Deformed Boson Scheme including Conventional $q$-Deformation in Time-Dependent Variational Method. I

The Case of Many-Body Systems Consisting of One Kind of Boson Operator

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Various aspects of the deformed boson scheme are investigated with the aim of applying it to the time-dependent variational method for many-boson systems. The case of the multi-boson states is also discussed. In analogy with the time-dependent Hartree-Fock theory in many-fermion systems, the classical aspect of the deformed boson is discussed.

§1. Introduction

The time-dependent variational method has played a central role in studies of many-body systems. One of the typical examples can be found in the time-dependent Hartree-Fock theory in canonical form. 1) This theory, first, starts with a Slater determinant parametrized by complex parameters. We can describe the time-evolution of many-fermion systems under a certain condition, which we call the canonicity condition. They can be expressed in terms of the canonical variables of classical mechanics and, finally, the time-evolution of the system is reduced to solving the Hamilton equation of motion. In addition to the case of many-fermion systems, we applied the above-mentioned idea to the time-evolution of many-boson systems in terms of coherent and squeezed states. This application has been reviewed in Ref. 2).

For the time-dependent variational method, selection of the trial function for the variation is the first task. In the case of many-boson systems, the simplest trial state may be the so-called boson coherent state. However, the boson coherent state is not always a unique trial function. The selection should be made with consideration of the Hamiltonian under investigation. In connection to this task, we proposed a possible generalization of the conventional boson coherent state. 3) With the help of this generalization, we described a boson system interacting with an external harmonic oscillator in terms of damped and amplified motion. 4) Another generalization obtained by using an appropriate mapping method was carried out 5) by two of the present authors (C. P. and J. da P.) with Brito in order to investigate the $su(2)$-Lipkin model from the point of view of the classical $q$-deformation of the $su(2)$- and $Os(1)$-algebras. Recently, an interesting idea was presented by Penson and Solomon. 6) In a certain theoretical framework, they found a possible general-
ization of the conventional boson coherent state. In particular, they investigated their form from the point of view of the deformed boson scheme. In this scheme, the function \([x]_q\) plays a central role. They mentioned that for historical reasons, some specific form of \([x]_q\) gained particular popularity, while another deformed boson scheme is possible and was in fact introduced earlier. In this deformed boson scheme treated in this paper, the ordinary \(q\)-deformation of boson operator is naturally contained. In this sense, the deformed boson scheme may be regarded as a kind of the generalization of the conventional \(q\)-deformation. After stressing that there exists an infinite number of possible deformations, they proposed a specific form for \([x]_q\). By changing the value of the parameter specifying the deformation, this function gives us various types of superpositions of many-boson states, which are different from that given in the most popular form. We would like to discuss the generalization of the boson coherent state in the framework of the deformed boson scheme. With the aid of this viewpoint, treatment of the time-dependent variational method in many-boson systems is expected to become more transparent than the treatment discussed in Ref. 3), where no reference is given to the deformed boson scheme.

The main aim of this series of the papers is to investigate extensively the generalization of the boson coherent state which has been used in our variational description of many-boson systems in the framework of the deformed boson scheme. In Part (I), the present paper, we treat the case of one kind of boson operator. By introducing a reasonable function for the boson number, the conventional boson coherent state is generalized. Further, we can find an operator whose eigenstate is just the above state. This operator is nothing but the deformed boson. For demonstration of the powerfulness of the generalization, three concrete examples are discussed; (1) the most popular form, (2) the form given in Ref. 6) and (3) a form consisting of the combination of (1) and (2). The idea for the multiboson states is presented in a manner different from that in Ref. 6); the use of the MYT boson mapping. The motivation comes from the work by Karpeshin, da Providência and Providência for the problem on anharmonic effects in large amplitude vibration of metal clusters.

Our deformed boson scheme is closely connected with classical mechanics and this connection is discussed in the framework of the canonicity condition which appears in the time-dependent Hartree-Fock theory. In the next section, the basic idea of our treatment is mentioned. Section 3 is devoted to discussing concrete examples of the deformations. In §4, the idea for the multiboson states is described. In §5, the classical aspect of the deformed boson scheme is mentioned, and in §6, short concluding remarks are given.

