{a}^{i\gamma \beta \gamma \Omega})
\]
\[
- N_{I} \left( ig[\bar{\Omega}, \Omega] I - i \bar{\psi}_{a}^{i\gamma \beta \gamma \Omega}A_{d}^{\beta}F_{a b}^{\beta}(W) I \right)
\]
\[
+ a_{I J} \left( -\frac{1}{4}F(W)^{I} \cdot F(W) J + \frac{1}{2}\nabla A M \cdot \nabla A M \right)
\]
\[
+ 2i\bar{\Omega}_{I}^{\gamma}b^{\gamma}F(W) J + i\bar{\psi}_{a}^{i}b(\gamma F(W) - 2\nabla M)^{I} \gamma \Omega_{j}^{J}
\]
\[
- 2(\bar{\Omega}_{I}^{\gamma}b^{\gamma}c^{a}b^{a})(\bar{\psi}_{a}^{i}b\gamma c^{J}(W) + 2(\bar{\Omega}_{I}^{\gamma}b^{\gamma}b^{a})(\bar{\psi}_{a}^{i}b\Omega_{J})
\]
\[
- N_{I J K} \left( -i\bar{\Omega}_{I}^{\gamma}b^{\gamma}F(W) K + \frac{2}{3}(\bar{\Omega}_{I}^{\gamma}b^{\gamma}c^{j}(W)\bar{\psi}_{a}^{i}b\gamma (W) + \frac{2}{3}(\bar{\psi}_{a}^{i}b\gamma (W))(\bar{\Omega}_{I}^{\gamma}b\gamma (W))
\]
\[
+ \frac{1}{8} \left( 2\bar{\psi}_{a}^{i}b + \bar{\zeta}_{a}b^{ab}c^{a} + a_{I J} \Omega_{I}^{\gamma}b^{a}c^{a} \right)^{2}
\]
\[
+ i\frac{1}{4}N_{I} F(W) I \left( 2\bar{\psi}_{a}^{i}b + \bar{\zeta}_{a}b^{ab}c^{a} + a_{I J} \Omega_{I}^{\gamma}b^{a}c^{a} \right)
\]
\[
+ \left( A_{i}^{\alpha}b\nabla_{a}A_{j}^{\alpha} + i a_{I J} \Omega_{I}^{\gamma}b\gamma (W) \right)^{2},
\]
\[
\mathcal{L}_{C S} = \frac{1}{8} e_{I J K} \epsilon^{\mu \nu \rho \sigma} W_{L} \left( F_{\mu \nu}^{I} b W_{\rho}^{K}(W) + \frac{1}{2} g[W_{\mu}, W_{\nu}]^{I} F_{\rho \sigma}^{K}(W)
\]
\[
+ \frac{1}{10} g^{2}[W_{\mu}, W_{\nu}]^{I} [W_{\rho}, W_{\sigma}]^{K} \right). \tag{2.4}
\]

Here \(\nabla\mu\) represents the derivative covariant only with respect to local-Lorentz and group transformations, and the metric \(a_{I J}\) in the vector multiplet kinetic term is
\[
a_{I J} \equiv -\frac{1}{2} \left( \frac{\partial^{2}}{\partial M^{I} \partial M^{J}} \ln N \right) = -\frac{1}{2N} \left( N_{I J} - N_{I} N_{J} \right). \tag{2.5}
\]

The barred group index \(\bar{\alpha}\) of the hypermultiplet is defined to be \(A_{i}^{\bar{\alpha}} \equiv A_{i}^{\beta}d_{\beta}^{\alpha}\) by using the hypermultiplet metric matrix \(d_{\alpha}^{\beta}\), which is given in standard form \(^{13}\) as
\[
d_{\alpha}^{\beta} = \begin{pmatrix}
12p \\
-12q
\end{pmatrix}, \quad p + q = r. \tag{2.6}
\]

Since this implies \(A_{i}^{\bar{\alpha}} A_{i}^{\alpha} = -\sum_{\alpha=1}^{2p} |A_{i}^{\alpha}|^{2} + \sum_{\alpha=2p+1}^{2(p+q)} |A_{i}^{\alpha}|^{2}\), the first \(2p\) components of the hypermultiplet carry a negative metric and are called a ‘compensator;’ which will be eliminated eventually by the gauge-fixing of suitable gauge symmetries. We consider only \(p = 1\) and \(p = 2\) cases explicitly in this paper.

The last part of the Lagrangian, \(\mathcal{L}_{aux}\), consists of the terms of the auxiliary fields which are written almost in the perfect square forms and vanish on shell, aside from the \(Y_{ij}^{I}\)-terms, to which additional contributions will appear later:
\[
e^{-1}L_{aux} = D'(A^{2}) + 8i\bar{\psi}_{a}^{i}b\Omega_{i}^{\alpha}c_{\alpha} + (Y_{ij}^{I}-\text{terms})
\]
\[
+ 2(v - v_{sol})^{a b}(v - v_{sol})_{a b} + (V_{I} - V_{sol})^{ij}(V_{I} - V_{sol})_{ij}
\]
\[ (Y_{ij}^I - \text{terms}) = -\frac{1}{2} N_{IJK} Y_{ij}^I Y_{ij}^J + Y_{ij}^I Y_{ij}^J, \]

where

\[ Y_{ij}^j = 2 A^{ij} (g_{ij}) \bar{A}^i J_{ij} + i N_{IJK} \bar{f}_{ij} Q_{ij} \]

\[ D' \equiv \frac{1}{8} D + \frac{3}{16} \bar{R} (M) - \frac{i_4 v^2 - i_3 \bar{v} \cdot \gamma \chi + 3 \bar{i} \cdot \bar{\gamma} \cdot \bar{R} (Q) + i \bar{v}_a \gamma_{i j} \bar{E}_{i j} \bar{v}_b, \]

\[ \chi_i \equiv \frac{1}{16} \gamma_i \gamma_i \bar{R} (Q) - \frac{1}{2} \gamma^a \Gamma_{i j} \bar{v}_a, \]

\[ \Gamma \equiv -\frac{1}{3} \varepsilon^i (\varepsilon) - \gamma^a V_{a j} \varepsilon^j + \gamma \cdot v \varepsilon^j. \]

The quantity \( \eta^i (\varepsilon) \) here is defined in Eq. (2.2). We have omitted the expressions for \( V_{\text{sol}}^{ij}, v_{\text{sol} a} \) and \( F_\alpha \), which are the same as those given in I, as those are not needed below.

§3. The invariant action for the four-form gauge field in the 5D bulk

A linear multiplet \( L \) consists of an \( SU(2) \) triplet boson \( L^{ij} \), an \( SU(2) \)-Majorana spinor \( \varphi^i \), a constrained vector \( E^a \), and a real auxiliary scalar \( N \). An invariant action formula exists for a pair of an Abelian vector multiplet \( V = (M, W_\mu, \Omega^i, Y^{ij}) \) and a linear multiplet \( L = (L^{ij}, \varphi^i, E^a, N) \), which is neutral or charged under the Abelian group of the vector multiplet \( V \).\(^{10,11} \)

\[ e^{-1} L_{VL} (V, L) = Y_{ij} L^{ij} + 2 i \Omega \varphi + 2 i \bar{v}_i \gamma_{a} \Omega J_{ij} \]

\[ - \frac{1}{2} W_a \left( E^a - 2 i \bar{v}_b \gamma_{a b} \varphi + 2 i \bar{v}_i (i_{a b c} \gamma_{j} \bar{v}_j) L_{ij} \right) \]

\[ + \frac{1}{2} \left( N - 2 i \bar{v}_b \gamma_{b} \varphi - 2 i \bar{v}_i (i_{a b c} \gamma_{j} \bar{v}_j) L_{ij} \right). \]

Now we consider the cases in which the linear multiplet \( L \) is neutral. It was observed in II that this Lagrangian density can be consistently written as the total derivative form of a 4-form field \( H_{\mu \nu \rho \sigma} \)

\[ 2 L_{VL} (V, L) = - \frac{1}{4} \varepsilon^{abcd} \hat{F}_{abcd} (E), \]

where \( E_{\mu \nu \rho} \) is an unconstrained 3-form field with which the ‘divergenceless’ constraint on \( E^a \) is solved in the form\(^{11} \)

\[ E^a = \frac{1}{4} \varepsilon^{abcd} \hat{F}_{abcd} (E), \]

\[ \hat{F}_{\mu \nu \rho \sigma} (E) = 4 \partial_{[\mu} E_{\nu \rho \sigma]} + 8 i \bar{v}_i \gamma_{\nu \rho \sigma} \varphi + 24 i \bar{v}_i [\gamma_{\nu \rho \sigma} L_{ij}. \]