§2. Basic scheme

We set out by investigating a space constructed from the boson operator \((\hat{c}, \hat{c}^*)\):

\[
[\hat{c}, \hat{c}^*] = 1 .
\]  

(2.1)

This boson space is spanned by the basis \(\{|n\}\) given by the form

\[
|n\rangle = (\sqrt{n!})^{-1}(\hat{c}^*)^n|0\rangle , \quad (n = 0, 1, 2, \cdots)
\]  

(2.2a)
\[ c|0\rangle = 0 \, . \] (2.2b)

For the operators \( \hat{c} \) and \( \hat{c}^* \), we have the relations
\[ \hat{c}^* \hat{c} = \hat{N} \, , \] (2.3a)
\[ \hat{c} \hat{c}^* = \hat{N} + 1 \, , \] (2.3b)

\[ \hat{c}|n\rangle = \sqrt{n}|n-1\rangle \, , \quad \hat{c}^*|n\rangle = \sqrt{n+1}|n+1\rangle \, , \] (2.4)

\[ [\hat{N}, \hat{c}] = -\hat{c} \, , \quad [\hat{N}, \hat{c}^*] = \hat{c}^* \, , \] (2.5)

\[ \hat{N}|n\rangle = n|n\rangle \, . \] (2.6)

The relation (2.3a) is the definition of the operator \( \hat{N} \) and, with the use of the relations (2.1), (2.2a) and (2.3a), the others are derived.

In this space, we introduce the following wave packet:
\[ |c\rangle = (\sqrt{\Gamma})^{-1} \sum_{n=0}^{\infty} f(n)(\sqrt{n!})^{-1} \gamma^n|n\rangle \, , \quad \langle c|c \rangle = 1 \] (2.7a)
\[ \Gamma = \sum_{n=0}^{\infty} f(n)^2(n!)^{-1}(|\gamma|^2)^n \, . \] (2.7b)

Here, \((\gamma, \gamma^*)\) denotes complex parameters. The quantity \( f(n) \) is a well-behaved function of \( n \) obeying
\[ f(n) = 1 \quad \text{for} \quad n = 0, 1 \, , \quad f(n) > 0 \quad \text{for} \quad n = 2, 3, \ldots \] (2.7c)

If \( f(n) = 1 \), the wave packet \(|c\rangle\) is nothing but a conventional boson coherent state. We can see that the infinite series (2.7b) is convergent in the domain \( 0 \leq |\gamma|^2 < \infty \).

In Ref. 3), we started with the state
\[ |c\rangle = (\sqrt{\Gamma})^{-1} \exp \left( \gamma \hat{c}^* \tilde{f}(\hat{N}) \right) |0\rangle \, . \] (2.8)

Here, \( \tilde{f}(\hat{N}) \) is a function of \( \hat{N} \), the inverse of which is well-defined. The function \( f(n) \) in the relation (2.7a) is related to \( \tilde{f}(n) \) through
\[ f(n) = \tilde{f}(0) \tilde{f}(1) \cdots \tilde{f}(n-1) \, , \quad (n = 1, 2, 3, \cdots) \]
\[ f(0) = 1 \, . \] (2.9)

Therefore, the state (2.8) is essentially the same as the state (2.7a). Our main interest is in applying the wave packet \(|c\rangle\) to the time-dependent variational method as a trial state:
\[ \delta \int \langle c|i\partial_t - \hat{H}|c \rangle dt = 0 \, . \] (2.10)

Here, \( \hat{H} \) denotes a Hamiltonian expressed in terms of \((\hat{c}, \hat{c}^*)\) and the variation is performed through the parameter \((\gamma, \gamma^*)\).
We note that the state (2.7a) can be rewritten in the following form:

\[ |c\rangle = \sqrt{I_0/\Gamma} \cdot f(\hat{N})|c^0\rangle, \]

\[ |c^0\rangle = \left(\frac{\sqrt{I_0}}{\Gamma}\right)^{-1} \exp(\gamma^*)|0\rangle, \quad I_0 = \exp(|\gamma|^2) . \tag{2.11b} \]

In relation to the wave packet \(|c\rangle\), we introduce the operators defined by

\[ \hat{\gamma} = f(\hat{N})\hat{c}(\hat{N})^{-1}, \quad \hat{\gamma}^* = f(\hat{N})^{-1}\hat{c}^* f(\hat{N}) . \tag{2.12} \]

For the sake of the property (2.7c), \(f(\hat{N})^{-1}\) can be defined. The form (2.11) and the definition (2.12) give the relation

\[ \hat{\gamma}|c\rangle = \gamma|c\rangle . \tag{2.13} \]

Here, we have used \(\hat{c}|c^0\rangle = \gamma|c^0\rangle\). The relation (2.13) shows that the state \(|c\rangle\) is, in some sense, a possible generalization of the conventional boson coherent state. With the use of \(\hat{\gamma}^*\), the state \(|n\rangle\) can be expressed in the form

\[ |n\rangle = \left(\sqrt{f(n)^{-2n!}}\right)^{-1}(\gamma^*^n)|0\rangle, \quad (n = 0, 1, 2, \ldots) \tag{2.14a} \]

\[ \hat{\gamma}|0\rangle = 0 . \tag{2.14b} \]