Equation (3.2) can be equivalently rewritten as

\[ MN + 2 Y_{ij} L^{ij} + 4 i \bar{\Omega} \varphi = - \frac{1}{3} \varepsilon^{abcd} \hat{F}_{abcd} (H), \]

\(^{10} \) Note, however, that we are not claiming that the action \( \int d^5 x L_{VL} (V, L) \) always vanishes. If the action \( \int d^5 x L_{VL} (V, L) \) is nonzero, it is merely implied that the 4-form field \( H_{\mu \nu \rho \sigma} \) does not vanish at infinity, so that the surface term remains finite.
using the covariant field strength \( \tilde{F}_{abced}(H) \) of the 4-form field \( H_{\mu
u\rho\sigma} \),

\[
\tilde{F}_{\lambda\mu\nu\rho\sigma}(H) = 5\partial[\lambda H_{\mu\nu\rho\sigma}] - 10F[\lambda\mu(W)E_{\nu\rho\sigma}] - 10i\bar{\psi}[\lambda\gamma_{\mu\nu\rho\sigma}]M\varphi \\
+ 20i\bar{\psi}[\lambda\gamma_{\mu\nu\rho\sigma}]Q^jL_{ij} - 40i\bar{\psi}[\lambda\gamma_{\mu\nu\rho\sigma}]\psi[\sigma]ML_{ij}.
\] (3.6)

Note the parallelism between Eqs. (3.3) and (3.5). Since the 4-form gauge field \( H_{\mu\nu\rho\sigma} \) has \( \left( \frac{4}{4} \right) = 1 \) degree of freedom in 5D, it can be regarded as a replacement of the scalar component \( N \) of the linear multiplet \( L \), just as the 3-form gauge field \( E_{\mu\nu\rho} \) (possessing \( \left( \frac{4}{3} \right) = 4 \) degrees of freedom in 5D) is the replacement of the constrained vector component \( E^a \) of \( L \). Therefore, the linear multiplet \( L = (L^{ij}, \varphi^i, E^a, N) \) can now be expressed as \( L = (L^{ij}, \varphi^i, E_{\mu\nu\rho}, H_{\mu\nu\rho\sigma}) \) by using the 3-form and 4-form gauge fields. The transformation \( \delta \equiv \delta Q(\varepsilon) + \delta S(\eta) + \delta E(\Lambda_{\mu\nu}) + \delta H(\Lambda_{\mu\nu\lambda}) \) of these gauge fields is given by

\[
\delta E_{\mu\nu\lambda} = 3\partial[\mu A_{\nu\lambda}] - 2i\bar{\varepsilon}\gamma_{\mu\nu\lambda}\varphi - 12i\bar{\psi}[\mu\nu\lambda]\psi^jL_{ij}, \\
\delta H_{\mu\nu\rho\sigma} = 4\partial[\mu A_{\nu\rho\sigma}] + 6F[\mu\nu(W)A_{\rho\sigma}] + 2i\bar{\varepsilon}\gamma_{\mu\nu\rho\sigma}M\varphi \\
- 4i\bar{\varepsilon}\gamma_{\mu\nu\rho\sigma}Q^jL_{ij} + 16i\bar{\varepsilon}\gamma_{\mu\nu\rho\sigma}\psi[\sigma]ML_{ij} + 4(\delta Q(\varepsilon)W_{[\mu]}E_{\nu\rho\sigma}].
\] (3.7)

It should, however, be kept in mind that this rewriting of the component field \( N \) in terms of the 4-form gauge field \( H_{\mu\nu\rho\sigma} \) is performed on the ‘background’ not only of the Weyl multiplet but also of the vector multiplet \( V \); that is, the \( H_{\mu\nu\rho\sigma} \) component of \( L \) depends on \( V \), and we use the notation \( H_{\mu\nu\rho\sigma}|V \) and

\[
L|V = (L^{ij}, \varphi^i, E_{\mu\nu\rho}, H_{\mu\nu\rho\sigma}|V)
\] (3.8)

to show explicitly the vector multiplet \( V \) used in this rewriting.

If we apply the invariant action formula \( \mathcal{L}_{VL}(V, L) \) in Eq. (3.1) using the vector multiplet \( V \) used in the rewriting \( N \rightarrow H_{\mu\nu\rho\sigma}|V \) of the linear multiplet \( L \), the action takes the total derivative form (3.2). However, we can use different vector multiplets, \( V_S \) and \( V_R \), for the former and the latter, respectively. Then the invariant action formula (3.1) gives the desired 4-form field action,

\[
\mathcal{L}_{4\text{-form}} = \mathcal{L}_{VL}(V_S, L|V_R) \\
= e\left((Y_{ij}^{ij} - GY_{ij}^{ij})L_{ij} + 2i(\bar{\varphi}^i - G\bar{\varphi}^i)L_{ij} + 2i\bar{\psi}_a^i\gamma_\alpha(\Omega^S_{ij} - G\Omega^R_{ij})L_{ij}\right) \\
- \frac{1}{4!}e^{\lambda\mu\nu\rho\sigma}\left\{(F_{\lambda\mu}(W_S) - GF_{\lambda\mu}(W_R))E_{\nu\rho\sigma} + \frac{1}{2}G\partial[\chi H_{\mu\nu\rho\sigma}\right\},
\] (3.9)

where \( G \equiv M_S/M_R \). This form is most easily obtained by rewriting as

\[
\mathcal{L}_{VL}(V_S, L|V_R) = \mathcal{L}_{VL}(V_S - GV_R, L|V_R) + G\mathcal{L}_{VL}(V_R, L|V_R).
\] (3.10)

Then, the first term can be calculated by the formula (3.1) substituting \( V_S - GV_R = (0, W_{Si} - GW_{Ri}, \Omega^S_{ij} - G\Omega^R_{ij}, Y^{ij}_S - GY^{ij}_R) \) and using Eq. (3.3), and the second term gives \( G \) times the total derivative form (3.2).
Equation (3.9) gives the desired 4-form field action, which gives an off-shell generalization of the corresponding action derived by BKVP. With this action, all the component fields of the linear multiplet $L_{[\mathcal{V}_R]}$ now play the role of Lagrange multipliers; in particular, the variation of the 4-form field $H_{\mu\nu\rho\sigma}$ constrains the field ratio $G(x) \equiv M_S(x)/M_R(x)$ to be a constant, which plays the role of a coupling constant $g_R$:

$$\partial_\mu G(x) = 0 \rightarrow G(x) = g_R \text{ (constant)}. \quad (3.11)$$

The other equations given by the variation of other components read

$$Y^{ij}_S = GY^{ij}_R, \quad \Omega_S j = G\Omega_R j, \quad F_{\lambda\mu}(W_S) = GF_{\lambda\mu}(W_R). \quad (3.12)$$

We now explain how the gauged $U(1)_R$ supergravity is constructed with these two vector multiplets $\mathcal{V}_S$ and $\mathcal{V}_R$ in the present formulation. We identify the vector multiplet $\mathcal{V}_R$ with the $U(1)_R$ gauge multiplet, which is generally given by a linear combination of the (Abelian) vector multiplets $\mathcal{V}^I$:

$$\mathcal{V}_R \equiv V_I \mathcal{V}^I, \quad \text{i.e.} \quad (M_R, W_{R\mu}, \cdots) = (V_I M^I, V_I W^I_{\mu}, \cdots). \quad (3.13)$$

We use the index $I$ to denote all the vector multiplets other than $\mathcal{V}_S$. In I, in which no $\mathcal{V}_S$ appears, the $U(1)_R$ gauge multiplet $\mathcal{V}_R \equiv V_I \mathcal{V}^I$ was made to couple only to the hypermultiplet compensator in the form

$$\mathcal{D}_\mu \mathcal{A}^a_i = \partial_\mu \mathcal{A}^a_i - V_{\mu ij} \mathcal{A}^a_j - g_R W_{R\mu}(i\vec{q} \cdot \vec{\sigma})^a_b \mathcal{A}^b_i, \quad (3.14)$$

where the indices $a$ and $b$ each take the values 1 and 2, the first two values of $\alpha = 1, 2, \cdots$, corresponding to the first compensator, and $\vec{q}$ is an arbitrary constant isovector of unit length, $|\vec{q}|^2 = 1$. Then, after the $SU(2)$ gauge of the index $i$ is fixed by the condition $\mathcal{A}^a_i \propto \delta^a_i$, the kinetic term $-|\mathcal{D}_\mu \mathcal{A}^a_i|^2$ of the compensator $\mathcal{A}^a_i$ gives a term quadratic in $V^N_{\mu ij} = V^\mu_{ij} - g_R W_{R\mu}(i\vec{q} \cdot \vec{\sigma})^{ij}$, so that the auxiliary $SU(2)$ gauge field $V^\mu_{ij}$ contained in the covariant derivative $\mathcal{D}_\mu$ in all the other places become replaced by $V^N_{\mu ij} + g_R W_{R\mu}(i\vec{q} \cdot \vec{\sigma})^{ij}$, thus yielding a universal coupling of $W_{R\mu}$ to the $U(1)_R$ subgroup of $SU(2)$ with index $i$. (See I for further details.)