Of course, \(|n\rangle\) is identical to \(|n\rangle\). Further, we have the relations

\[ \hat{\gamma}^*\hat{\gamma} = \hat{N}f(\hat{N})^{-2}f(\hat{N} - 1)^2 , \]

\[ \hat{\gamma}\hat{\gamma}^* = (\hat{N} + 1)f(\hat{N} + 1)^{-2}f(\hat{N})^2 , \]

\[ \hat{\gamma}|n\rangle = \sqrt{nf(n)^{-2}f(n - 1)^2}|n - 1\rangle , \]

\[ \hat{\gamma}^*|n\rangle = \sqrt{(n + 1)f(n + 1)^{-2}f(n)^2}|n + 1\rangle . \tag{2.16} \]

\[ [\hat{N}, \hat{\gamma}] = -\hat{\gamma} , \quad [\hat{N}, \hat{\gamma}^*] = \hat{\gamma}^* , \tag{2.17} \]

\[ \hat{N}|n\rangle = n|n\rangle . \tag{2.18} \]

The above relations can be interpreted in terms of the deformed boson scheme. We define the function \([x]_q\) in the form \(^{\ast}\)

\[ [x]_q = xf(x)^{-2}f(x - 1)^2 , \quad (x = n, \hat{N} , \ n \geq 1) \]

\[ [0]_q = 0 . \tag{2.19} \]

Then, \([n]_q!\) is given by

\[ [n]_q! = nf(n)^{-2}f(n - 1)^2 \cdot (n - 1)f(n - 1)^{-2}f(n - 2)^2 \cdots 1 \cdot f(1)^{-2}f(0)^2 \]

\[ = f(n)^{-2}n! , \quad (n \geq 1) \]

\[ [0]_q! = 1 . \tag{2.20} \]

\(^{\ast}\) We should adopt the notation \([x]_f\) instead of \([x]_q\) because the deformation is characterized by the function \(f\). However, in this paper (I), since the function \(f\) is specified by the parameter \(q\), the notation \([x]_q\) is used similar to the notation for the conventional theory of \(q\)-deformation. In part (II), we will use \([x]_f\).
For \( n = 0, 1 \), the relation (2.20) gives us \([0]_q! = [1]_q! = 1\). Thus, the relations (2.14)–(2.18) are summarized in the framework of the deformed boson scheme. Of course, \((\hat{\gamma}, \hat{\gamma}^*)\) denotes the deformed boson. The relations (2.15) and (2.19) give us the following commutation relation:

\[
[\hat{\gamma}, \hat{\gamma}^*] = [\hat{N} + 1]_q - [\hat{N}]_q
= \Delta_{\hat{N}}(\hat{\gamma}^*\hat{\gamma}) .
\] (2.21)

Here, \(\Delta_{\hat{N}}(\hat{\gamma}^*\hat{\gamma})\) denotes the difference with \(\Delta_{\hat{N}} = 1\). Concerning the relation between \(\hat{\gamma}\hat{\gamma}^*\) and \(\hat{\gamma}^*\hat{\gamma}\), there exist infinite possibilities such as shown in the form

\[
\hat{\gamma}^*\hat{\gamma}^* - [f(\hat{N})^4 f(\hat{N} + 1)^{-2} f(\hat{N} - 1)^{-2} + F(\hat{N})] \hat{\gamma}^*\hat{\gamma}
= f(\hat{N} + 1)^{-2} f(\hat{N})^2 - F(\hat{N}) \hat{N} f(\hat{N})^{-2} f(\hat{N} - 1)^2 .
\] (2.22)

The actual form depends on the choice of the function \(F(\hat{N})\).

§3. **Concrete examples**

In this section, we discuss some concrete examples for the choice of the function \(f(x) (x = n, \hat{N})\).

3.1. **The most popular form**

This case starts with the well-known form

\[
[x]_q = (q^x - q^{-x})/(q - q^{-1}) , \quad (x = n, \hat{N})
\] (3.1)

where we assume that \(q\) is a real number satisfying \(0 < q \leq 1\). The function \(f(x)\) can be determined through the relation (2.19):

\[
x f(x)^{-2} f(x - 1)^2 = (q^x - q^{-x})/(q - q^{-1}) .
\] (3.2)

Since \(x = n (n = 0, 1, 2, \cdots)\), the relation (3.2) gives us the following recursion formula:

\[
f(n) = \sqrt{n(q - q^{-1})/(q^n - q^{-n})} f(n - 1) .
\] (3.3)

By solving the above relation successively, we have

\[
f(n) = \begin{cases} 1 , & (n = 0, 1) \\ \frac{n! \prod_{k=2}^{n} (q - q^{-1})/(q^k - q^{-k})}{\sqrt{n! \prod_{k=2}^{n} (q^{(k-1)/2} - q^{-(k-1)/2})}} = \sqrt{n! \prod_{k=2}^{n} \left( \sum_{m=-(k-1)/2}^{(k-1)/2} (q^2)^m \right)} , & (n = 2, 3, 4, \cdots) \end{cases}
\] (3.4)

Of course, \(f(\hat{N})\) is obtained by replacing \(n\) with \(\hat{N}\). As \(f(n)\), if we choose the form (3.4), our scheme is reduced to the most popular form. Since \(f(n)\) is symmetric with respect to \(q^1\) and \(q^{-1}\), it is sufficient to investigate the case in the domain \(0 < q \leq 1\).
This is the reason why we assumed that \( 0 < q \leq 1 \) in Eq. (3.1). For \( n = 2, 3, 4, \cdots \), two special cases of the form (3.4) are as follows:

\[
\begin{align*}
  f(n) &\to 0 \quad \text{for} \quad q \to 0, \\
  f(n) &\equiv 1 \quad \text{for} \quad q = 1.
\end{align*}
\]

(3.5a) (3.5b)

In these cases, the state \( |c\rangle \) can be expressed as

\[
|c\rangle \to (\sqrt{\Gamma})^{-1}(|0\rangle + \gamma|1\rangle) = (1 + |\gamma|^2)^{-1/2} \cdot (1 + \gamma \hat{c}^*)|0\rangle \quad \text{for} \quad q \to 0,
\]

(3.6a)

\[
|c\rangle \to (\sqrt{\Gamma})^{-1} \sum_{n=0}^{\infty} (\sqrt{n!})^{-1} \gamma^n |n\rangle = \exp(-|\gamma|^2/2) \cdot \exp(\gamma \hat{c}^*)|0\rangle \quad \text{for} \quad q = 1.
\]

(3.6b)

The state (3.6a) is the simplest mixture and the state (3.6b) is nothing but the conventional boson coherent state. Therefore, the state \( |c\rangle \) in the general case is an intermediate mixture of the above two cases. We have a relation familiar from the deformed boson scheme,

\[
\hat{\gamma} \hat{\gamma}^* - q^{-1} \hat{\gamma}^* \hat{\gamma} = q^{\hat{N}},
\]

(3.7)

if \( F(\hat{N}) \) in the relation (2.22) is chosen as

\[
F(\hat{N}) = q^{-1} - f(\hat{N})^4 f(\hat{N}+1)^{-2} f(\hat{N}-1)^{-2}.
\]

(3.8)

3.2. The form presented by Penson and Solomon

Our scheme is reduced to the form proposed by Penson and Solomon, 6) if \( f(n) \) is chosen as

\[
 f(n) = q^{n(n-1)/4}, \quad \text{i.e.,} \quad f(n) = q^{(n-1)/2} f(n-1).
\]

(3.9)

As was stressed by them, if \( 0 < q \leq 1 \), the infinite series (2.7b) is convergent for any value of \( |\gamma|^2 \). For \( f(n) \) given in Eq. (3.9), we have \( f(0) = f(1) = 1 \), and for \( n = 2, 3, 4, \cdots \), the same result as shown in Eq. (3.5). Therefore, we obtain the same forms as those given in Eq. (3.6) and the state \( |c\rangle \) in the general case is also in the intermediate mixture of the above two cases. However, the mechanism of the mixture may be different from the most popular form. In this case, \([n]_q\) and \([n]_q!\) are given by

\[
 [n]_q = nq^{-(n-1)},
\]

(3.10)

\[
 [n]_q! = q^{-n(n-1)/2} n!.
\]

(3.11)

For \( n = 0, 1 \), the relation (3.11) gives us \([0]_q! = [1]_q! = 1\). If \( F(\hat{N}) = 0 \), the relation (2.22) is reduced to

\[
\hat{\gamma} \hat{\gamma}^* - q^{-1} \hat{\gamma}^* \hat{\gamma} = q^{-\hat{N}}.
\]

(3.12)

This is different from the previous form (3.7).
The above case can be easily extended by putting \( f(n) \) in the form

\[
f(n) = q^{C(n,r)/2}, \quad C(n,r) = n!/(n-r)!r! \quad \text{for} \quad n \geq r,
\]

\[
f(n) = 1 \quad \text{for} \quad n = 0, 1, 2, \ldots, r-1.
\]  \(3.13\)

For \( n = r, r+1, \ldots, f(n) \) is reduced to

\[
f(n) \to 0 \quad \text{for} \quad q \to 0,
\]

\[
f(n) = 1 \quad \text{for} \quad q = 1.
\]  \(3.14\)

Therefore, for the above two cases, \( |c\rangle \) can be expressed as

\[
|c\rangle \to (\sqrt{T})^{-1} \sum_{n=0}^{r-1} (\sqrt{n!})^{-1} \gamma^n |n\rangle
\]

\[
= \left( \sum_{n=0}^{r-1} (n!)^{-1} (|\gamma|^2)^n \right)^{-1/2} \cdot \sum_{n=0}^{r-1} (n!)^{-1} (\gamma \hat{c}^*)^n |0\rangle \quad \text{for} \quad q \to 0,
\]  \(3.15a\)

\[
|c\rangle = (\sqrt{T})^{-1} \sum_{n=0}^{\infty} (\sqrt{n!})^{-1} \gamma^n |n\rangle
\]

\[
= \exp(-|\gamma|^2/2) \cdot \exp(\gamma \hat{c}^*)|0\rangle \quad (= |c_0\rangle) \quad \text{for} \quad q = 1.
\]  \(3.15b\)

We can see that the form (3.16) is extended from that given in Eq. (3.6).