Here, on the other hand, $\mathcal{V}_S$ eventually becomes the $U(1)_R$ gauge multiplet $\mathcal{V}_R$ times the ‘coupling constant’ $G = g_R$ by the equations of motion (3.12), and thus the $\mathcal{V}_S$ may be called a ‘pre-$U(1)_R$ gauge multiplet’. In the present formulation, therefore, it is this pre-$U(1)_R$ gauge multiplet $\mathcal{V}_S$ that we make couple to the hypermultiplet compensator at the beginning. Explicitly, we have

$$\mathcal{D}_\mu \mathcal{A}^a_i = \partial_\mu \mathcal{A}^a_i - V_{\mu ij} \mathcal{A}^a_j - W_{S\mu}(i\vec{q} \cdot \vec{\sigma})^a_b \mathcal{A}^b_i, \quad (3.15)$$

which reduces to the previous Eq. (3.14) after the equations of motion (3.12) are used. We assume that the pre-$U(1)_R$ gauge multiplet $\mathcal{V}_S$ does not have its own kinetic term; that is, $M_S$ is not contained in $\mathcal{N}(M)$. The kinetic term of the $U(1)_R$ gauge multiplet, of course, exists when $\det \mathcal{N}_{IJ} \neq 0$, which we assume throughout this paper.
§4. Compactifying on orbifold and the brane action

We now compactify the fifth direction, \( y \equiv x^4 \), on the orbifold \( S^1/Z_2 \), the two fixed planes of which are placed at \( y = 0 \) and \( y = \pi R \equiv \tilde{y} \). We must first know the properties of the fields under the \( Z_2 \) parity transformation, \( y \to -y \). The parity quantum number \( \Pi \) is defined by

\[
\Phi(-y) = \Pi(\Phi) \Phi(y) \quad (4.1)
\]

for boson fields \( \Phi \), and, as discussed by BKVP,\(^7\) by

\[
\psi^i(-y) = \Pi(\psi) \gamma_5 M^i_j \psi^j(y), \quad (\bar{\psi}^i(-y) = \Pi(\psi) M^i_j \bar{\psi}^j(y) \gamma_5), \quad (4.2)
\]

for \( SU(2) \)-Majorana spinor fermions \( \psi^i \) (\( i = 1, 2 \)). Consistency with the reality condition \( \bar{\psi}^i = (\psi_i)^\dagger \gamma_0 = \psi_i^T C \) requires

\[
(M^i_j)^* = \epsilon_{ik} M^k_l \epsilon^{lj}, \quad \text{or} \quad M^* = -\sigma_2 M \sigma_2, \quad (4.3)
\]

and we can take

\[
M^i_j = (\sigma_3)^i_j = \delta^i_j (1)^{i+1} \quad (4.4)
\]

without loss of generality. For fermion components \( \zeta^\alpha (\alpha = 1, 2, \cdots, 2r) \) of hypermultiplets, the parity is similarly defined by

\[
\zeta^\alpha(-y) = \Pi(\zeta) \gamma_5 M^\alpha_\beta \zeta^\beta(y), \quad (4.5)
\]

and so the reality condition implies \((M^\alpha_\beta)^* = \rho^\alpha_\gamma M^\gamma_\delta \rho^\delta_\beta \). Thus we can take \( M^\alpha_\beta = \sigma_3 \otimes 1_r \) in the standard representation in which \( \rho^\alpha_\beta = \epsilon \otimes 1_r \).

The parity is determined by demanding the invariance of the action and the consistency of both sides of the supersymmetry transformation rules. We find, for the Weyl multiplet fields and \( Q- \) and \( S \)-transformation parameters \( \varepsilon \) and \( \eta \),

\[
\Pi(e^\alpha_\mu) = \Pi(e^4_y) = +1, \quad \Pi(e^4_\mu) = \Pi(e^\rho_\mu) = -1, \quad \Pi(\psi^\mu_\mu) = \Pi(\varepsilon) = +1, \quad \Pi(\psi^\mu_\mu) = \Pi(\eta) = -1, \\
\Pi(b^\mu_\mu) = \Pi(V^3_\mu) = \Pi(V^{1,2}_\mu) = \Pi(v^4_\mu) = \Pi(\chi) = \Pi(D) = +1, \quad \Pi(b^\mu_\mu) = \Pi(V^3_\mu) = \Pi(V^{1,2}_\mu) = \Pi(v^{ab}_\mu) = -1, \quad (4.6)
\]

where the underlined indices \( \mu \) and \( a \) represent the four-dimensional parts of the five-dimensional curved index \( \mu \) and flat index \( a \). The fifth directions of \( \mu \) and \( a \) are denoted by \( y \) and \( 4 \), respectively. The (real) ‘isovector’ components \( \vec{t} = (t^{1,2,3}) \) is generally defined for any symmetric \( SU(2) \) tensor \( t^{ij} \) [satisfying hermiticity \( t^{ij} = (t_{ij})^* \)] as

\[
t^{i,j} \equiv t^{ik} \epsilon^{kj} \equiv i \vec{t} \cdot \vec{\sigma}^j. \quad (4.7)
\]

For the vector multiplet \( V = (M, W_\mu, \Omega^i, Y^{ij}) \), we have

\[
\Pi(M) = \Pi(W_y) = \Pi(Y^{1,2}) = \Pi_V, \quad \Pi(\Omega) = \Pi(W_\mu) = \Pi(Y^3) = -\Pi_V. \quad (4.8)
\]
We define the parity \( \Pi_V \) of vector multiplet \( V \) to be the parity \( \Pi(M) \) of the first component scalar \( M \). Normally, the parity \( \Pi_V \) of vector multiplets must be +1 in five dimensions, since they appear in the action via the homogeneous cubic function \( \mathcal{N}(M) \), which should have even parity +1. However, if a certain subset of the vector multiplets appear in \( \mathcal{N}(M) \) only in the terms quadratic in them, then there is a choice of +1 or −1 for their parity assignment.

For linear multiplet \( L = (L^i, \varphi^i, E^α, N) \), we find

\[
\begin{align*}
\Pi(L^{1,2}) &= \Pi(\varphi) = \Pi(N) = \Pi(E^A) = \Pi(E_{\mu\nu}) = \Pi_L, \\
\Pi(L^3) &= \Pi(E^a) = \Pi(E_{\mu\nu}) = -\Pi_L.
\end{align*}
\]

The parity of the 4-form field \( H_{\mu\nu\rho\sigma} \) also depends on the parity \( \Pi_V \) of the vector multiplet \( V \) used in the rewriting \( N \rightarrow H_{\mu\nu\rho\sigma} \):

\[
\Pi(H_{\mu\nu\rho\sigma}) = -\Pi_V \Pi_L, \quad \Pi(H_{\nu\mu\rho\sigma}) = \Pi_V \Pi_L.
\]

The hypermultiplet \( H^α = (\mathcal{A}^α, \zeta^α, \mathcal{F}^α_i) \) (\( α = 1, 2, \cdots, 2r \)) splits into \( r \) pairs \( (H^{2α-1}, H^{2α}) \) (\( α = 1, 2, \cdots, r \)) in the standard representation, in which \( ρ_{αβ} = \epsilon \otimes 1_r \), and then the reality condition of the scalar components \( \mathcal{A}^α_i \), with \( \epsilon^{ij} A^β_j ρ_{βα} = -(\mathcal{A}^α_i)^* \), implies that the \( 2 \times 2 \) matrix \((\mathcal{A}^{2α-1}, \mathcal{A}^{2α})\) with \( i = 1, 2 \) for each fixed \( α \) (which can be identified with a quaternion*) has the form

\[
\begin{pmatrix}
\mathcal{A}^{2α-1}_1 & \mathcal{A}^{2α-1}_2 \\
\mathcal{A}^{2α}_1 & \mathcal{A}^{2α}_2
\end{pmatrix} = \mathcal{A}^0_α 1_2 - \sum_{k=1}^{3} i \mathcal{A}^k_α \sigma_k = \begin{pmatrix}
\mathcal{A}^0_α - i \mathcal{A}^3_α & -i \mathcal{A}^1_α - \mathcal{A}^2_α \\
-i \mathcal{A}^1_α + \mathcal{A}^2_α & \mathcal{A}^0_α + i \mathcal{A}^3_α
\end{pmatrix},
\]

with \( \mathcal{A}^i_α (i = 0, \cdots, 3) \) real. The components \( \mathcal{F}^i_α (i = 0, \cdots, 3) \) are defined similarly. Then, if the component \( \mathcal{A}^0_α \) has parity \( Π_α \), we have

\[
\Pi(\mathcal{A}^{0,3}_α) = \Pi(\mathcal{F}^{1,2}_α) = Π_α, \quad Π(\mathcal{A}^{1,2}_α) = Π(\mathcal{F}^{0,3}_α) = -Π_α.
\]