3.3. Modified forms

First, we treat the following two forms:

\[
f(n) = \left[ 1 - \exp(-C(n,r)^{-1} q (1-q)^{-1}) \right]^{1/2},
\]  \(3.17\)

\[
f(n) = \sqrt{2} \left[ 1 + \exp(C(n,r) q^{-1} (1-q)) \right]^{-1/2}.
\]  \(3.18\)

Here, \( C(n,r) \) is defined in Eq. (3.13). The parameter \( q \) is in the domain \( 0 < q \leq 1 \).

For the above two forms, we can show the following relation:

\[
f(n) = 1 \quad \text{for} \quad n = 0, 1, 2, \ldots, r-1.
\]  \(3.19\)

Further, for \( n = r, r+1, \ldots, f(n) \) is reduced to

\[
f(n) \to 0 \quad \text{for} \quad q \to 0,
\]

\[
f(n) = 1 \quad \text{for} \quad q = 1.
\]  \(3.20\)

The relations (3.19) and (3.20) are exactly the same as those shown in Eqs. (3.14) and (3.15). Therefore, for the cases (3.17) and (3.18), we have the form (3.16). There exists an infinite number of possible deformations. The forms (3.17) and (3.18) may be two of the possibilities. If \( r = 2 \), the forms correspond to those shown in the relations (3.4) and (3.9).

The next modification takes the form of the products of various \( f(n) \) appearing already in some places. An example is the product of two \( f(n) \) shown in the relations...
(3.4) and (3.9). In this case, the state $|c\rangle$ is in the intermediate mixture of the cases (3.6a) and (3.6b). The relations (3.3) and (3.9) give us the following relation for the producted new $f(n)$:

$$f(n) = \sqrt{n(1-q^{-2})/(1-q^{-2n})} f(n-1). \quad (3.21)$$

By substituting the relation (3.21) into $[n]_q$ given in Eq. (2.19), $[n]_q$ is obtained in the form

$$[n]_q = (1-q^{-2n})/(1-q^{-2}). \quad (3.22)$$

The deformation deduced in the last equation is studied in Ref. 8) for the case $Q = q^2$. Then, the relations (2.15) in this case become

$$\hat{\gamma}^* \hat{\gamma} = [\hat{N}]_q = (1-q^{-2\hat{N}})/(1-q^{-2}), \quad (3.23a)$$

$$\hat{\gamma}^* \hat{\gamma} = [\hat{N} + 1]_q = (1-q^{-2(\hat{N}+1)})/(1-q^{-2}). \quad (3.23b)$$

The above relations give us

$$\hat{\gamma}^* - q^{-2} \hat{\gamma}^* \hat{\gamma} = 1. \quad (3.24)$$

This should be compared with the relations (3.7) and (3.12). The form is quite simple.

The third modification is related to the product of $f(n)$ appearing already, and a new function $g(n)$, i.e., $f_M(n) = f(n)g(n)$. Of course, $g(n)$ obeys the same condition as that given in the relation (2.7c). As $f(n)$, we can adopt, for example, $f(n)$ shown in the relation (3.21). Then, the state $|c\rangle$ can be expressed as

$$|c\rangle = (\sqrt{\Gamma})^{-1} \sum_{n=0}^{\infty} f_M(n)(\sqrt{n!})^{-1} \gamma^n |n\rangle. \quad (3.25)$$

The state $|c\rangle$ in the limit $q \to 0$ and the case $q = 1$ can be written as

$$|c\rangle = (1 + |\gamma|^2)^{-1/2}(1 + \gamma \hat{c}^*)|0\rangle \quad \text{for} \quad q \to 0, \quad (3.26a)$$

$$|c\rangle = \left(\sqrt{\Gamma}\right)^{-1} \sum_{n=0}^{\infty} g(n)(\sqrt{n!})^{-1} \gamma^n |n\rangle,$$

$$\Gamma = \sum_{n=0}^{\infty} g(n)^2 (n!)^{-1} (|\gamma|^2)^n \quad \text{for} \quad q = 1. \quad (3.26b)$$

It is important to see that for $q = 1$, the state $|c\rangle$ is not the conventional boson coherent state.