In the present gauged \( U(1)_R \) supergravity, the vector multiplet \( V_\Sigma \) couples to the hypermultiplet compensator in the form (3-15). In order for the two terms \(-V_{\mu\nu} \mathcal{A}^{αij}\) and \(-W_{\mu\nu}(i\vec{q} \cdot \vec{σ})^α b_{\mathcal{A}^i_A} \) on the RHS of Eq. (3-15) to have the same \( Z_2 \) parity property, it must be the case that \( i\vec{q} \cdot \vec{σ} \) commutes or anti-commutes with \( σ_3 \), and \( Π(W_{\mathcal{A}^i_A}) = +1 \) when \( i\vec{q} \cdot \vec{σ}, σ_3 = 0 \) and \( Π(W_{\mathcal{A}^i_A}) = -1 \) when \( i\vec{q} \cdot \vec{σ}, σ_3 = 0 \). This implies that the parity \( Π_S \) of the vector multiplet \( V_\Sigma \) (i.e., \( Π(M_\Sigma) \)) should be

\[
Π_S = \begin{cases}
-1 & \text{when } i\vec{q} \cdot \vec{σ} = σ_3, \\
+1 & \text{when } i\vec{q} \cdot \vec{σ} = σ_1 \cos θ + σ_2 \sin θ.
\end{cases}
\]

In order for the 4-form field action \( \mathcal{L}_{VL}(V_\Sigma, L|V_R) \) in Eq. (3-9) to be invariant under \( Z_2 \)-parity, we must have \( Π_S Π_L = +1 \).

As discussed by BKVP, we wish to have the pullback of the component \( H_{\mu\nu\rho\sigma} \) on the brane nonvanishing so that \( Π(H_{\mu\nu\rho\sigma}) = +1 \). This implies \( Π_R Π_L = -1 \), and hence \( Π_R = -Π_S \) and \( Π(G) = -1 \).

*) The quaternion is given by \( q = A^0_α + i A^1_α + j A^2_α + k A^3_α \).
We allow the cubic term in $M_R$ to exist in $\mathcal{N}(M)$, thus allowing the Chern-Simons term of the gauge field $W_{R\mu}$ in the five-dimensional bulk. We therefore assign $\Pi_R = +1$, and then we have $\Pi_S = -1$ and $\Pi_L = -1$. Note that, since $\Pi_S = -1$, the vector multiplet $V_S$ or its first component $M_S$ can appear at most only in quadratic form in $\mathcal{N}(M)$. For simplicity, we assume that $V_S$ does not appear in $\mathcal{N}(M)$ at all, as stated above.

With these parity quantum numbers kept in mind, we consider the transformation rules (3.7) of $H_{\mu\nu\rho\sigma}$ and

$$\delta M = 2i\bar{\epsilon}\Omega, \quad \delta L^{ij} = 2i\bar{\epsilon}^{(i}\varphi^{j)}, \quad (4.14)$$

for the first scalar components $M$ and $L^{ij}$ of vector and linear multiplets. Keeping all the even parity fields nonvanishing on the brane on the RHSs of these equations, we find the following transformation rules on the brane:

$$\delta H_{\mu\nu\rho\sigma} = 4\partial_{[\mu}A_{\nu\rho\sigma]} + 2i\bar{\epsilon}\gamma_{\mu\nu\rho\sigma}\varphi M_R - 4\bar{\epsilon}\gamma_{\mu\nu\rho}\gamma_5\Omega_R L^3 + 16\bar{\epsilon}\gamma_5\gamma_{[\mu\nu\rho}[\psi_\sigma]} M_R L^3,$$

$$\delta M_R = 2i\bar{\epsilon}\Omega, \quad \delta L^3 = \bar{\epsilon}\gamma_5\varphi. \quad (4.15)$$

Here we have used the identity $\bar{\epsilon}\gamma_5\varphi = -2\bar{\epsilon}^{(1}\varphi^{2)}$, which holds on the brane for $SU(2)$ Majorana spinors $\bar{\epsilon}^i$ and $\varphi^i$ with $\Pi(\bar{\epsilon}) = +1$ and $\Pi(\varphi) = -1$. Now, it is clear that the brane action

$$S_{\text{brane}} = \int d^5x (A_1\delta(y) + A_2\delta(y - \tilde{y})) \left[ \frac{1}{4!} e^{\mu\nu\rho\sigma} H_{\mu\nu\rho\sigma} + 2e(4) M_R L^3 \right] \quad (4.16)$$

is superconformal invariant, where $e(4) \equiv e/e_y$ is the determinant of the four-dimensional vierbein on the brane.

§5. Relation between the bulk cosmological constant and brane tensions

Now the action of our total system is given by the bulk action $L_0$ [Eq. (2.4)] plus the 4-form field action $L_{4\text{-form}}$ [Eq. (3.9)] plus the brane action $L_{\text{brane}}$ [Eq. (4.16)]. The latter two parts $L_{4\text{-form}} + L_{\text{brane}}$ give an off-shell generalization of the BKVP action \(^7\) for realizing the odd parity coupling ‘constant’ (field) $G(x)$.

Indeed, variation of the components $\varphi^i$ and $E_{\mu\rho}$ of the linear multiplet still gives the equations in Eq. (3.12): $\Omega_{S\ j} = G\Omega_{R\ j}$ and $F_{\lambda\mu}(W_S) = GF_{\lambda\mu}(W_R)$. The latter implies that $W_S^\mu = GW_R^\mu$ up to a gauge transformation if $G$ is a constant. This equation clearly shows that the field $G(x)$ plays the role of the $U(1)_R$ gauge coupling constant. On the other hand, the equation obtained by variation of the 4-form field $H_{\mu\nu\rho\sigma}$ is now changed in the presence of branes into

$$\partial_{\mu}G = 2\delta^y_{\mu} (A_1\delta(y) + A_2\delta(y - \tilde{y})). \quad (5.1)$$

The integrability condition of this equation on the orbifold $S^1/Z_2$ requires the condition

$$A_1 = -A_2 \equiv g_R, \quad (5.2)$$
and then the solution of \( G(y) \) is given by

\[
G = g_R \epsilon(y),
\]

with the periodic sign function \( \epsilon(y) \) (with period \( 2\tilde{y} \)) defined as

\[
\epsilon(y) \equiv \begin{cases} 
+1 & \text{for } 0 < y < \tilde{y}, \\
-1 & \text{for } -\tilde{y} < y < 0, \\
0 & \text{for } y = 0, \tilde{y}.
\end{cases}
\]

That is, the sign of the coupling ‘constant’ \( G \) changes across the branes and is constant away from the branes.