§4. Multiboson states

One of the motivations of the present work comes from the investigation of a problem how to describe the time-evolution of multiboson coherent states in terms of the time-dependent variational method.\textsuperscript{10} First, we prepare another space constructed from the boson operator $(\hat{c}, \hat{c}^*)$, which is independent of $(\hat{c}, \hat{c}^*)$. In this
boson space, we introduce the following state:

\[ |c\rangle = \left( \sqrt{\Gamma} \right)^{-1} f_m(\tilde{N}) \exp( \sqrt{\gamma} \tilde{c}^* ) |0\rangle , \quad (4.1a) \]

\[ f_m(\tilde{N}) = (1/m)(1 - e^{2\pi i \tilde{N}})/(1 - e^{2\pi i \tilde{N}/m}) , \quad (\tilde{N} = \tilde{c}^* \tilde{c}) \quad (4.1b) \]

\[ m = 2, 3, 4, \ldots . \quad (4.2) \]

The state \(|c\rangle\) can be rewritten as

\[ |c\rangle = \left( \sqrt{\Gamma} \right)^{-1} \sum_{n=0}^{\infty} \left( \sqrt{(mn)!} \right)^{-1} \gamma^n |mn\rangle , \quad (4.3) \]

\[ |mn\rangle = \left( \sqrt{(mn)!} \right)^{-1} (\tilde{c}^*)^mn |0\rangle , \quad (4.4) \]

\[ \Gamma = \sum_{n=0}^{\infty} ((mn)!)^{-1} (|\gamma|^2)^n . \quad (4.5) \]

The operator \((\tilde{c}^*)^m\) is the building block of the state \(|c\rangle\) and we call the state \(|c\rangle\) the multiboson coherent state. It may be self-evident that the infinite series \((4.5)\) is convergent in the domain \(0 \leq |\gamma|^2 < \infty.\)

The structure of the state \((4.3)\) is different from that of the state \((2.7a)\) except \(m = 1\) and, then, we cannot apply the basic scheme presented in §2 to the state \((4.3)\) directly. In order to make it possible, we adopt the basic idea of the MYT boson mapping method;\(^9\) the state \(|n\rangle\) given in the relation \((2.2a)\) is the image of the state \(|mn\rangle\) given in the relation \((4.4)\), i.e., we set up the correspondence

\[ |mn\rangle \sim |n\rangle . \quad (4.6) \]

The above correspondence permits us to introduce the mapping operator \(U\) in the form

\[ U = \sum_{n=0}^{\infty} |n\rangle\langle mn| . \quad (4.7) \]

Then, the image of \(|c\rangle\), which we denote \(|c\rangle\), can be given in the form

\[ |c\rangle = U|c\rangle = \left( \sqrt{\Gamma} \right)^{-1} \sum_{n=0}^{\infty} f(n) \left( \sqrt{n!} \right)^{-1} \gamma^n |n\rangle , \quad (4.8) \]

\[ f(n) = \sqrt{n!} \left( \sqrt{(mn)!} \right)^{-1} . \quad (4.9) \]

As is clear from the relations \((4.8)\) and \((4.9)\), we can apply the basic scheme to the state \((4.8)\). It is interesting to see that the state \((4.8)\) can be rewritten as

\[ |c\rangle = \left( \sqrt{\Gamma} \right)^{-1} \exp \left( \left( \sqrt{m^m} \right)^{-1} \gamma \tilde{c}^* \left( \sqrt{F_m(\tilde{N})} \right)^{-1} \right) |0\rangle , \quad (4.10) \]

\[ F_m(\tilde{N}) = \prod_{p=1}^{m-1} (\tilde{N} + p/m) . \quad (4.11) \]
The images of $\tilde{c}^*\tilde{c}$ and $(\tilde{c}^*)^m$, which we denote as $(\tilde{c}^*\tilde{c})_c$ and $(\tilde{c}^*)_c^m$, respectively, can be expressed in the forms

\[
(\tilde{c}^*\tilde{c})_c = U\tilde{c}^*\tilde{c}U^\dagger = m\tilde{c}^* = m\hat{N}, \quad (4.12a)
\]

\[
(\tilde{c}^*)_c^m = U(\tilde{c}^*)^m U^\dagger = \sqrt{mm^*}\tilde{c}^*\sqrt{F_m(\hat{N})}. \quad (4.12b)
\]

With the use of $f(n)$ shown in Eq. (4.9), we have

\[
f(\hat{N} + 1)^{-1}f(\hat{N}) = \sqrt{mm^*}\sqrt{F_m(\hat{N})}. \quad (4.13)
\]

Then, $\hat{\gamma}$ and $\hat{\gamma}^*$ are given by

\[
\hat{\gamma} = \sqrt{mm^*}\sqrt{F_m(\hat{N})}\tilde{c}, \quad \hat{\gamma}^* = \sqrt{mm^*}\tilde{c}^*\sqrt{F_m(\hat{N})}. \quad (4.14)
\]