Let us now discuss the relation between the cosmological constant in the 5D bulk and the brane tensions of the boundary planes. First, note that the auxiliary fields \( Y^{ij} \) are contained in the action in the form [see (2.7) and (3.9)]

\[
-\frac{1}{2} \mathcal{N}_{IJ} Y_{ij} Y^{ij} + Y_{ij} \mathcal{A}_i (2g t_S)^{ab} \mathcal{A}_j^2 + (Y^S - G Y^R)_{ij} L^{ij},
\]

where \( a, b = 1, 2 \) are the indices of the first compensator \( \mathcal{A}_i^a \) to which only the vector multiplet \( V_S \) couples, \( (g t_S)^{ab} = (i \sigma_3)^{ab} \) and the last term results from \( L_4 \) form. The vector multiplet \( V_S \) is special, since it is not contained in \( \mathcal{N}(M) \) and so has no kinetic term. Then the auxiliary field component \( Y_S \) appears in the action only linearly and plays the role of a Lagrange multiplier, yielding a constraint equation:

\[
L^{ij} = -2A^i_a (i \sigma_3)^{ab} A^j_b \equiv L_{sol}^{ij}, \quad \text{or equivalently}
\]

\[
L^1 = L^2 = 0 \quad \text{and} \quad L^3 = -2iA^i_a (i \sigma_3)^{ab} A^j_b \equiv L_{sol}^3.
\]

Then Eq. (5.5) can be rewritten as

\[
-\frac{1}{2} \mathcal{N}_{IJ} (Y - Y_{sol})^{ij} (Y - Y_{sol}^{ij}) + \frac{1}{2} \mathcal{N}_{IJ} Y_{sol}^{ij} Y_{sol}^{ij} + \mathcal{Y}^{ij} = (\mathcal{N}^{-1})^{IJ} (Y_{sol}^{ij} - G V_{IJ} L_{sol}^{ij}).
\]

After the auxiliary fields \( Y_S, \varphi, E_{\mu\nu\rho}, H_{\mu\nu\rho\sigma} \) and \( Y^I \) are eliminated, the brane action now takes the form

\[
S_{brane} = -g_R \int d^5x (\delta(y) - \delta(y - \tilde{y})) \left[ 3 \epsilon(4) W L_{sol}^3 \right],
\]

where \( W \equiv -(3/2) M_R = -(3/2) V_I M^I \) is the ‘superpotential’ introduced in Eq. (5.15) below. This shows that the brane tensions of the planes at \( y = 0 \) and \( y = \tilde{y} \) are given by \( \mp 3g_R W L_{sol}^3 \), respectively.

Next, we turn to the computation of the scalar potential in the 5D bulk. For this purpose, it is better to discuss separately the two cases of one and two compensators. As noted above, the hypermultiplet scalars \( \mathcal{A}^a_\alpha \ (\alpha = 1, 2, \ldots, 2r) \) (or fermions \( \zeta^a \)) have the metric matrix \( d_{\alpha \beta} \) given in (2.6) in their kinetic term, and so the first 2p

\(^{*} \text{We adopt the convention that the usual expression for the Pauli matrix } \vec{\sigma} \text{ applies to the matrix with index position } \vec{\sigma}^{ij}. \text{ The indices } i \text{ and } j \text{ are raised or lowered by using the } \epsilon \text{ tensor, so that, for instance, } \vec{\sigma}^{ij} = \epsilon^{jk} \vec{\sigma}^{i k} = -\vec{\sigma}^{i k} \epsilon^{kj} \text{ denotes } -\vec{\sigma} \epsilon \text{ as a matrix.}\)
components $A_i^a$ with $a = 1, 2, \cdots, 2p$, corresponding to $p$ quaternions, have negative metric and are called compensators, while the remaining components $A_i^\alpha$, which we denote with underlined superscript $\alpha$, have positive metric and represent usual matter fields. We consider the simplest two cases of one ($p = 1$) and two ($p = 2$) compensators separately.

5.1. \( p = 1 \) case

First we consider the most common $p = 1$ case, containing a single compensating hypermultiplet $H^a$ with $a = 1, 2$. Independently of $p$, we always fix the $SU(2)$ gauge by imposing the condition

$$A_i^a(x) = a(x) \delta_i^a, \quad a(x) : \text{real positive}$$

on the first compensator scalars $A_i^1$ and $A_i^2$, and so $L_{\text{sol}}^3$ in Eq. (5.6) takes the form

$$L_{\text{sol}}^3 = -2iA_i^a(\iota \sigma_3)^{ab}A_b^2 = -2(\sigma_3)^{11}a^2(x) = -2a^2(x).$$

In the present case of $p = 1$, the equation of motion $A^2 + 2 = 0^*$ for $A^2 \equiv A_i^a d_\alpha^\beta A^\beta_\beta = -|A_i^a|^2 + |A_i^\alpha|^2$ determines the magnitude $a(x)$ to be

$$a(x) = \sqrt{1 + \frac{1}{2}|A_i^\alpha|^2}.$$

The scalar potential $V$ in the bulk is given by

$$V = -\frac{1}{2}N_{IJ}Y^I_{\text{sol}ij}Y^J_{\text{sol}ij} \bigg|_{\text{bosonic}} - A_i^\alpha (gM)^2 \alpha^\beta A_i^\beta,$$

where use has been made of $(\mathcal{N}^{-1})^{IJ} = -(1/2)(a^{IJ} - M^IM^J)$ with $a^{IJ}$ denoting the inverse of $a_{IJ}$ in Eq. (2.5), and taking Eq. (5.7) into account,

$$P_{ij}^j \equiv -\frac{1}{2}GV_I L_{\text{sol}ij}^{ij} + \frac{1}{2}Y_{ij}^{ij} \bigg|_{\text{bosonic}} = GV_I a^2 (\iota \sigma_3)^{ij} + A_i^\alpha (gM)^2 A_i^\beta,$$

$$Q_i^a \equiv \delta_G (M) A_i^a = M_s (gt_1)^a b A_i^b = GM_R (\iota \sigma_3)^a b A_i^b = GV_I M^I a (\iota \sigma_3)^a,$$

$$Q_i^\alpha \equiv \delta_G (M) A_i^\alpha = M^I (gt_1)^\alpha b A_i^b.$$

If the hypermultiplet fields $A_i^\alpha$ other than the compensator $A_i^a$ are assumed not to be charged, i.e., $gt_1 = 0$, then the potential reduces to the simple form

$$V = \frac{3}{2}g_R \left\{ 3 |a|^4 g^{xy} \frac{\partial W}{\partial \varphi^x} \frac{\partial W}{\partial \varphi^y} - (|a|^4 + 3 |a|^2) W^2 \right\}.$$

Here $\varphi^x (x = 1, \cdots, n)$ are $n$ independent scalar fields with which $n + 1$ vector multiplet scalar fields $M^I$, constrained by the $D$ gauge condition $\mathcal{N}(M) = 1$.

---

*Note that the four compensating bosons $A_i^a (a = 1, 2, i = 1, 2)$ carrying negative metric in this $p = 1$ case are eliminated by the three gauge-fixing conditions of $SU(2)$, (5.9), and this equation of motion $A^2 = -2N$, with dilatation gauge condition $\mathcal{N} = 1$. The target manifold spanned by the hypermultiplet scalars constrained by these conditions becomes $USp(2, 2q)/USp(2) \times USp(2q)$.\textsuperscript{14}
are parametrized.\textsuperscript{8}) We have used the relations \( a^{IJ} = g^{xy} h_x^I h_y^J + h^I h^J \) and \( h^I = -\sqrt{2/3} M^I \) given in Eqs. (I 7.3) and (I 7.1) of I and the definitions\textsuperscript{7})

\[
W \equiv \sqrt{\frac{2}{3}} V_I h^I = -\frac{2}{3} V_I M^I , \quad \frac{\partial W}{\partial \varphi^x} = -\frac{2}{3} V_I h^I_x = -\frac{2}{3} V_I M^I_{x} . \tag{5.15}
\]

If the system contains no physical vector multiplets (i.e., \( n = 0 \)), so that there appear no scalars \( \varphi^x \) and only the graviphoton with \( I = 0 \) exists as a vector field, then, the ‘superpotential’ \( W \) reduces to a constant, and the bulk scalar potential \( V \) to \( V = -(3/2) g_R^2 (|a|^4 + 3 |a|^2) W^2 \). This should be compared with the brane tensions \( \pm 6 g_R a^2 W \) of the planes at \( y = 0 \) and \( \bar{y} \). If the system has no matter-hypermultiplets either (i.e., \( q = 0 \)), then, \( a \) becomes 1, and the bulk scalar potential further reduces to \( V = -6 g_R^2 W^2 \) and the brane tensions to \( \pm 6 g_R W \), yielding exactly the same relations required by Randall and Sundrum.\textsuperscript{1)}