Penson and Solomon have also investigated the multiboson coherent state in their framework. Let us discuss their form in the present scheme. Their state, which we denote $|c\rangle\rangle$, is expressed as

\[
|c\rangle\rangle = (\sqrt{T})^{-1}\sum_{n=0}^{\infty}\sqrt{(mn)!} (n!)^{-1}q^{n(n-1)/4}\gamma^n |mn\rangle, \quad (4.15)
\]

\[
\Gamma = \sum_{n=0}^{\infty}(mn)!{(n!)^{-2}q^{n(n-1)/2}(\gamma^2)^n}. \quad (4.16)
\]

If $q < 1$, the infinite series (4.16) is convergent, but, if $q = 1$, the series (4.16) is convergent only for the case $m = 2$. However, judging from the spirit of the present treatment, the state (4.8) should be compared with the state (4.15) at $q = 1$. In this case, both are different from each other. Our state is normalizable, but theirs is not. However, we can introduce the effect of $q$ in terms of the product of $f(n)$ given in the relation (4.9) and $f(n)$ discussed in §3, for example, such form as

\[
f(n) = \sqrt{n!} \left(\sqrt{(mn)!} \right)^{-1} q^{n(n-1)/4}. \quad (4.17)
\]

Thus, for $0 < q \leq 1$, we have the state $|c\rangle$, which is normalizable and through the inverse process, we have $|c\rangle\rangle$, i.e.,

\[
|c\rangle\rangle = U^\dagger|c\rangle. \quad (4.18)
\]

§5. Classical aspect of the deformed boson scheme

We return to the time-dependent variational equation (2.10). In order to perform the variation, it is necessary to calculate the two quantities $\langle c|i\partial_t|c\rangle$ and $\langle c|\hat{H}|c\rangle$. \cite{2}

Concerning the former, the time-derivative is obtained through the parameter $(\gamma, \gamma^*)$, and the result is as follows:

\[
\langle c|i\partial_t|c\rangle = (i/2)(\hat{\gamma}\gamma^* - \hat{\gamma}^*\gamma)\Gamma'/\Gamma. \quad (5.1)
\]
Here, $\Gamma$ is given in the relation (2.7b), and $\Gamma'$ is given by
\[
\Gamma' = d\Gamma/d|\gamma|^2 = \sum_{n=0}^{\infty} f(n+1)^2(n!)^{-1}(|\gamma|^2)^n \geq 0 .
\]
(5.2)
The expression (5.1) can be changed into the form
\[
\langle c| i\partial_t |c\rangle = (i/2)(\dot{z}z^* - \dot{z}^*z) .
\]
(5.3)
The quantity $(z, z^*)$ is defined as
\[
z = \gamma \sqrt{\Gamma'/\Gamma} , \quad z^* = \gamma^* \sqrt{\Gamma'/\Gamma} .
\]
(5.4)
The expectation value $\langle c| \hat{H} |c\rangle$ is obtained as a function of $(\gamma, \gamma^*)$, i.e., $(z, z^*)$:
\[
\langle c| \hat{H} |c\rangle = H .
\]
(5.5)
The variation (2.10) gives us the following equations:
\[
i\dot{z} = \partial H/\partial z^* , \quad i\dot{z}^* = -\partial H/\partial z .
\]
(5.6)
These equations are equivalent to the Hamilton equation of motion. The quantity $(z, z^*)$ can be regarded as the canonical variable in boson type. The above formulation is in parallel with the TDHF theory in canonical form. By solving Eq. (5.6) under appropriate initial conditions, we can determine the time-dependence of $(z, z^*)$, i.e., $(\gamma, \gamma^*)$. Then, we have the state $|c\rangle$ as a function of time. It is noted that the above process can be performed if $(\gamma, \gamma^*)$ is expressed in terms of $(z, z^*)$ in the relation (5.4). However, in general, it may be impossible to get an analytical form. Then, we will try to describe the system under investigation in terms of $(\gamma, \gamma^*)$. For this purpose, we must investigate the physical meaning of the variable $(\gamma, \gamma^*)$.