5.2. \( p = 2 \) case

Next we turn to the \( p = 2 \) case, for which the manifold spanned by the hypermultiplet scalar fields becomes \( SU(2, q)/SU(2) \times SU(q) \times U(1) \), and, when \( q = 1 \), this just corresponds to the manifold of the universal hypermultiplet\textsuperscript{15}) appearing in the reduction of the heterotic M-theory on \( S^1/Z_2 \) to five dimensions.\textsuperscript{6}) In this \( p = 2 \) case, as explained in I in detail, we need to introduce another special vector multiplet \( V_T \), which possesses no kinetic term either and couples to the hypermultiplets via the charge \( T = \sigma_3 \otimes 1_{2+q} \), that is, the hypermultiplet \( H^\alpha \) with odd (even) \( \alpha \) carries +1 (−1) charge. In this case the auxiliary field component \( Y_T^{ij} \) again appears in the action as a multiplier field and gives the three constraints

\[
\mathcal{A}_{a i} (\sigma_3)_{\alpha b} A_b^i + \mathcal{A}_{a'i} (\sigma_3)_{\alpha' b'} A_{b'}^{i'} = \mathcal{A}_{a i} (\sigma_3)_{\alpha' \beta} A_{b'}^{\beta}, \tag{5.16}
\]

where the primed indices \( a', b' = 3, 4 \) are used to denote those of the second compensator \( A_{b'}^{i'} \) and the indices \( \alpha \) and \( \beta \) of \( (\sigma_3)_{\alpha \beta} \) should generally be understood to be 1 and 2 when they are odd and even, respectively. The equation of motion \( A^2 = A_i^a d_\alpha^\beta A^{i}_\beta = -2 \) gives

\[
- |A_i|^2 - |A_i'|^2 + |A_i|^2 = -2 . \tag{5.17}
\]

If we impose the \( SU(2) \) gauge-fixing condition Eq. (5.9) and require \( A_{a'=3}^{i=2} \) to be real, as the \( U(1)_T \) gauge-fixing, the solution of these constraints (5.16) and (5.17) is shown in the Appendix to be given in terms of two \( q \)-component complex vectors \( \phi_1 \) and \( \phi_2 \) as

\[
\mathcal{A}_{a=1}^{i=1} = a \left[ \frac{1 - |\phi_2|^2}{2(1 - |\phi_1|^2 - |\phi_2|^2 + |\phi_1|^2 |\phi_2| - |\phi_1|^2 \phi_2|^2)} \right] \equiv a, \tag{5.18}
\]

\[
\mathcal{A}_{a'=3}^{i=1} = a \frac{\phi_2 \cdot \phi_1}{1 - |\phi_2|^2}, \quad \mathcal{A}_{a'=3}^{i=2} = \sqrt{\frac{1}{2(1 - |\phi_2|^2)}} \equiv b, \tag{5.19}
\]

\[
\mathcal{A}_{a=\text{odd}}^{i=1} = a \left( \phi_1 + \frac{\phi_2 \cdot \phi_1}{1 - |\phi_2|^2} \phi_2 \right), \quad \mathcal{A}_{a=\text{odd}}^{i=2} = b \phi_2 ,
\]
where $|\phi|^2 \equiv \phi^\dagger \phi$ and we have shown only the components $A^\alpha_i$ with odd $\alpha = 2\hat{\alpha} - 1$, whose complex conjugates essentially give the even $\alpha = 2\hat{\alpha}$ components, $A^{2\hat{\alpha}}_i = (A^{2\hat{\alpha} - 1}_j)^\ast e_{ji}$, because of the quaternionic nature (4.11) of the hypermultiplet scalars $A^\alpha_i$.

Moreover, consider the case $A$ vector multiplets and hence no scalars.

The scalar potential in the bulk is now given by

$$
\mathcal{V} = (a^{I^I} - M^I M^J) P^{ij}_I (P^{ij}_I)^\ast - |Q^\alpha_i|^2 - |Q^\alpha_i|^2 + |Q^\alpha_i|^2,
$$

(5.20)

where $P^{ij}_I$ is the same as before, while the $Q^\alpha_i$ are now

$$
Q^\alpha_i \equiv \delta_G(M) A^\alpha_i = (M_S + M_T)(i\sigma_3)^a_b A^b_i = (G V_I M^I + M_T) a(i\sigma_3)^a_i,
$$

$$
Q_i^\alpha' \equiv \delta_G(M) A^\alpha_i' = M_T(i\sigma_3)^a'_{b'} A^b_i,
$$

$$
Q_i^\alpha \equiv \delta_G(M) A^\alpha_i = \left( M_T(i\sigma_3)^a_{b\dagger} + M^I (gt_I)^{a}_{b\dagger} \right) A^b_i.
$$

(5.21)

If we assume again that the hypermultiplet matter fields $A^\alpha_i$ other than the compensators $A^\alpha_i$ and $A^\alpha_i'$ are not charged (i.e., $gt_I = 0$), then the potential takes a simpler form. The contribution of the three $|Q^\alpha_i|^2$ terms is evaluated as

$$
-|Q_i^\alpha|^2 - |Q_i^\alpha'|^2 + |Q_i^\alpha|^2

= -2(M_S + M_T)^2 |a|^2 - M_T^2 |A^\alpha_i|^2 + M_T^2 |A^\alpha_i|^2

= -2M_S^2 |a|^2 - 4M S M_T |a|^2 + M_T^2 (-|A^\alpha_i|^2 - |A^\alpha_i'|^2 + |A^\alpha_i|^2)

= -2M_S^2 |a|^2 - 4M S M_T |a|^2 - 2M_T^2,
$$

(5.22)

where we have inserted Eq. (5.17) in the last step. Since $V_T$ does not have its own kinetic term and the scalar component $M_T$ appears only here in the action, $M_T$ can be eliminated by using its equation of motion, $M_T = -M_S |a|^2$. Then this contribution (5.22) reduces to

$$
-2M_S^2 |a|^2 + 2M_T^2 |a|^4.
$$

(5.23)

The first term here represents the same contribution as in the previous $p = 1$ case, which together with the first $|P_i^{ij}|^2$ term gives the same expression as for the previous potential (5.14) [although $a$ here is given by Eq. (5.18) and is different from the previous one (5.11)]. Thus the second term is the additional new contribution in this $p = 2$ case,

$$
2M_S^2 |a|^4 = 2g_R^2 (V_I M^I)^2 |a|^4 = \frac{9}{2} g_R^2 W^2 |a|^4.
$$

(5.24)

Adding this to the previous potential (5.14), we find the scalar potential in the $p = 2$ case as,

$$
\mathcal{V} = \frac{3}{2} g_R^2 \left\{ 3 |a|^4 g_{xy} \frac{\partial W}{\partial \varphi^x} \frac{\partial W}{\partial \varphi^y} - (3 |a|^2 - 2 |a|^4) W^2 \right\}.
$$

(5.25)

Again, let us consider the special case in which the system contains no physical vector multiplets and hence no scalars $\varphi^x$, and thus the case in which $W$ is constant. Moreover, consider the case $q = 1$; that is, the case that the system contains only
a single physical hypermultiplet, which corresponds to the universal hypermultiplet. Then, \( \phi_1 \) and \( \phi_2 \) are single component complex fields and can be rewritten in terms of more commonly used variables, a complex \( \xi \) and real \( V \) and \( \sigma \):\(^{16}\)

\[
\phi_1 = \frac{2 \xi}{1 + S}, \quad \phi_2 = \frac{1 - S}{1 + S}, \quad S \equiv V + \bar{\xi} \xi + i \sigma. \tag{5.26}
\]

Then, \( a \) in Eq. (5.18) reduces to

\[
a = \sqrt{\frac{1}{2} + \frac{|\xi|^2}{2V}}. \tag{5.27}
\]

We can easily see that the result (5.25) with this \( a \) in (5.27) agrees with the result of Falkowski, Lalak and Pokorski.\(^{4}\) Actually, the present result (5.25) reproduces only a part of their result, the part proportional to their \( \beta \) parameter, but the part proportional to their \( \alpha \) parameter is missing. The reason for this is clear. Up to here we have tacitly assumed that our pre-\( U(1)_R \) gauge multiplet \( V_S \) couples only to the first compensator. More precisely, the \( U(1)_S \) charge \( g_{\alpha} \) to which \( V_S \) couples has been chosen to be

\[
gt_S = i \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \quad \text{on} \quad \mathcal{A}_{\text{odd}} \equiv \left( \begin{array}{c} \mathcal{A}^{a=1} \iota_i \\ \mathcal{A}^{a=3} \iota_i \\ \mathcal{A}^{a=5} \iota_i \end{array} \right). \tag{5.28}
\]