For the above investigation, first, we note the relation (2.13), from which we have
\[
\langle c| \hat{\gamma} |c\rangle = \gamma , \quad \langle c| \hat{\gamma}^* |c\rangle = \gamma^* ,
\]
(5.7)
\[
\langle c| \hat{\gamma} \hat{\gamma}^* |c\rangle = \gamma^* \gamma .
\]
(5.8)
Further, $\langle c| \hat{N} |c\rangle$ is given as
\[
\langle c| \hat{N} |c\rangle = \gamma^* \gamma \Gamma'/\Gamma = z^*z (= N) .
\]
(5.9)
With the use of the relation (5.4), we can prove the relation:
\[
[N, \gamma]_P = -\gamma , \quad [N, \gamma^*]_P = \gamma^* ,
\]
(5.10)
\[
[\gamma, \gamma^*]_P = dN(\gamma^* \gamma) .
\]
(5.11)
Here, $dN$ denotes the differential with respect to $N$ and $[A, B]_P$ expresses the Poisson bracket:\footnote{The definition of the Poisson bracket may in general be considered as $\{A, B\}_P = \partial A/\partial z \cdot \partial B/\partial (iz^*) - \partial A/\partial (iz^*) \cdot \partial B/\partial z$. The relation to our definition (5.12) is simply obtained as $i\{A, B\}_P = [A, B]_P$.}
\[
[A, B]_P = \partial A/\partial z \cdot \partial B/\partial z^* - \partial A/\partial z^* \cdot \partial B/\partial z .
\]
(5.12)
The relations (5.10) and (5.11) should be compared with the relations (2.17) and (2.21), together with the relations (5.7)–(5.9). Through the relation between the commutator and the Poisson bracket and, further, through the relation between the difference and the differential (in old quantum theory), we can conclude that the form presented in this section is the classical counterpart of the deformed boson scheme presented in §2, and we have the correspondence

$$\hat{\gamma} \sim \gamma, \quad \hat{\gamma}^* \sim \gamma^*, \quad \hat{N} \sim N. \quad (5.13)$$

Then, we can say that $(\gamma, \gamma^*)$ is the classical counterpart of the deformed boson $(\hat{\gamma}, \hat{\gamma}^*)$.

The time-evolution of the classical counterpart of the deformed boson and/or deformed $su(2)$-generators has been investigated in the context of the $q$-deformed Lipkin model. In our case, the Hamilton equation of motion (5.6) can be rewritten in terms of $(\gamma, \gamma^*)$ as follows:

$$i\dot{\gamma} = \frac{\partial H}{\partial \gamma^*} \left( \frac{\partial N}{\partial |\gamma|^2} \right)^{-1}, \quad i\dot{\gamma}^* = -\frac{\partial H}{\partial \gamma} \left( \frac{\partial N}{\partial |\gamma|^2} \right)^{-1}, \quad (5.14)$$

$$\frac{\partial N}{\partial |\gamma|^2} = \frac{\Gamma'}{\Gamma} + |\gamma|^2 \left( \frac{\Gamma'}{\Gamma} \right)'.$$

(5.15)

If $(\Gamma'/\Gamma)' = 0$, the equation of motion (5.14) is reduced to the conventional Hamilton equation of motion. The solution of Eq. (5.14) gives us the time-dependence of $(\gamma, \gamma^*)$, and, then, we obtain the state $|c\rangle$ as a function of $t$. For the calculation of $H (= \langle c|\hat{H}|c\rangle)$, it is necessary to give the expectation value of $(\hat{c}^*)^r(\hat{c}^*)^s(\hat{c})^s$ and its hermite conjugate. The expectation value is given in the form

$$\langle c|(\hat{c}^*)^r(\hat{c}^*)^s(\hat{c})^s|c\rangle = (\gamma^*)^r(\gamma^2)^s \cdot \Gamma_r(s)/\Gamma. \quad (5.16)$$

Here, $\Gamma_r(s)$ is defined by

$$\Gamma_r(s) = \sum_{n=0}^{\infty} f(n + r + s)f(n + s)(n!)^{-1}(\gamma^2)^n. \quad (5.17)$$

In the case $r = 0$, $\Gamma_0(s)$ is given as

$$\Gamma_0(s) = \sum_{n=0}^{\infty} f(n + s)^2(n!)^{-1}(\gamma^2)^n = (d/d|\gamma|^2)^s \Gamma = \Gamma^{(s)}. \quad (5.18)$$

In the case $r > 0$, it is impossible to give a compact expression for $\Gamma_r(s)$ and its derivative like that given in the relation (5.18). However, with the use of the Cauchy (Schwarz) inequality, we find that $\Gamma_r(s)$ obeys

$$\Gamma_r(s) \leq \sqrt{\Gamma^{(r+s)}(s)}. \quad (5.19)$$

Therefore, if the trial state $|c\rangle$ is not too different from the boson coherent state, the relation (5.19) may be used as the equality. Namely, the equality may be assumed in (5.19).
§6. Concluding remarks

In this paper, we presented our basic idea for the deformed boson scheme in the case of one kind of boson operator. This idea suggests that the time-dependent variational method is also workable in terms of the variables which correspond to the deformed bosons. However, the deformed boson scheme realizes its real ability in the case of two kinds of boson operators. In this case, we know two algebras: the $su(2)$- and the $su(1,1)$-algebras represented in terms of the Schwinger boson representation.\(^{11}\) In such systems, we investigated the time-evolutions of the system in the framework of coherent and squeezed states.\(^2\) For this description, it would be interesting to give the deformed algebraic foundation because our treatment is systematic and general, although there are already studies on this topic such as that concerning the $su(2)$-algebra in the Schwinger boson representation.\(^{12}\) This is the topic we consider in Part (II) of this work.

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