However, the isometry group of the hypermultiplet manifold is \( U(2, q = 1) \), which is given by the \( 3 \times 3 \) matrices \( U_{2,1} \) acting on this \( 3 \times 2 \) matrix \( \mathcal{A}_{\text{odd}} \) from the left (and its complex conjugate \( (U_{2,1})^* \) on the even row elements \( \mathcal{A}_{\text{even}} \equiv (\mathcal{A}^{2\alpha-i}) \)). When the system is compactified on \( S^1/Z_2 \), the isometry group is reduced to the subgroup \( U(1) \times U(1, q = 1) \), since it should commute with the \( Z_2 \) parity transformation under which \( \mathcal{A}^{a=1} \iota_i \times \mathcal{A}^{a=3} \iota_i \times \mathcal{A}^{a=5} \iota_i \) acquires the signs \((+1, -1, -1)\). [This parity assignment corresponds to \( \Pi(\phi_1) = -1 \) and \( \Pi(\phi_2) = +1 \) (or, \( \Pi(\xi) = -1 \) and \( \Pi(S) = +1 \)) and is consistent with Eq. (4.12) and the expressions of \( \mathcal{A}^{2\alpha-i} \) in Eqs. (5.18) and (5.19).] The \( U(1)_S \) charge \( g_{\alpha} \) to which \( V_S \) couples can actually be any \( U(1) \) generator of this isometry group \( U(1) \times U(1, q = 1) \). The \( U(1)_S \) generator which was chosen by Falkowski et al. is, given, in our terminology, by

\[
gt_S = i \alpha \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{array} \right) + i \beta \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right). \tag{5.29}
\]

The first generator with coefficient \( \alpha \) corresponds to the isometry in \( U(1, q = 1) \) shifting the field \( \sigma = \text{Im} \ S \) in Eq. (5.26), and the second generator with coefficient \( \beta \) is the original \( U(1) \) charge in Eq. (5.28). Also in this case, one can compute the bulk cosmological constant and brane tensions in the same manner as above. It is convenient to use the \( 3 \times 2 \) matrix \( \Phi \equiv \mathcal{A}_{\text{odd}} \) consisting of the odd row elements (5.28) and the notation defined in the Appendix, with which we find the formulas (\( W_x \equiv \partial W/\partial \phi^x \))

\[
L_{\text{sol}}^{i,j} = 4 \left( \Phi^i \eta (gt_S) \Phi - \frac{1}{2} \mathbf{1}_2 \text{tr}(\Phi^i \eta gt_S \Phi) \right)^i_j.
\]
\[
\langle a^{IJ} - M^I M^J \rangle P_{ij}^{ij} (P_{ij}^{ij})^* = \frac{3}{16} g_R^2 \left( 3 g_{xy} W_{x,y} W_{x,y} - W^2 \right) \text{tr}(L_{3\text{sol}}^+ L_{3\text{sol}}),
\]
\[
-|Q_1|^2 - |Q_2|^2 + |Q_3|^2 = -2 M_S^2 \left\{ \text{tr}(\Phi^+ \eta (gt_S)^2 \Phi) + \left[ \text{tr}(\Phi^+ \eta g t_S \Phi) \right]^2 \right\}
+ 2 \left( M_T + i M_S \text{tr}(\Phi^+ \eta g t_S \Phi) \right)^2.
\]

The last square term vanishes if we use the \( M_T \) equation of motion. Inserting the expression (5.29) for the \( U(1)_S \) generator \( gt_S \) in this case and expressions (5.18) and (5.19) for \( \Phi = A_{\text{odd}} \), and using \( M_S^2 = (9/4) g_R^2 W^2 \) and parametrization (5.26), it is straightforward to find
\[
L_{3\text{sol}}^3 = -\beta \left( 1 + \frac{|\xi|^2}{V} \right) - \frac{\alpha}{V} \left( \frac{V - |\xi|^2}{V + |\xi|^2} \right),
\]
\[
\mathcal{V} = \frac{9}{2} g_R g_{xy} \partial W \partial W \left[ \frac{1}{4} \left\{ \beta \left( 1 + \frac{|\xi|^2}{V} \right) - \frac{\alpha}{V} \right\}^2 + \frac{\alpha \beta}{V} \right]
+ \frac{3}{2} g_R^2 W^2 \left\{ \frac{1}{2} \left[ \beta \left( 1 + \frac{|\xi|^2}{V} \right) - \frac{\alpha}{V} \right]^2 - \frac{3}{2} \beta^2 \left( 1 + \frac{|\xi|^2}{V} \right) - \frac{\alpha \beta}{V} \right\}. \tag{5.31}
\]

We see that the brane tension \( \mp 3 g_R W L_{3\text{sol}}^3 \) with this \( L_{3\text{sol}}^3 \) coincides with that of Falkowski et al.,\(^4\) provided that the \( \xi \) terms are eliminated by assigning odd parity to \( \xi \). Their tension parameter \( \Lambda \) is identified with our \( 3 g_R W \). If the system is reduced to the \( n = 0 \) case, where \( W \) is constant, then this scalar potential \( \mathcal{V} \) also agrees with theirs, aside from the term proportional to \( \alpha \beta \), which we believe is their error, perhaps typographical.

\section{Discussion}

We have given an off-shell formulation of the odd-parity ‘coupling constant’ field \( G(x) \) and 4-form multiplier field \( H_{\mu \nu \rho \sigma} \). This was achieved by rewriting a neutral linear multiplet \( L \) in terms of 3-form and 4-form gauge fields. In particular, we need a vector multiplet background in rewriting the auxiliary scalar component \( N \) of \( L \) into the 4-form gauge field \( H_{\mu \nu \rho \sigma} \), and we obtain the 4-form field action in the five-dimensional bulk by applying the invariant action formula for the product of an Abelian vector multiplet \( V \) and the linear multiplet \( L \). Using here a vector multiplet different from the above-mentioned background vector multiplet, we can obtain the ‘coupling constant’ field \( G(x) \) as the ratio of the two scalar components \( M \) of these two vector multiplets. All the components of this linear multiplet \( L \) now become Lagrange multiplier fields and, in particular, the 4-form gauge field component becomes the multiplier requiring that \( G \) be a constant and change sign across the branes.

We have presented this formulation in a rather general system of Yang-Mills and hypermultiplet matters and discussed the relation between the bulk cosmological constant and the brane tensions of the two boundary planes. This result agrees with those of other authors when the system is reduced to some special cases.

It is interesting that our approach suggests that this parity-odd coupling constant...
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formulation cannot be generalized to the case of non-Abelian gauge coupling. This is because there does not appear a ratio of two scalar components, which is group singlet to be identified with the coupling constant field if both of or one of the two vector multiplets is non-Abelian.

We have not discussed another approach taken by Altendorfer et al. 2) As discussed by Falkowski 4) and BKVP, 7) this approach requires that the $U(1)_R$ generator $i\vec{q} \cdot \vec{\sigma}$ anti-commute with the matrix $M_{ij} = (\sigma_3)^{ij}$, which is used in defining the $Z_2$ parity of the fermions. The off-shell formulation for this approach is not difficult and actually was given by Zucker 17) for the case of linear multiplet compensator in the bulk. Half components of the bulk compensator yield a compensator multiplet on the brane. Both the linear multiplet and hypermultiplet compensator in the bulk induce a chiral multiplet compensator on the brane, which we denote by $\Sigma_0$. Generally, in the 4D superconformal framework, 18) the cosmological constant (without breaking supersymmetry) is supplied from the superpotential term $[\Sigma^3]_F$ of the chiral compensating multiplet $\Sigma$ with Weyl weight 1, provided that the $F$-component of $\Sigma$ develops a non-zero VEV. The brane tension term here, therefore, should be produced by including a superpotential $F$ term $[\Sigma^n_0]_F$ on the brane. (Here $n$ is a suitable power such that $n$ times the Weyl weight of $\Sigma_0$ is 3.) But it is easy to see that the $F$-component of this chiral compensator $\Sigma_0$ develops a non-zero VEV if and only if the $U(1)_R$ generator $i\vec{q} \cdot \vec{\sigma}$ anti-commutes with $\sigma_3$. Moreover, this superpotential term can be multiplied by any function $g(S)$ of (Weyl weight 0) matter chiral multiplets $S_i$ existing on the brane as $[\Sigma^n_0 g(S)]_F$, and then it will clearly yield a brane tension of arbitrary magnitude which has no relation to the bulk cosmological term.

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Appendix A

Parametrization of Hypermultiplet Manifold

We concentrate on only the odd $\alpha$ components $A^{\alpha=2\bar{a}-1}_i$, since the even $\alpha$ components $A^{2\bar{a}}_i$ are given by the odd components by the reality as $A^{2\bar{a}}_i = (A^{2\bar{a}-1}_i)^* \epsilon_{ij}$. It was shown in I that the solution to the constraints (5.16) and (5.17) can be given in the form

$$A_{\text{odd}} \equiv \begin{pmatrix} A^{q=1}_{a=1} \\ A^{q=1}_{a'=3} \\ A^{\alpha=\text{odd}}_{a=1} \\ A^{\alpha=\text{odd}}_{a'=3} \end{pmatrix} = U_{2,q} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

(A.1)
with a unitary matrix $U_{2,q} \in U(2, q)$, where $\mathcal{A}_{\text{odd}}$ is a $q$-component complex vector for each $i = 1, 2$. This unitary matrix $U_{2,q}$ can be parametrized in the form\(^{19)}

$$
U_{2,q} = e^{\phi_i X^i} e^{\alpha^i (\phi, \phi^\dagger) X^i} e^{\beta_m (\phi, \phi^\dagger) S_m},
$$

$$
\phi_i X^i \equiv \begin{pmatrix} 0 & 0 & 0 \\ \phi_1 & 0 & 0 \\ \phi_2 & 0 & 0 \end{pmatrix}, \quad \alpha^i X^i \equiv \begin{pmatrix} 0 & 0 & \alpha^1 \\ 0 & 0 & \alpha^2 \\ 0 & 0 & 0 \end{pmatrix}
$$

(A-2)

by using two independent $q$-component vectors $\phi_1$ and $\phi_2$, where the $S_m$ are the generators of the ‘unbroken’ subgroup $H = U(2) \times U(q)$, and the $\beta_m$ are complex generally. (Note that the broken generators $X^i$, $X^i$, and their coefficients $\phi_i$ and $\alpha^i$ are all $q$-components for each $i = 1, 2$.) Actually, the first factor $e^{\phi_i X^i}$ parametrizes the complex group coset $G^C/\hat{H}$ (where $G^C$ is the complex extension of $G$, and $\hat{H}$ is the complex subgroup whose generators are given by $\{X^i, S_m\}$), and it is just the basic variable in the non-linear realization theory of BKMU\(^{20)}$ in supersymmetric theory. The coefficients $\alpha^i (\phi, \phi^\dagger)$ and $\beta_m (\phi, \phi^\dagger)$ are not independent parameters, but are determined to be functions of $\phi_i$ and $\phi_i^\dagger$ by the requirement that $U_{2,q}$ be a unitary matrix belonging to $U(2, q)$. It is not so easy to find the explicit form of $\alpha^i (\phi, \phi^\dagger)$ and $\beta_m (\phi, \phi^\dagger)$, but here, we fortunately can avoid the computation. Since $U_{2,q}$ acts in Eq. (A-1) on the $(2 + q) \times 2$ matrix whose last $q$ rows are all zero and whose first two rows give a $2 \times 2$ unit matrix, the third columns of the matrices $e^{\alpha^i (\phi, \phi^\dagger) X^i}$ and $e^{\beta_m (\phi, \phi^\dagger) S_m}$ are irrelevant, so that $e^{\alpha^i (\phi, \phi^\dagger) X^i}$ can be replaced by 1, and $e^{\beta_m (\phi, \phi^\dagger) S_m}$ can be replaced by a $2 \times 2$ matrix $U^C$ acting from the right:

$$
\mathcal{A}_{\text{odd}} = U_{2,q} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = e^{\phi_i X^i} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} U^C = \begin{pmatrix} U^C \\ (\phi_1 \phi_2) U^C \end{pmatrix}.
$$

(A-3)

Now, $U^C$ is determined by requiring that this form of $\mathcal{A}_{\text{odd}}$ satisfy the constraints (5.16) and (5.17), which is equivalent to the original condition that $U_{2,q}$ belongs to $U(2, q)$. Then, imposing the $SU(2)$ and $U(1)_T$ gauge conditions on $U^C$ so that it takes the form

$$
U^C = \begin{pmatrix} a & 0 \\ \mathcal{A}_{11} & b \end{pmatrix}, \quad (a, b : \text{real, positive})
$$

(A-4)

we can find the solution of $U^C$ and hence of $\mathcal{A}_{\text{odd}}$, as given in Eq. (5.19) in the text.

The hypermultiplet scalar part of the Lagrangian is given as follows if written in the form before eliminating the auxiliary $SU(2)$ gauge field $V^{ij}_\mu$:

$$
e^{-1} \mathcal{L}_{\text{Hyp}} = +D^\mu \mathcal{A}_i^\alpha D_\mu \mathcal{A}_i^\alpha - |D_\mu \mathcal{A}_i^a|^2 - |D_\mu \mathcal{A}_i^{a'}|^2 + |D_\mu \mathcal{A}_i^{\alpha_\beta}|^2,
$$

(A-5)

where

$$
D_\mu \mathcal{A}_i^a = \partial_\mu \mathcal{A}_i^a + \mathcal{A}_i^a V_j^i - (W_{S\mu} + W_{T\mu})(i\sigma_3)^a_{\beta} \mathcal{A}_i^\beta,
$$

$$
D_\mu \mathcal{A}_i^{a'} = \partial_\mu \mathcal{A}_i^{a'} + \mathcal{A}_j^{a'} V_j^i - W_{T\mu}(i\sigma_3)^{a'}_{\beta} \mathcal{A}_i^\beta,
$$

$$
D_\mu \mathcal{A}_i^{\alpha_\beta} = \partial_\mu \mathcal{A}_i^{\alpha_\beta} + \mathcal{A}_j^{\alpha_\beta} V_j^i - W_{T\mu}(i\sigma_3)^{\alpha_\beta}_{\gamma} \mathcal{A}_i^\gamma - W_{\mu}(gt)^{\alpha_\beta}_{\gamma} \mathcal{A}_i^\gamma.
$$

(A-6)
If we neglect the $U(1)_R$ gauge interaction and other matter gauge interaction with generators $gt_I$ by setting $g_R = g = 0$, then the Lagrangian (A.5) can simply be rewritten in the following form by using the $(2 + q) \times 2$ complex matrix $\Phi = A_{\text{odd}}$:

\[
e^{-1} \mathcal{L}_{\text{Hyp}} = 2 \text{tr}[(D^\mu \Phi)^\dagger \eta (D_\mu \Phi)], \quad D_\mu \Phi = \partial_\mu \Phi + \Phi A_\mu, \quad (A.7)
\]

\[
(A_\mu)^i_j = (V_\mu)^i_j - iW_T \delta^i_j, \quad \eta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \quad (A.8)
\]

$A_\mu$ now gives a $U(2)$ gauge field, which comes from the combination of the $SU(2)$ and $U(1)_T$ symmetries. The four constraints (5.16) and (5.17) can be rewritten as

\[
\Phi^\dagger \eta \Phi = -\frac{1}{2} \mathbf{1}_2. \quad (A.8)
\]

The Lagrangian (A.7) with constraints (A.8) clearly describes a nonlinear sigma model of the Grassmannian manifold $U(2,q)/U(2) \times U(q)$.\textsuperscript{21} If the $U(2)$ auxiliary gauge field $A_\mu$ is eliminated, the Lagrangian can be rewritten as

\[
e^{-1} \mathcal{L}_{\text{Hyp}} = 2 \text{tr}[(\partial^\mu \Phi)^\dagger \eta (\partial_\mu \Phi)] + 4 \text{tr}[(\Phi^\dagger \eta \partial_\mu \Phi)^2]. \quad (A.9)
\]

If the expression (5.19) of $A_{\text{odd}}$ derived above is substituted for $\Phi$ here, then it is easy to confirm that this Lagrangian can be written in the form

\[
e^{-1} \mathcal{L}_{\text{Hyp}} = \frac{\partial K(\phi, \bar{\phi})}{\partial \phi_I \partial \bar{\phi}_J} \partial_\mu \phi_I \partial^\mu \bar{\phi}_J,
\]

\[
K(\phi, \bar{\phi}) = -\ln \det \left[ \begin{pmatrix} 1 & \phi_1^\dagger \\ 0 & 1 \end{pmatrix} \eta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] = -\ln \left[ (1 - |\phi_1|^2)(1 - |\phi_2|^2) - |\phi_1^\dagger \cdot \phi_2|^2 \right], \quad (A.10)
\]

with $|\phi|^2 \equiv \phi^\dagger \cdot \phi$. We see that this $K(\phi, \bar{\phi})$ is just the well-known Zumino-Aoyama form\textsuperscript{22,23} of the Kähler potential corresponding to the (Kählerian) Grassmannian manifold $U(2,q)/U(2) \times U(q)$, giving a special example of the general form presented by BKMU.\textsuperscript{20}

If the $U(1)_R$ gauge field is retained in the calculation, we see that the derivative $\partial_\mu$ in this Lagrangian is replaced by the $U(1)_R$-covariant derivative $\nabla_R \mu = \partial_\mu - \delta_R(GW_R \mu)$ with non-linear $U(1)_R$ transformation $\delta_R$.

References

22) B. Zumino, Phys. Lett. 87B (1979), 293.